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Relation Between Second and Third Coefficients of Taylor's Expansion of Functions Belonging to A Subclass of Univalent Functions

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Abstract: The objective of this paper is to present new class and various subclasses of that class of univalent functions. We discuss approximations on the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$, and the Fekete–Szego problem is also considered for the new class and its subclasses of functions introduced. We denote these classes by$KS^*(f, f', \alpha,)$ and will be defined as$KS^*(f, f', \alpha) = \left\{ f\,(z); \frac{zf'(z)}{f(z)}\left[1 - \alpha + \alpha\,\frac{\{zf'(z)\}'}{f'(z)}\right] = \left(\frac{1+w(z)}{1+w(z)}\right); z \in E \right\}$. $KS^*(f, f', \alpha, A, B\ \delta)$will give its various subclasses for different values of the parameters $A, B\ \delta$ and will be defined as$KS^*(f, f', \alpha, A, B\ \delta) = \left\{ f\,(z); \frac{zf'(z)}{f(z)}\left[1 - \alpha + \alpha\,\frac{\{zf'(z)\}'}{f'(z)}\right] = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}; z \in E \right\}$. We obtain a Fekete–Szegö inequality for certain normalized analytic function belonging to these classes and subclasses defined on the open unit disk. As a special case of this result, the Fekete–Szegö inequality for this class of functions defined through extremal function which makes the inequality justified strongly is obtained. We establish the coefficient inequality proved by Fekete and Szegö [5] in 1933 by using the analytic functions of the formf(z) = z $+\sum_{k=2}^{\infty} a_k\, z^k$belonging to the newly established class of analytic functions.

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I. Introduction

It is well known that the coefficient problems in geometric function theory have ever demanding attention. Bieberbach was the first to prove the Coefficient result $|a_2| \leq 2$ for functions $f \in \mathcal{S}$ i.e. class of Starlike functions and equality holds if and only if f is a rotation of the Koebe function. Having proved that $|a_2| \leq 2$, Bieberbach conjectured that the coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq n\ for\ n \geq 2.$ Strict inequality holds for all values of n unless f is the Koebe function or one of its rotations. Loewner proved the Bieberbach conjecture for $n = 3$, Garabedian and Schiffer proved it for $n = 4$, Pederson and Schiffer proved it for $n = 4$and Pederson proved the Bieberbach conjecture for $n = 6$. Since the Bieberbach conjecture was difficult to settle, several authors have considered different classes defined by different geometric conditions. Notable among them are the classes of convex function, Starlike function and Close-to-convex function. At last, Louis de Branges proved the Bieberbach conjecture completely.

In this paper, we will be dealing with geometric function theory that is abranch of complex analysis which deals with the regular functions geometrically. The pillar of this theory is Riemann Mapping Theorem which was proved in 19th century. It initiated its roots in the work of great mathematician Koebe [10] in 1907, who stated that "An analytic function which is univalent has properties of conformal mapping i.e. angle preserving property". From this theorem, Bieberbach conjecture was proved. This was given by L. Bieberbach[2] in 1916 but proved finally by Louis De Branges [3] in 1985 and while tackling with this conjecture, an equality arises, which is called FeketeSzegö Inequality given by Fekete and Szegö [5].

The inequality which is for the function *f (z)*∈*A* and based on Bieberbach conjecture,is named as FeketeSzegö Inequality, which states that if *f(z)* is a function of type

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which is univalent in *E*, then

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu \; if \mu \le 0 \\ 1 + 2\,exp\left(\frac{-2\mu}{1-\mu}\right) \; if \; 0 \le \mu \le 1 \\ 4\mu - 3 \; if \mu \ge 1 \end{cases}$$

This is an inequality which is related to univalent analytic functions $[8], [16], [18] - [40]$ and gives the necessary condition to map the unit disk of a complex plane injectively to the complex plane. It gives the relation between second and third coefficient of univalent analytic function.

In order to prove our result, let us explain some classes and some basic results related to our work:-

*A*consists all those functions f which are analytic in open unit disc $E = \{ z \in C: |z| < 1\}$ and are of the form$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with normalization conditions $f(0) = 0,\; f'(0) = 1.$

S be the family of functions *f*which are univalent in the open disk $\{ z \in C: |z| < 1\}$ with conditions $f(0) = 0, f'(0) = 1$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$

$S^*(\phi)$ is the class of functions in $f \in S$,for which$\frac{z f'(z)}{f(z)} \prec \phi$ (z), given by Ma and Minda [12] and is known as class of starlike functions.

