Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 44 No. 8
August 2023

On Product and Edge Product Cordial Labeling of some Graphs

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Abstract-All the Graphs consider in this article are finite, simple and undirected In this paper we investigate the product cordial and edge product cordial labeling for some snake related graphs.

Keywords: Product Cordial graph, Edge Product Cordial graph, Double Quadrilateral Snake, Alternate Double Quadrilateral Snake
AMS Subject Classification (2010): 05C78.

1. Introduction

Product Cordial labeling was introduced by M. Sundaram, R. Ponraj and S. Somasundaram [6] in the year 2004. It is a binary vertex labeling in which the edge label is obtained as the product of its end vertices labels. This idea makes the Product Cordial labeling quite interesting in the sense that, assigning a label zero to any vertex would induce a zero labeling of all the edges incident on this vertex. The Edge Product Cordial labeling was introduced by S. K. Vaidya and C. M. Barasara [7] in the year 2012 which was inspired from the Product Cordial labeling.

2. Preliminaries

Definition 2.1

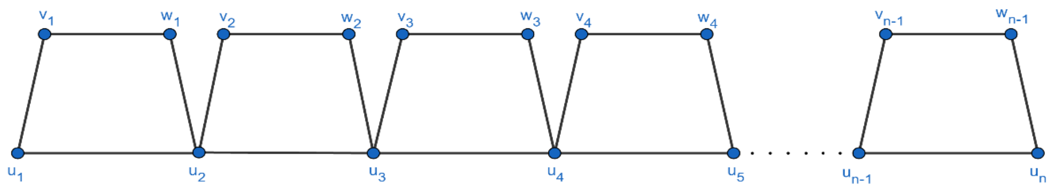
Product Cordial labeling : Consider a graph $G=(V,E)$. Consider a vertex labeling function $f:V(G)\rightarrow\{0,1\}$ which induces an edge labeling function $f^*:E(G)\rightarrow\{0,1\}$ defined as $f^*(uv) = f(u)f(v)$. Such a labeling function f is called a Product Cordial labeling of graph G, if it satisfies the conditions $|v_f(0)-v_f(1)|\leq 1$ and $|e_f(0)-e_f(1)|\leq 1$, where $v_f(i)$ denotes the number of vertices of G labeled using f and $e_f(i)$ denotes the number of edges of G labeled using f^*; $i\in\{0,1\}$. A graph which admits a Product Cordial labeling is called a Product Cordial graph.

Definition 2.2.

Edge Product Cordial labeling : Consider a graph $G=(V,E)$. Consider an edge labeling function $f:E(G)\rightarrow\{0,1\}$ which induces a vertex labeling function $f^*:V(G)\rightarrow\{0,1\}$ which defines the labeling of a vertex v as the product of the labels of all edges incident on v. Such a labeling function f is called an Edge Product Cordial labeling of graph G, if it satisfies the conditions $|v_f(0)-v_f(1)|\leq 1$ and $|e_f(0)-e_f(1)|\leq 1$, where $v_f(i)$ denotes the number of vertices of G labeled using f^* and $e_f(i)$ denotes the number of edges of G labeled using f; $i\in\{0,1\}$. A graph which admits an Edge Product Cordial labeling is called an Edge Product Cordial graph.

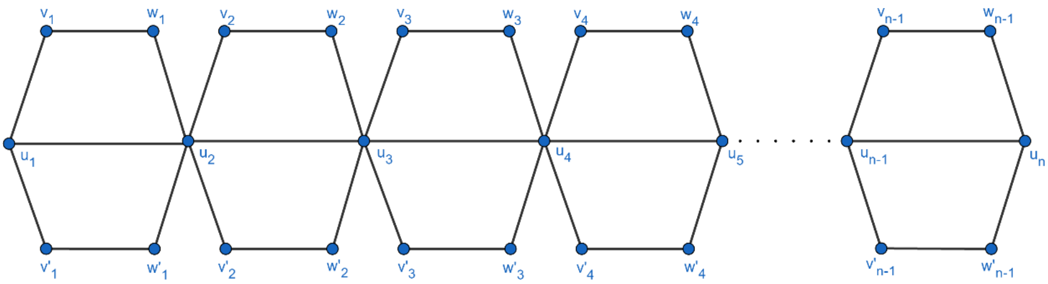
Definition 2.3.

Quadrilateral snake : Consider a path $u_1,u_2,u_3,\ldots,u_n$. Consider a graph obtained by joining successive vertices pair u_i and u_{i+1}, of the path, to new vertices v_i and w_i respectively, and connecting them with an edge v_iw_i for $i\in\{1,2,\ldots,n-1\}$. Such a graph is called a quadrilateral snake, and is denoted by QS_n. Thus, in a quadrilateral snake, every edge of the considered path gets replaced by a cycle C_4. Demonstrated below, is the figure of a quadrilateral snake with n vertices.