$t(z)$be a family of analytic functions in the open unit disk *E*, having functions of the form$t(z) = \sum_{n=1}^{\infty} c_n z^n$, it is a class of bounded analytic function denoted by U, if the conditions $t(0) = 0$and $|t(z)| < 1\, hold$. The necessary conditions for any function to be bounded analytic function are $|c_1| \le 1$, $|c_2| \le 1-|c_1|^2$; which were given by Miller et. al. [13].

Let *u (z)* and *v(z)* are two analytic functions in *E.* If there existsaSchwarzian function *F(z)* (analytic in *E*) in such a way that $|F(z)| < 1$, $F(0) = 0$ and $u(z) = v(F(z))$; $z \in E$ then the function *u*(*z*) is subordinate to *v(z)* written as $u(z) \prec v(z)$ and this concept (called subordination) was given by Lindelof [11].

We introduce more subclasses$KS^*(f, f', \alpha)$,$KS^*(f, f', \alpha, \delta)$, $KS^*(f, f', \alpha, A, B)$, $KS^*(f, f', \alpha, A, B\ \delta)$of $KS^*(f, f', \alpha)$containingfunctions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; defined as

$$KS^*(f, f', \alpha) = \left\{ f(z); \frac{zf'(z)}{f(z)} \left[1 - \alpha + \alpha \frac{\{zf'(z)\}'}{f'(z)} \right] = \left(\frac{1 + w(z)}{1 - w(z)} \right); z \in E \right\}.$$

$$KS^*(f, f', \alpha, \delta) = \left\{ f(z); \frac{zf'(z)}{f(z)} \left[1 - \alpha + \alpha \frac{\{zf'(z)\}'}{f'(z)} \right] = \left(\frac{1 + w(z)}{1 - w(z)} \right)^{\delta}; z \in E \right\}.$$

$$KS^*(f, f', \alpha, A, B) = \left\{ f(z); \frac{zf'(z)}{f(z)} \left[1 - \alpha + \alpha \frac{\{zf'(z)\}'}{f'(z)} \right] = \left(\frac{1 + Aw(z)}{1 + Bw(z)} \right); z \in E \right\}.$$

$$KS^*(f, f', \alpha, A, B\ \delta) = \left\{ f(z); \frac{zf'(z)}{f(z)} \left[1 - \alpha + \alpha \frac{\{zf'(z)\}'}{f'(z)} \right] = \left(\frac{1 + Aw(z)}{1 + Bw(z)} \right)^{\delta}; z \in E \right\}.$$

II. Main Results

THEOREM1:- Let *f (z)*$\in KS^*(f, f', \alpha)$ and ϕ *(z)* = $\frac{1+w(z)}{1-w(z)}$;*w(z)* is a Schwarzian function , then

$$|(3\alpha + 1)a_3 - \mu(2\alpha + 1)^2 a_2^2| \le \begin{cases} \frac{2\alpha+3}{2\alpha+1} - 4\mu; \mu \le \frac{1}{2(2\alpha+1)}; \\ 1; \frac{1}{2(2\alpha+1)} \le \mu \le \frac{\alpha+1}{2\alpha+1}; \\ 4\mu - \frac{2\alpha+3}{2\alpha+1}; \mu \ge \frac{\alpha+1}{2\alpha+1}. \end{cases}$$

The result is sharp

PROOF: - By definition of $KS^*(f, f', \alpha)$, given by (1.1)

and using $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \dots$,

$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$

$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots$ we get

$\{1 + (2\alpha + 1)a_2 z + \{2(3\alpha + 1)a_3 - (2\alpha + 1)a_2^2\}z^2 + \dots\} = 1 + 2z + 2(c_2 + c_1^2)z^2 + \dots$

Comparing like coefficients, one can easily obtain

$a_2 = \frac{2c_1}{2\alpha+1}anda_3 = \frac{1}{(3\alpha+1)}\left\{c_2 + \frac{2\alpha+3}{2\alpha+1} c_1^2\right\}$

Using these values of a_2 and a_3 , one can construct

$$(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2 = c_2 + \left(\frac{2\alpha+3}{2\alpha+1}- 4\mu\right)c_1^2$$

After applying mode on both sides, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le |c_2| + \left|\frac{2\alpha+3}{2\alpha+1}- 4\mu\right||c_1^2|$$

Using $|c_2| \le 1-|c_1|^2$, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 1 + \left\{\left|\frac{2\alpha+3}{2\alpha+1}- 4\mu\right|-1\right\}|c_1|^2$$

Case 1:- If $\mu \le \frac{2\alpha+3}{4(2\alpha+1)}$. In this case, we obtain

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 1+\left\{\frac{2}{2\alpha+1}-4\mu\right\}|c_1|^2$$

Subcase – 1 (a):- When $\mu \le \frac{2}{4(2\alpha+1)}$

By using $|c_1| \le 1$, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le \frac{2\alpha+3}{2\alpha+1}- 4\mu \tag{1.2}$$

Subcase – 1 (b):- When $\mu \ge \frac{2}{4(2\alpha+1)}$. It gives us

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 1 \tag{1.3}$$

Case – 2:- If $\boldsymbol{\mu} \ge \frac{2\alpha+3}{4(2\alpha+1)}$.

We can easily obtain

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 4\mu-\frac{4(\alpha+1)}{2\alpha+1}$$

Subcase-2 (a):- When $\mu \ge \frac{\alpha+1}{2\alpha+1}$. It yields

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 1 \tag{1.4}$$

Subcase – 2 (b):- When $\mu \le \frac{\alpha+1}{2\alpha+1}$. Solving, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le 4\mu-\frac{2\alpha+3}{2\alpha+1} \tag{1.5}$$

Combining (1.2), (1.3), (1.4) and (1.5) , we get the required result.

Corollary 2:- Putting $\alpha = 0$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} 3-4\mu;\ \mu \le \frac{1}{2}; \\ 1;\ \frac{1}{2} \le \mu \le 1; \\ 4\mu-3;\ \mu \ge 1. \end{cases}$$

which is the required resultofFeketeSzego inequality for class of starlike functions.

THEOREM3:- Let $f(z) \in KS^*(f, f', \alpha, \delta)$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$;*w(z)* is a Schwarzian function , then

$$\left|\frac{3\alpha+1}{\delta}a_3-\mu(2\alpha+1)^2a_2^2\right| \le \begin{cases} \frac{\delta+2\delta(\alpha+1)}{2\alpha+1}-4\delta^2\mu;\ \mu \le \frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)\delta^2}; \\ 1;\ \frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)\delta^2} \le \mu \le \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)\delta^2}; \\ 4\delta^2\mu-\frac{\delta+2\delta(\alpha+1)}{2\alpha+1};\ \mu \ge \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)\delta^2}. \end{cases}$$

The result is sharp

PROOF: - By definition of $KS^*(f, f', \alpha, \delta)$, given by (1.1)

and using *w(z)* = $c_1z + c_2z^2 + c_3z^3 + \dots$,

f(z) = $z + a_2z^2 + a_3z^3 + \dots$

f ′ (z) = $1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots$ we get

$$\{1+(2\alpha+1)a_2z+\{2(3\alpha+1)a_3-(2\alpha+1)a_2^2\}z^2 + \dots\} = 1+2\delta z+ 2\delta(c_2+\delta c_1^2)z^2+\dots$$

Comparing like coefficients, one can easily obtain

$a_2 = \frac{2\delta c_1}{2\alpha+1}$ and $a_3 = \frac{\delta}{(3\alpha+1)}\left\{c_2 + \frac{\delta+2\delta(\alpha+1)}{2\alpha+1}c_1^2\right\}$

Using these values of a_2 and a_3 , one can construct

$$\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2 = c_2 + \left(\frac{\delta+2\delta(\alpha+1)}{2\alpha+1} - 4\delta^2\mu\right)c_1^2$$

After applying mode on both sides, we get

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le |c_2| + \left|\frac{\delta+2\delta(\alpha+1)}{2\alpha+1} - 4\delta^2\mu\right||c_1^2|$$

Using $|c_2| \le 1 - |c_1|^2$, we get

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 1 + \left\{\left|\frac{\delta+2\delta(\alpha+1)}{2\alpha+1} - 4\delta^2\mu\right| - 1\right\}|c_1|^2$$

Case 1:- If $\mu \le \frac{A\delta-2\delta B(\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$. In this case, we obtain

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 1 + \left\{\frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{2\alpha+1} - 4\delta^2\mu\right\}|c_1|^2$$

Subcase – 1 (a):- When $\mu \le \frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)}$

By using $|c_1| \le 1$, we get

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le \frac{\delta+2\delta(\alpha+1)}{2\alpha+1} - 4\delta^2\mu \qquad (3.1)$$

Subcase – 1 (b):- When $\mu \ge \frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)}$. It gives us

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 1 \qquad (3.2)$$

Case2:- If $\mu \ge \frac{\delta+2\delta(\alpha+1)}{4(2\alpha+1)\delta^2}$.