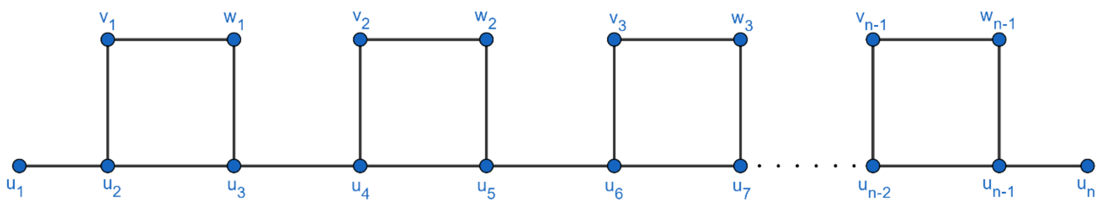
Definition 2.4.

Double quadrilateral snake : Two quadrilateral snakes that have a common path, considered together, give rise to a double quadrilateral snake, which is denoted as $D(QS_n)$. Demonstrated below, is the figure of a double quadrilateral snake with n vertices.



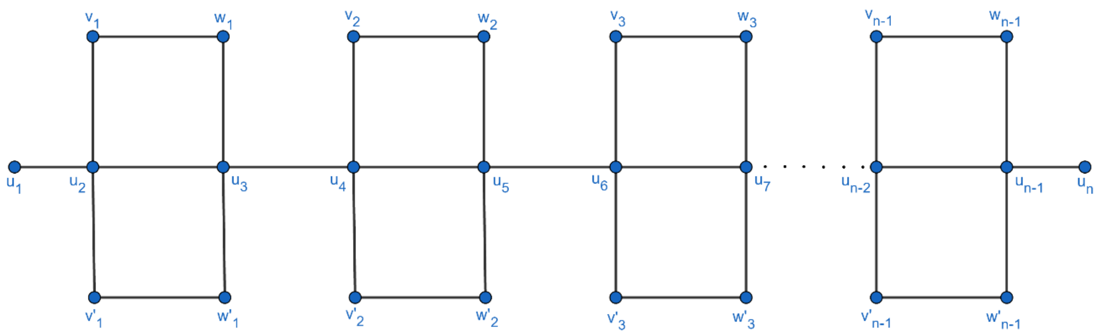
Definition 2.5.

Alternate quadrilateral snake : A graph obtained by replacing every alternate edge of a path $u_1, u_2, u_3, \dots, u_n$ by a cycle C_4, such that each successive pair of vertices of the path is adjacent, is called an alternate quadrilateral snake and is denoted by $A(QS_n)$. Further, by joining successive vertices pair u_i and u_{i+1}, to new vertices v_j and w_j respectively, where if $i \in \left\{1, 3, 5, \dots, 2\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}$ and $j = \frac{i+1}{2}$, then the alternate quadrilateral snake starts from a cycle; and if $i \in \left\{2, 4, 6, \dots, 2\left\lfloor\frac{n}{3}\right\rfloor + 2\right\}$ and $j = \frac{i}{2}$, then the alternate quadrilateral snake starts from a pendant vertex. Demonstrated below, is the figure of an alternate quadrilateral snake with even number of vertices, staring from a pendant vertex.



Definition 2.6.

Double alternate quadrilateral snake : Two alternate quadrilateral snakes that have a common path, considered together, give rise to a double alternate quadrilateral snake, which is denoted as $DA(QS_n)$. Demonstrated below, is the figure of a double alternate quadrilateral snake with even number of vertices, staring from a pendant vertex.



3. Main Results

Theorem 3.1: A path deleted double quadrilateral snake admits a Product Cordial labeling, when n is odd.

Proof : Consider a path P_n and a double quadrilateral snake generated by it. Delete all the edges of path P_n and call this graph as G. Thus, graph G becomes a series of $n-1$ cycles each of order six. Consider the vertex set and the edge set of G respectively as

$V(G) = \{u_i, v_j, w_j, u'_j, w'_j\ ; 1 \le i \le n, 1 \le j \le n-1\}$ and

$E(G) = \{v_j w_j, v'_j w'_j\ ; 1 \le j \le n-1\} \cup \{u_i v_j, u_i v'_j\ ; i = j\ , 1 \le i \le n-1\} \cup \{u_i w_j, u_i w'_j\ ; i = j+1, 1 \le i \le n\}$

Note that, $|V| = 5n-4$ and $|E| = 6n-6$

Define a vertex labeling function $f: V(G) \to \{0,1\}$ as below.

Case I : $n \equiv 1(\mathrm{mod}\ 2)$

$$f(u_i) = \begin{cases} 1\ ; 1 \le i \le \frac{n+1}{2} \\ 0\ ;\ \text{otherwise} \end{cases}$$

$$f(v_i) = f(v_i{}') = \begin{cases} 1\ ; 1 \le i \le \frac{n-1}{2} \\ 0\ ; \text{otherwise} \end{cases}$$

$$f(w_i) = f(w_i{}') = \begin{cases} 1\ ;\ 1 \le i \le \frac{n-1}{2} \\ 0\ ;\ \text{otherwise} \end{cases}$$

We now check for the conditions of Product Cordial labeling for f.

$|v_f(0)| + 1 = |v_f(1)| = \frac{5n-3}{2} \Rightarrow |v_f(0) - v_f(1)| = 1 \le 1$

$|e_f(0)| = |e_f(1)| = 3n-3 \Rightarrow |e_f(0) - e_f(1)| = 0 \le 1$

Thus, G satisfies the conditions for Product Cordial labeling in this case.

Case II : $n \equiv 0(\mathrm{mod}\ 2)$

In order to satisfy the conditions for Product Cordial labeling of G, we need to assign a label 0 to exactly $\frac{5n-4}{2}$ vertices because there is a total of $5n-4$ vertices in this case. Now, if we assign a label 0 to any vertex, it will induce a label 0 to all the edges which are incident on that vertex. Therefore, to minimize the number of edges with label 0, we assign a label 0 to the first $\frac{5n-4}{2}$ vertices in the left half of the G and assign a label 1 to the remaining $\frac{5n-4}{2}$ vertices in the right half of the G.

The above labeling pattern produces $3n+4$ edges with label 0 and $3n+2$ edges with label 1.

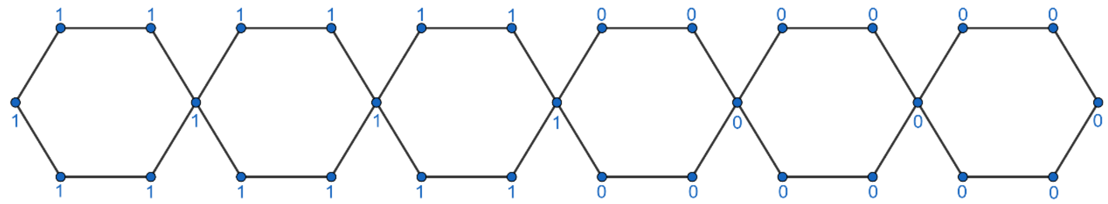
Consequently, we observe that $|e_f(0) - e_f(1)| = |(3n+4) - (3n+2)| = 2 \nleq 1$. For any other vertex labeling pattern of G, it can be easily verified that the edge labeling condition is not satisfied. In fact, the absolute difference between $e_f(0)$ and $e_f(1)$ will increase for any other pattern.

Thus, G does not satisfy the conditions for Product Cordial labeling in this case.

Thus, we observe from the above two cases that f satisfies the conditions for Product Cordial labeling of G, only when n is odd.

Hence, G admits a Product Cordial labeling, when n is odd.

Illustration 3.2 : A Product Cordial labeling of a path deleted double quadrilateral snake $D(QS_7)$ is shown in the below figure.



The path deleted double quadrilateral snake - $D(QS_7)$

Theorem 3.3 : A graph obtained from an alternate quadrilateral snake by deleting those alternate edges of the path, which leave the graph connected, admits a Product Cordial labeling, except for $n \equiv 0(\mathrm{mod}\ 4)$ when it starts from a pendant vertex; and for $n \equiv 2(\mathrm{mod}\ 4)$ when it starts from a cycle.

Proof : Consider a graph G as defined in the hypothesis. We will denote this graph as G_c if it starts from a cycle and as G_v if it starts from a pendant vertex.

Consider the vertex sets and the edge sets of the graphs G_c and G_v as follows.

$V(G_c) = \left\{u_i, v_j, v_j', w_j, w_j' \,;\, 1 \le i \le n, 1 \le j \le \left\lceil\frac{n-1}{2}\right\rceil\right\}$
$V(G_v) = \left\{u_i, v_j, v_j', w_j, w_j' \,;\, 1 \le i \le n, 1 \le j \le \left\lfloor\frac{n-1}{2}\right\rfloor\right\}$
$E(G_c) = \{u_{2i}u_{2i-1} \,;\, 1 \le i \le n\} \cup \left\{u_i v_j, u_i v_j' \,;\, j = \frac{i+1}{2}, i = 1,3,5,\dots\right\}$
$\cup \left\{u_i w_j, u_i w_j' \,;\, j = \frac{i}{2}, i = 2,4,6,\dots\right\} \cup \left\{v_j w_j, v_j' w_j' \,;\, 1 \le j \le \left\lceil\frac{n-1}{2}\right\rceil\right\}$
$E(G_v) = \{u_{2i-1}u_{2i} \,;\, 1 \le i \le n\} \cup \left\{u_i v_j, u_i v_j' \,;\, j = \frac{i}{2}, i = 2,4,6,\dots\right\}$
$\cup \left\{u_i w_j, u_i w_j' \,;\, j = \frac{i+1}{2}, i = 2,4,6,\dots\right\} \cup \left\{v_j w_j, v_j' w_j' \,;\, 1 \le j \le \left\lfloor\frac{n-1}{2}\right\rfloor\right\}$