We can easily obtain

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 4\delta^2\mu - \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{2\alpha+1}$$

Subcase-2 (a):- When $\mu \ge \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)}$. It yields

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 1 \qquad (3.3)$$

Subcase – 2 (b):- When $\mu \le \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)}$. Solving, we get

$$\left|\frac{3\alpha+1}{\delta}a_3 - \mu(2\alpha+1)^2 a_2^2\right| \le 4\delta^2\mu - \frac{\delta+2\delta(\alpha+1)}{2\alpha+1} \qquad (3.4)$$

Combining (3.1), (3.2), (3.3) and (3.4) , we get the required result.

Corollary 4:- Putting $\delta = 1$, the result becomes

$$|(3\alpha+1)a_3 - \mu(2\alpha+1)^2 a_2^2| \le \begin{cases} \frac{2\alpha+3}{2\alpha+1} - 4\mu; \mu \le \frac{1}{2(2\alpha+1)}; \\ 1; \frac{1}{2(2\alpha+1)} \le \mu \le \frac{\alpha+1}{2\alpha+1}; \\ 4\mu - \frac{2\alpha+3}{2\alpha+1}; \mu \ge \frac{\alpha+1}{2\alpha+1}. \end{cases}$$

which is the required resultof FeketeSzego inequality in Theorem 1 above.

Corollary 5:- Putting $\alpha = 0$, the result becomes

$$\left|\frac{1}{\delta}a_3 - \mu a_2^2\right| \le \begin{cases} \delta(3-4\delta\mu); \mu \le \frac{3\delta-1}{4\delta^2}; \\ 1; \frac{3\delta-1}{4\delta^2} \le \mu \le \frac{3\delta+1}{4\delta^2}; \\ \delta(4\delta\mu-3); \mu \ge \frac{3\delta+1}{4\delta^2}. \end{cases}$$

Corollary 6:- Putting $\alpha = 0, \delta = 1$, the result becomes

$$|a_3-\mu a_2^2| \leq \begin{cases} 3-4\mu; \mu \leq \frac{1}{2}; \\ 1; \frac{1}{2} \leq \mu \leq 1; \\ 4\mu-3; \mu \geq 1. \end{cases}$$

which is the required resultof FeketeSzego inequality for class of starlike functions.

THEOREM7:- Let $f(z) \in KS^*(f, f', \alpha, A, B)$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$;*w(z)* is a Schwarzian function , then

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq \begin{cases} \frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu; \mu \leq \frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}; \\ 1; \frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)(A-B)^2} \leq \mu \leq \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}; \\ (A-B)^2\mu-\frac{A-2B(\alpha+1)}{2\alpha+1}; \mu \geq \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}. \end{cases}$$

The result is sharp

PROOF: - By definition of $KS^*(f, f', \alpha, A, B)$, given by (1.1)
and using *w(z)* = $c_1z + c_2z^2 + c_3z^3 + \dots$,
f(z) = $z + a_2z^2 + a_3z^3 + \dots$
f ′ (z) = $1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots$ we get

$$\{1+(2\alpha+1)a_2z+\{2(3\alpha+1)a_3-(2\alpha+1)a_2^2\}z^2+\dots\} = 1+(A-B)z+(A-B)(c_2-Bc_1^2)z^2+\dots$$

Comparing like coefficients, one can easily obtain

$a_2 = \frac{(A-B)c_1}{2\alpha+1}$ and $a_3 = \frac{(A-B)}{2(3\alpha+1)}\left\{c_2+\frac{A-2B(\alpha+1)}{2\alpha+1}c_1^2\right\}$

Using these values of a_2 and a_3 , one can construct

$$\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2 = c_2+\left(\frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu\right)c_1^2$$

After applying mode on both sides, we get

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq |c_2|+\left|\frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu\right||c_1^2|$$

Using $|c_2| \leq 1-|c_1|^2$, we get

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1+\left\{\left|\frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu\right|-1\right\}|c_1|^2$$

Case 1:- If $\mu \leq \frac{A-2B(\alpha+1)}{(2\alpha+1)(A-B)^2}$. In this case, we obtain

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1+\left\{\frac{A-2B(\alpha+1)-(2\alpha+1)}{2\alpha+1}-(A-B)^2\mu\right\}|c_1|^2$$

Subcase – 1 (a):- When $\mu \leq \frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)(A-B)^2}$

By using $|c_1| \leq 1$, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \leq \frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu \quad (7.1)$$

Subcase – 1 (b):- When $\mu \geq \frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)(A-B)^2}$. It gives us

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 \quad (7.2)$$

Case – 2:- If **μ** $\geq \frac{A-2B(\alpha+1)}{(2\alpha+1)(A-B)^2}$.