Note that,

$|V(G_c)| = \begin{cases} 3n & ;\, n \equiv 0 (\text{mod } 2) \\ 3n-2 & ;\, n \equiv 1 (\text{mod } 2) \end{cases}$ and $|E(G_c)| = \begin{cases} \frac{7n-2}{2} \,;\, n \equiv 0 (\text{mod } 2); n > 2 \\ \frac{7n-7}{2} \,;\, n \equiv 1 (\text{mod } 2); n > 2 \end{cases}$

$|V(G_v)| = \begin{cases} 3n-4 \,;\, n \equiv 0 (\text{mod } 2) \\ 3n-2 \,;\, n \equiv 1 (\text{mod } 2) \end{cases}$ and $|E(G_v)| = \begin{cases} \frac{7n-12}{2} \,;\, n \equiv 0 (\text{mod } 2); n > 2 \\ \frac{7n-7}{2} \,;\, n \equiv 1 (\text{mod } 2); n > 2 \end{cases}$

Case I : If the graph starts from a cycle
Define a vertex labeling function $f: V(G_c) \to \{0, 1\}$ as below.
Subcase I(a) : $n \equiv 0 (\text{mod } 4)$

$$f(u_i) = \begin{cases} 1 \,;\, 1 \le i \le \frac{n}{2} \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$f(w_j) = f(w_j') = f(v_j) = f(v_j') = \begin{cases} 1 \,;\, 1 \le j \le \frac{n}{4} \\ 0 \,;\, \text{otherwise} \end{cases}$$

Subcase I(b) : $n \equiv 1 (\text{mod } 4)$

$$f(u_i) = \begin{cases} 1 \,;\, 1 \le i \le \frac{n+1}{2} \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$f(w_j) = f(w_j') = f(v_j) = f(v_j') = \begin{cases} 1 \,;\, 1 \le j \le \frac{n-1}{4} \\ 0 \,;\, \text{otherwise} \end{cases}$$

Subcase I(c) : $n \equiv 3 (\text{mod } 4)$

$$f(u_i) = \begin{cases} 1 \,;\, 1 \le i \le \frac{n-1}{2} \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$f(v_j) = f(v_j') = \begin{cases} 1 \,;\, 1 \le j \le \frac{n+1}{4} \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$f(w_j) = \begin{cases} 1 \,;\, 1 \le j \le \frac{n+1}{4} \\ 0 \,;\, \text{otherwise} \end{cases}$$

$$f(w_j') = \begin{cases} 1 \,;\, 1 \le j \le \frac{n-3}{4} \\ 0 \,;\, \text{otherwise} \end{cases}$$

Subcase I(d) : $n \equiv 2 (\text{mod } 4)$

In order to satisfy the conditions for Product Cordial labeling of G_c, we need to assign a label 0 to $\frac{3n}{2}$ vertices, because there is a total of $3n$ vertices in this case. If we assign a label 0 to any vertex, it will induce a label 0 to all edges which are incident on that vertex.

Therefore, to minimize the number of edges with label 0, we assign a label 0 to the first $\frac{3n}{2}$ vertices in the left half of G_c and a label 1 to the remaining

$\frac{3n}{2}$ vertices in the right half of G_c. This labeling pattern produces $\frac{7n+2}{4}$ edges with label 0 and $\frac{7n-6}{4}$ edges with label 1. In view of this labeling pattern, the vertex labeling condition $|v_f(0) - v_f(1)| \leq 1$ gets satisfied, but the edge labeling condition does not, because

$|e_f(0) - e_f(1)| = \left|\frac{7n+2}{4} - \frac{7n-6}{4}\right| = 2 \nleq 1$

For any other pattern of vertex labeling of G_c, it can be easily verified that the edge labeling condition is not satisfied. In fact, the absolute difference between $e_f(0)$ and $e_f(1)$ will increase for any other pattern.

Thus, G_c satisfies the conditions for Product Cordial labeling in subcases I(a), I(b) and I(c) but not in I(d).

Case II : If the graph starts from a pendant vertex

Define a vertex labeling function $f: V(G_v) \to \{0, 1\}$ as below.

Subcase II(a) : $n \not\equiv 0(\text{mod } 4)$

$$f(u_i) = \begin{cases} 1\,; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 0\,; \text{otherwise} \end{cases}$$

$$f(w_j) = f(w_j') = f(v_j) = f(v_j') = \begin{cases} 1\,; 1 \leq i \leq \left\lceil \frac{n-2}{4} \right\rceil \\ 0\,; \text{otherwise} \end{cases}$$

Subcase II(b) : $n \equiv 0(\text{mod } 4)$

In order to satisfy the conditions for Product Cordial labeling of G_v, we need to assign a label 0 to exactly $\frac{3n-4}{2}$ vertices, because there is a total of $3n-4$ vertices in this case. If we assign a label 0 to any vertex, it will induce a label 0 to all edges which are incident on that vertex. Therefore, to minimize the number of edges with label 0, we assign a label 0 to the first $\frac{3n-4}{2}$ vertices in the left half of G_2 and a label 1 to the remaining $\frac{3n-4}{2}$ vertices in the right half of G_v. This labeling pattern produces $\frac{7n-8}{4}$ edges with label 0 and $\frac{7n-16}{4}$ edges with label 1.