We can easily obtain

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq (A-B)^2\mu-\frac{A-2B(\alpha+1)+(2\alpha+1)}{2\alpha+1}$$

Subcase-2 (a):- When $\mu \geq \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}$. It yields

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 \quad (7.3)$$

Subcase – 2 (b):- When$\mu \le \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}$. Solving, we get

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \le (A-B)^2\mu-\frac{A-2B(\alpha+1)}{2\alpha+1} \quad (7.4)$$

Combining (7.1), (7.2), (7.3) and (7.4) , we get the required result.

Corollary 8:- Putting α= 0, the result becomes

$$\left|\frac{2}{(A-B)}a_3-\mu a_2^2\right| \le \begin{cases} A-2B-(A-B)^2\mu;\ \mu \le \frac{A-2B-1}{(A-B)^2}; \\ 1;\frac{A-2B-1}{(A-B)^2} \le \mu \le \frac{A-2B+1}{(A-B)^2}; \\ (A-B)^2\mu-A-2B;\ \mu \ge \frac{A-2B+1}{(A-B)^2}. \end{cases}$$

which is the required result of FeketeSzego inequality for the class $S^*(A,B)$.

Corollary 9:- Putting $A=1, B=-1$, the result becomes

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le \begin{cases} \frac{2\alpha+3}{2\alpha+1}-4\mu;\ \mu \le \frac{1}{2(2\alpha+1)}; \\ 1;\frac{1}{2(2\alpha+1)} \le \mu \le \frac{\alpha+1}{2\alpha+1}; \\ 4\mu-\frac{2\alpha+3}{2\alpha+1};\ \mu \ge \frac{\alpha+1}{2\alpha+1}. \end{cases}$$

which is the required resultof FeketeSzego inequality in Theorem 1 above.

Corollary 10:- Putting $\alpha=0, A=1, B=-1$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} 3-4\mu;\ \mu \le \frac{1}{2}; \\ 1;\frac{1}{2} \le \mu \le 1; \\ 4\mu-3;\ \mu \ge 1. \end{cases}$$

which is the required resultof FeketeSzego inequality for class of starlike functions.

THEOREM11:- Let *f (z)*$\in KS^*(f, f', \alpha, A, B\ \delta)$ and ϕ *(z)* = $\frac{1+w(z)}{1-w(z)}$*;w(z)* is a Schwarzian function , then

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right|$$

$$\le \begin{cases} \frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu;\ \mu \le \frac{A\delta-2\delta B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}; \\ 1;\frac{A\delta-2\delta B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2} \le \mu \le \frac{A\delta-2\delta B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}; \\ \delta^2(A-B)^2\mu-\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1};\ \mu \ge \frac{A\delta-2\delta B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}. \end{cases}$$

The result is sharp

PROOF: - By definition of $S^*(f, f', \alpha, A, B\ \delta)$, given by (1.1)

and using *w(z)* = $c_1z + c_2z^2 + c_3z^3 + \dots$,

f(z) = $z + a_2z^2 + a_3z^3 + \dots$

f ′ (z) = $1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots$ we get

$\{1+(2\alpha+1)a_2z+\{2(3\alpha+1)a_3-(2\alpha+1)a_2^2\}z^2+\dots\} = 1+\delta(A-B)z+\delta(A-B)(c_2-\delta Bc_1^2)z^2 +$
…

Comparing like coefficients, one can easily obtain

$a_2 = \frac{\delta(A-B)c_1}{2\alpha+1}anda_3 = \frac{\delta(A-B)}{2(3\alpha+1)}\left\{c_2+\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}c_1^2\right\}$

Using these values of a_2 and a_3 , one can construct

$$\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2 = c_2+\left(\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu\right)c_1^2$$

After applying mode on both sides, we get

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq |c_2| + \left|\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu\right||c_1^2|$$

Using $/c_2/ \leq 1-|c_1|^2$, we get

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 + \left\{\left|\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu\right|-1\right\}|c_1|^2$$

Case 1:- If $\mu \leq \frac{A\delta-2\delta B(\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$. In this case, we obtain

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 + \left\{\frac{A\delta-2\delta B(\alpha+1)-(2\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu\right\}|c_1|^2$$

Subcase – 1 (a):- When $\mu \leq \frac{A\delta-2\delta B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$

By using $|c_1| \leq 1$, we get

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \leq \frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1}-\delta^2(A-B)^2\mu \qquad (11.1)$$

Subcase – 1 (b):- When $\mu \geq \frac{A\delta-2\delta B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$. It gives us

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 \qquad (11.2)$$

Case – 2:- If $\mu \geq \frac{A\delta-2\delta B(\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$.