In view of this labeling pattern, the vertex labeling condition $|v_f(0) - v_f(1)| \leq 1$ gets satisfied, but the edge labeling condition does not, because

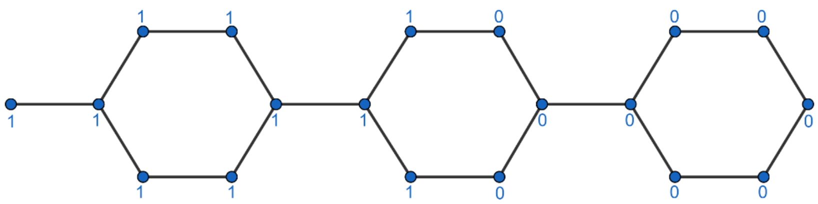
$|e_f(0) - e_f(1)| = \left|\frac{7n-8}{4} - \frac{7n-16}{4}\right| = 2 \nleq 1$

For any other pattern of vertex labeling of G_v, it can be easily verified that the edge label condition is not satisfied. In fact, the absolute difference between $e_f(0)$ and $e_f(1)$ will increase for any other pattern.

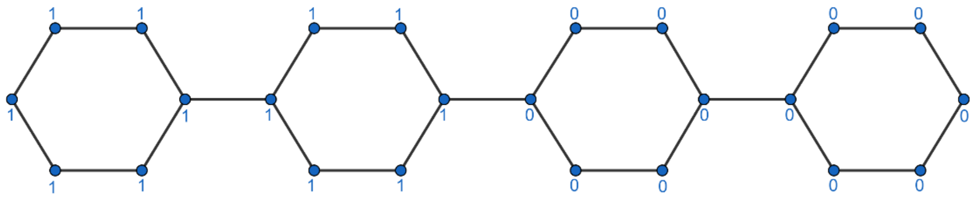
Thus, G_v satisfies the conditions for Product Cordial labeling in subcase II(a), but not in II(b).

Hence, G admits a Product Cordial labeling except for $n \equiv 0(\text{mod } 4)$, if it starts from a pendant vertex ; and for $n \equiv 2(\text{mod } 4)$, if it starts from a cycle.

Illustration 3.4: Product Cordial labelings of path deleted double alternate quadrilateral snakes with 7 and 8 vertices, starting from a pendant vertex and a cycle respectively, are shown in the below figures.



A path deleted double alternate quadrilateral snake - $A(QS_7)$



A path deleted double alternate quadrilateral snake - $A(QS_8)$

Theorem 3.5: A path deleted double quadrilateral snake admits Edge Product Cordial labeling when n is odd.

Proof : Consider a path P_n and a double quadrilateral snake generated by it. Delete all the edges of path P_n and call this graph as graph G. Thus graph G becomes a series of $n-1$ cycles of order six. Consider the vertex set and the edge set of G respectively as

$V(G) = \{u_i, v_j, w_j, u_j', w_j' \,; 1 \le i \le n, 1 \le j \le n-1\}$ and

$E(G) = \{v_j w_j, v_j' w_j' \,; 1 \le j \le n-1\} \cup \{u_i v_j, u_i v_j' \,; 1 \le i = j \le n-1\} \cup \{u_i w_j, u_i w_j' \,; 1 < i = j-1 \le n\}$

Note that, $|V| = 5n-4$ and $|E| = 6n-6$

Define an edge labeling function $f: E(G) \to \{0, 1\}$ as below.

Case I : $n \equiv 1(\text{mod } 2)$

$$f(v_j w_j) = \begin{cases} 0\,;\ 1 \le j \le \frac{n-1}{2} \\ 1\,;\ \text{otherwise} \end{cases}$$

$$f(v_j' w_j') = \begin{cases} 0\,;\ 1 \le j \le \frac{n-1}{2} \\ 1\,;\ \text{otherwise} \end{cases}$$

$$f(u_i v_j) = f(u_i v_j') = \begin{cases} 0\,;\ 1 \le i = j \le \frac{n-1}{2} \\ 1\,;\ \text{otherwise} \end{cases}$$

$$f(u_i w_j) = f(u_i w_j') = \begin{cases} 0\,;\ 1 < i = j-1 \le \frac{n-1}{2} \\ 1\,;\ \text{otherwise} \end{cases}$$

We now check for the conditions of an Edge Product Cordial labeling for f.

$|v_f(0)| = |v_f(1)| + 1 = \frac{5n-3}{2} \Rightarrow |v_f(0) - v_f(1)| = 1 \le 1$

$|e_f(0)| = |e_f(1)| = 3n-3 \Rightarrow |e_f(0) - e_f(1)| = 0 \le 1$

Thus, G satisfies the conditions for Edge Product Cordial labeling in this case.

Case II : $n \equiv 0(\text{mod } 2)$

In order to satisfy the edge labeling condition for G, we need to assign a label 0 to $3n-3$ edges, which induces $\frac{5n-2}{2}$ vertices with label 0 and $\frac{5n-6}{2}$ vertices with label 1. Consequently, we get $|v_f(0) - v_f(1)| = \left|\frac{5n-2}{2} - \frac{5n-6}{2}\right| = 2 \nleq 1$.

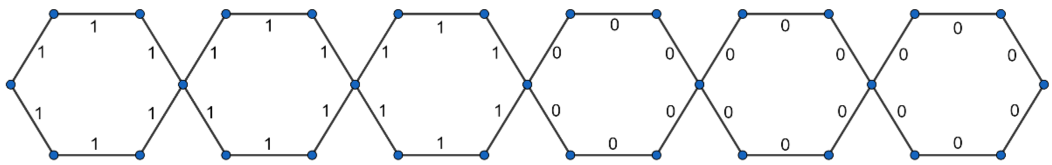
Thus, G does not satisfy the conditions for Edge Product Cordial labeling in this case.

Thus, we observe from the above two cases that f satisfies the conditions for Edge Product Cordial labeling of G only when n is odd.

Hence, G admits an Edge Product Cordial labeling when n is odd.

Illustration 3.6: An Edge Product Cordial labeling of a path deleted double quadrilateral snake $DQ(S_7)$ is shown in the below figure.

A path deleted double quadrilateral snake - $DQ(S_7)$



Theorem 3.7: A graph obtained from double alternate quadrilateral snake by deleting those alternate edges of the path P_n, which leave the graph connected, admits an Edge Product Cordial labeling, except for $n \equiv 0(\text{mod } 4)$ when it starts from a pendant vertex; and for $n \equiv 2(\text{mod } 4)$ when it starts from a cycle.

Proof : Consider a graph G as defined in the hypothesis. We will denote this graph as G_c, if it starts from a cycle and as G_v, if it starts from a pendant vertex.

Consider the vertex sets and the edge sets of the graphs G_c and G_v as follows.

$V(G_c) = \left\{u_i, v_j, v_j', w_j, w_j' \; ; 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil \right\}$

$V(G_v) = \left\{u_i, v_j, v_j', w_j, w_j' \; ; 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil \right\}$

$E(G_c) = \{u_{2i}u_{2i-1} \; ; 1 \leq i \leq n\} \cup \left\{u_i v_j, u_i v_j' \; ; j = \frac{i+1}{2}, i = 1, 3, 5, \dots \right\}$
$\cup \left\{u_i w_j, u_i w_j' \; ; j = \frac{i}{2}, i = 2, 4, 6, \dots \right\} \cup \left\{v_j w_j, v_j' w_j' \; ; 1 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil \right\}$

$E(G_v) = \{u_{2i-1}u_{2i} \; ; 1 \leq i \leq n\}$
$\cup \left\{u_i v_j, u_i v_j' \; ; j = \frac{i}{2}, i = 2, 4, 6, \dots \right\}$
$\cup \left\{u_i w_j, u_i w_j' \; ; j = \frac{i+1}{2}, i = 2, 4, 6, \dots \right\} \cup \left\{v_j w_j, v_j' w_j' \; ; 1 \leq j \leq \left\lceil \frac{n-1}{2} \right\rceil \right\}$

Note that,

$|V(G_c)| = \begin{cases} 3n & ; n \equiv 0 (\text{mod } 2) \\ 3n-2 ; & n \equiv 1 (\text{mod } 2) \end{cases}$ and

$|E(G_c)| = \begin{cases} \frac{7n-2}{2} \; ; \; n \equiv 0 (\text{mod } 2); \; n > 2 \\ \frac{7n-7}{2} \; ; \; n \equiv 1 (\text{mod } 2); \; n > 2 \end{cases}$

$|V(G_v)| = \begin{cases} 3n-4 \; ; \; n \equiv 0 (\text{mod } 2) \\ 3n-2 \; ; \; n \equiv 1 (\text{mod } 2) \end{cases}$ and

$|E(G_v)| = \begin{cases} \frac{7n-12}{2} \; ; \; n \equiv 0 (\text{mod } 2); \; n > 2 \\ \frac{7n-7}{2} \; ; \; n \equiv 1 (\text{mod } 2); \; n > 2 \end{cases}$

Case I : If the graph starts from a cycle

Define an edge labeling function $f: E(G_c) \to \{0, 1\}$ as below.