We can easily obtain

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq \delta^2(A-B)^2\mu-\frac{A\delta-2\delta B(\alpha+1)+(2\alpha+1)}{2\alpha+1}$$

Subcase-2 (a):- When $\mu \geq \frac{A\delta-2\delta B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$. It yields

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq 1 \qquad (11.3)$$

Subcase – 2 (b):- When $\mu \leq \frac{A\delta-2\delta B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}$. Solving, we get

$$\left|\frac{2(3\alpha+1)}{\delta(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq \delta^2(A-B)^2\mu-\frac{A\delta-2\delta B(\alpha+1)}{2\alpha+1} \qquad (11.4)$$

Combining (11.1), (11.2), (11.3) and (11.4) , we get the required result.

Corollary 12:- Putting $\delta = 1$, the result becomes

$$\left|\frac{2(3\alpha+1)}{(A-B)}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq \begin{cases} \frac{A-2B(\alpha+1)}{2\alpha+1}-(A-B)^2\mu;\ \mu \leq \frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)\delta^2(A-B)^2}; \\ 1;\frac{A-2B(\alpha+1)-(2\alpha+1)}{(2\alpha+1)(A-B)^2} \leq \mu \leq \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}; \\ (A-B)^2\mu-\frac{A-2B(\alpha+1)}{2\alpha+1};\ \mu \geq \frac{A-2B(\alpha+1)+(2\alpha+1)}{(2\alpha+1)(A-B)^2}. \end{cases}$$

which is the required result of FeketeSzego inequality proved in Theorem 5.

Corollary 13:- Putting $A = 1, B = -1$, the result becomes

$$\left|\frac{3\alpha+1}{\delta}a_3-\mu(2\alpha+1)^2a_2^2\right| \leq \begin{cases} \frac{\delta+2\delta(\alpha+1)}{2\alpha+1}-4\delta^2\mu;\ \mu \leq \frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)\delta^2}; \\ 1;\frac{\delta+2\delta(\alpha+1)-(2\alpha+1)}{4(2\alpha+1)\delta^2} \leq \mu \leq \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)\delta^2}; \\ 4\delta^2\mu-\frac{\delta+2\delta(\alpha+1)}{2\alpha+1};\ \mu \geq \frac{\delta+2\delta(\alpha+1)+(2\alpha+1)}{4(2\alpha+1)\delta^2}. \end{cases}$$

which is the required result of FeketeSzego inequality proved in Theorem 1.

Corollary 14:- Putting $A = 1, B = -1, \delta = 1$, the result becomes

$$|(3\alpha+1)a_3-\mu(2\alpha+1)^2a_2^2| \le \begin{cases} \frac{2\alpha+3}{2\alpha+1}-4\mu; \mu \le \frac{1}{2(2\alpha+1)}; \\ 1; \frac{1}{2(2\alpha+1)} \le \mu \le \frac{\alpha+1}{2\alpha+1}; \\ 4\mu-\frac{2\alpha+3}{2\alpha+1}; \mu \ge \frac{\alpha+1}{2\alpha+1}. \end{cases}$$

which is the required resultof FeketeSzego inequality in Theorem 1 above.

Corollary 15:- :- Putting $A=1, B=-1, \delta=1, \alpha=0$, the result becomes

$$|a_3-\mu a_2^2| \le \begin{cases} 3-4\mu;\ \mu\le\frac{1}{2}; \\ 1;\ \frac{1}{2}\le\mu\le\ 1; \\ 4\mu-3;\ \mu\ge 1\ . \end{cases}$$

which is the required result of FeketeSzego inequality for Starlike functions.

Conclusion: A subclass of analytic functions which take a broad view of some well-known subclasses of analytic and univalent functions was demarcated. The better estimates for the Fekete-Szegö functional for the defined class were obtained along with extremalfunctions. The study combines existing results and attains new outcomes in geometric function theory. Forthcoming researches can be done to acquire the geometric properties.

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