Subcase I(a) : $n \not\equiv 2(\text{mod } 4), n \neq 3$

$f(u_i u_{i+1}) = \begin{cases} 1; \; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \; i \equiv 0 (\text{mod } 2) \\ 0; \; \left\lceil \frac{n}{2} \right\rceil < i \leq n, \; i \equiv 0 (\text{mod } 2) \end{cases}$

$f(v_j w_j) = f(v_j' w_j') = \begin{cases} 1; \; 1 \leq j \leq \left\lceil \frac{n}{4} \right\rceil \\ 0; \; \left\lceil \frac{n}{4} \right\rceil < j \leq n \end{cases}$

If $i \equiv 1(\text{mod } 2)$ and $j = \frac{i+1}{2}$, then

$f(u_i v_j) = f(u_i v_j') = \begin{cases} 1; \; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \; 1 \leq j \leq \left\lceil \frac{n}{4} \right\rceil \\ 0; \; \left\lceil \frac{n}{2} \right\rceil < i \leq n, \; \left\lceil \frac{n}{4} \right\rceil < j \leq n \end{cases}$

If $i \equiv 0(\text{mod } 2)$ and $j = \frac{i}{2}$, then

$f(u_i w_j) = f(u_i w_j') = \begin{cases} 1; \; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \; 1 \leq j \leq \left\lceil \frac{n}{4} \right\rceil \\ 0; \; \left\lceil \frac{n}{2} \right\rceil < i \leq n, \; \left\lceil \frac{n}{4} \right\rceil < j \leq n \end{cases}$

We now check for the conditions of Edge Product Cordial labeling for f.

If $n \equiv 0(\text{mod } 4)$, then

$|v_f(0)| = |v_f(1)| = \frac{3n}{2} \Rightarrow \left|v_f(0) - v_f(1)\right| = 0 \leq 1$

$\left|e_f(0)\right| + 1 = |e_f(1)| = \frac{7n}{4} \Rightarrow \left|e_f(0) - e_f(1)\right| = 1 \leq 1$

If $n \equiv 1(\text{mod } 4)$, then

$\left|v_f(0)\right| = \left|v_f(1)\right| + 1 = \frac{3n-1}{2} \Rightarrow \left|v_f(0) - v_f(1)\right| = 1 \leq 1$

$\left|e_f(0)\right| = |e_f(1)| = \frac{7n-7}{4} \Rightarrow \left|e_f(0) - e_f(1)\right| = 0 \leq 1$

If $n \equiv 3(\text{mod } 4)$, then

- for $n \neq 3$

$|v_f(0)| = |v_f(1)| = \frac{3n-1}{2} \Rightarrow \left|v_f(0) - v_f(1)\right| = 0 \leq 1$

$\left|e_f(0)\right| + 1 = |e_f(1)| = \frac{7n-5}{4} \Rightarrow \left|e_f(0) - e_f(1)\right| = 1 \leq 1$

- for $n = 3$

$f(u_2 w_1) = f(u_2 w_1') = f(u_2 u_3) = 0$

$f(v_1 w_1) = f(v_1' w_1') = f(u_1 v_1) = f(u_1 v_1') = 1$

We now check for the conditions of the Edge Product Cordial labeling for f.

$\left|v_f(0)\right| = \left|v_f(1)\right| + 1 = 4 \Rightarrow \left|v_f(0) - v_f(1)\right| = 1 \leq 1$

$\left|e_f(0)\right| = \left|e_f(1)\right| = 4 \Rightarrow \left|e_f(0) - e_f(1)\right| = 0 \leq 1$

Subcase I(b) : $n \equiv 2(\text{mod } 4)$

In order to satisfy the conditions for Edge Product Cordial labeling of G_c, we need to assign labels 1 and 0 to equal number of edges as we have even number of edges in this subcase. If we arrange

$e_f(0) = e_f(1) = \frac{7n-4}{4}$, then resulting vertex labeling arrangement will be $v_f(0) = v_f(1) + 2$, since this labeling pattern produces $\frac{3n+2}{2}$ vertices with label 0 and $\frac{3n-2}{2}$ vertices with label 1.

In view of this labeling pattern, the edge labeling condition $|e_f(0) - e_f(1)| \leq 1$ gets satisfied, but the vertex labeling condition does not, because $|v_f(0) - v_f(1)| = \left|\frac{3n+2}{2} - \frac{3n-2}{2}\right| = 2 \nleq 1$.

For any other pattern of edge labeling of G_c, it can be easily verified that the vertex labeling condition is not satisfied. In fact, the absolute difference between $v_f(0)$ and $v_f(1)$ will increase for any other pattern.

Thus, G_c satisfies the conditions for Edge Product Cordial labeling in subcase I(a) but not in I(b).

Case II : If the graph starts from a pendant vertex

Define a vertex labeling function $g: E(G_v) \to \{0, 1\}$ as below.

Subcase II(a) : $n \not\equiv 0(\text{mod } 4), n \neq 3$

$$g(u_i v_j) = g(u_i v_j') = \begin{cases} 0; 1 \leq j \leq \left\lceil \frac{n-2}{4} \right\rceil, i = 2j \\ 1; \left\lceil \frac{n-2}{4} \right\rceil < j \leq \left\lceil \frac{n-2}{2} \right\rceil, i = 2j \end{cases}$$

$$g(u_i w_j) = g(u_i w_j') = \begin{cases} 0; 1 \leq j \leq \left\lfloor \frac{n-1}{4} \right\rfloor, i = 2j+1 \\ 1; \left\lfloor \frac{n-1}{4} \right\rfloor < j \leq \left\lceil \frac{n-2}{2} \right\rceil, i = 2j+1 \end{cases}$$

$$g(v_j w_j) = g(v_j' w_j') = \begin{cases} 0; 1 \leq j \leq \left\lfloor \frac{n-1}{4} \right\rfloor \\ 1; \left\lfloor \frac{n-1}{4} \right\rfloor < j \leq \left\lceil \frac{n-2}{2} \right\rceil \end{cases}$$

$$g(u_{2i-1} u_{2i}) = \begin{cases} 0; 1 \leq i \leq \left\lceil \frac{n-2}{4} \right\rceil \\ 1; \left\lceil \frac{n-2}{4} \right\rceil < i \leq n \end{cases}$$

We now check for the conditions of Edge Product Cordial labeling for f.

If $n \equiv 1(\text{mod } 4)$, then

$$|v_f(0)| = |v_f(1)| + 1 = \frac{3n-1}{2} \Rightarrow |v_f(0) - v_f(1)| = 1 \leq 1$$

$$|e_f(0)| = |e_f(1)| = \frac{7n-7}{4} \Rightarrow |e_f(0) - e_f(1)| = 0 \leq 1$$

If $n \equiv 2(\text{mod } 4)$, then

$$|v_f(0)| = |v_f(1)| = \frac{3n-4}{2} \Rightarrow |v_f(0) - v_f(1)| = 0 \leq 1$$

$$|e_f(0)| + 1 = |e_f(1)| = \frac{7n-10}{4} \Rightarrow |e_f(0) - e_f(1)| = 1 \leq 1$$

If $n \equiv 3(\text{mod } 4)$, then

- for $n \neq 3$

$$|v_f(0)| = |v_f(1)| + 1 = \frac{3n-1}{2} \Rightarrow |v_f(0) - v_f(1)| = 1 \leq 1$$

$$|e_f(0)| + 1 = |e_f(1)| = \frac{7n-5}{4} \Rightarrow |e_f(0) - e_f(1)| = 1 \leq 1$$

- for $n = 3$

$$f(u_2 v_1) = f(u_2 v_1') = f(u_1 u_2) = 0$$

$$f(u_3 w_1) = f(u_3 w_1') = f(v_1 w_1) = f(v_1' w_1') = 1$$

We now check for the conditions of the Edge Product Cordial labeling for f.

$$|v_f(0)| = |v_f(1)| + 1 = 4 \Rightarrow |v_f(0) - v_f(1)| = 1 \leq 1$$

$$|e_f(0)| + 1 = |e_f(1)| = 4 \Rightarrow |e_f(0) - e_f(1)| = 1 \leq 1$$

Subcase II(b) : $n \equiv 0(\text{mod } 4)$

In order to satisfy the conditions for Edge Product Cordial labeling of G_v, we need to assign labels 1 and 0 to equal number of edges as we have even number of edges in this subcase.

If we arrange $e_f(0) = e_f(1) = \frac{7n-12}{4}$, then resulting vertex labeling arrangement will be $v_f(0) = v_f(1) + 2$, since this labeling pattern produces $\frac{3n-2}{2}$ vertices with label 0 and $\frac{3n-6}{2}$ vertices with label 1.

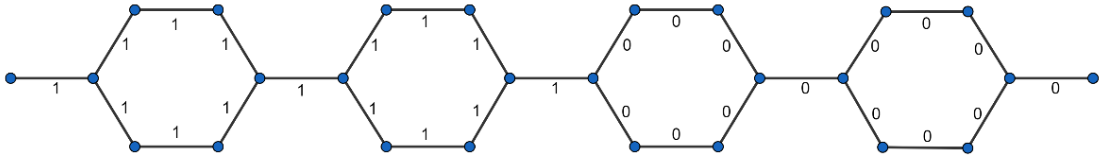
In view of this labeling pattern, the edge labeling condition $|e_f(0) - e_f(1)| \leq 1$ gets satisfied, but the vertex labeling condition does not, because $|v_f(0) - v_f(1)| = \left|\frac{3n-2}{2} - \frac{3n-6}{2}\right| = 2 \nleq 1$.

For any other pattern of edge labeling of G_v, it can easily verify that the vertex label condition is not satisfied. In fact, the absolute difference between $v_f(0)$ and $v_f(1)$ will increase for any other pattern.

Thus, G_v satisfies the conditions for Edge Product Cordial labeling in subcase II(a) but not in II(b).

Hence, G admits an Edge Product Cordial labeling, except for $n \equiv 0(\text{mod } 4)$, if it starts from a vertex; and for $n \equiv 2(\text{mod } 4)$, if it starts from a cycle.

Illustration 3.8: An Edge Product Cordial labeling of a path deleted double alternate quadrilateral snake $A(QS_{10})$ starting from a pendant vertex is shown in the below figure.



A path deleted double alternate quadrilateral snake $A(QS_{10})$

4. Concluding Remarks

In this paper, we have proved that a path deleted double quadrilateral snake and a path deleted double alternate quadrilateral snake are Product Cordial and Edge Product Cordial graphs.

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