Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 44 No. 10
October 2023

Some Results on Commutativity of Some 2-Torsion Free Non-Associative Ring with Unity

Dr. Madana Mohana Reddy Y.
Department of Mathematics
Rayalaseema University
Kurnool-518007 (A.P)

Abstract: - In this paper, we prove that some results oncommutativity of primitive rings with some identities

Key Words: Commutative ring, Non associative primitive ring,Central

1.Introduction

In this paper, we first study some commutativity theorems of non-associative primitive rings with some identities in thecenter. We show that some preliminary results that we need inthe subsequent discussion and prove some commutativity theorems of non-associative rings and also non-associativeprimitive ring with $(\alpha\beta)^2 - \alpha\beta \in Z(R)$ or $(\alpha\beta)^2 - \beta\alpha \in Z(R)$ $\forall \alpha, \beta$ in R then R is is commutative. We also prove that if R is a non-associative primitive ring with identity $(\alpha\beta)^2 - \beta(\alpha^2\beta) \in Z(R)$ for all α, β in R then R is commutative. Also we prove that if R is an alternative prime ring with identity $\beta(\alpha\beta^2)\alpha - (\beta\alpha^2)\beta \in Z(R)$ for all α, β in R, then R is commutative. Some commutativity theorems for certain non-associative rings, which are generalization for the results of Johnsen and others and R.N. Gupta, are proved in this paper. Johensen, Outcalt and Yaqub proved that if a non-associative ring R satisfy the identity$(\alpha\beta)^2 = \alpha^2\beta^2$ for all α, β in R, then R is commutative.

The generalization of this result proved by R.D. Giri and othersstates that if R is a non-associative primitive ring satisfies theidentity $(\alpha\beta)^2 - \alpha^2\beta^2 \in Z(R)$, where Z(R) denoted the center, then R is commutative.

A modification of Johnsen`s identity viz., $(\alpha\beta)^2 = (\beta\alpha)^2$ for all α, β in R for a non - associative ring R which has no element of additive order 2, is commutative was proved by R.N. Gupta [1]. R.D. Giri and others [2] generalized Gupta`s result by taking $(\alpha\beta)^2 - (\beta\alpha)^2 \in Z(R)$.

I. Main Results

Theorem 2.1 : If R is a 2-torsion free non-associative ring with unity satisfying $(\alpha\beta)^2 = (\beta\alpha)^2$, then R is commutative.

Proof : Let $\alpha, \beta \in R$.

Then $[\alpha(1+\beta)^2] = [(1+\beta)\alpha]^2$

i.e., $(\alpha + \alpha\beta)^2 = (\alpha + \beta\alpha)^2$

i.e., $\alpha^2 + \alpha(\alpha\beta) + (\alpha\beta)\alpha + (\alpha\beta)^2 = \alpha^2 + \alpha(\beta\alpha) + (\beta\alpha)\alpha + (\beta\alpha)^2$

i.e., $\alpha(\alpha\beta) + (\alpha\beta)\alpha = \alpha(\beta\alpha) + (\beta\alpha)\alpha$……………………2.1

substituting α by (1 + α) in 2.1., we get

$(1+\alpha)(\beta+\alpha\beta) + (\beta+\alpha\beta)(1+\alpha) = (1+\alpha)(\beta+\beta\alpha) + (\beta+\beta\alpha) + (\beta+\beta\alpha)(1+\alpha)$By simplifying, ,

$\beta + \alpha\beta + \alpha\beta + \alpha(\alpha\beta) + \beta + \beta\alpha + \alpha\beta + (\alpha\beta)\alpha = \beta + \beta\alpha + \alpha\beta + \alpha(\beta\alpha) + \beta + \beta\alpha + \beta\alpha + \beta\alpha + (\beta\alpha)\alpha$.

Using 2.1, we get

$2(\alpha\beta - \beta\alpha) = 0$, i.e., $\alpha\beta = \beta\alpha$. Hence R is commutative.

Theorem 2.2 : If R is a 2 – torsion free non-associativeprimitive ring with unity such that $(\alpha\beta)^2 - (\beta\alpha)^2 \in Z(R)$, for all α, β in R , then R iscommutative.

Proof : Given $(\alpha\beta)^2 - (\beta\alpha)^2 \in Z(R)$……………….……2.2

Replacing β by (β+1) in 2.2, and using

2.2, we obtain

$\alpha(\alpha\beta) + (\alpha\beta)\alpha - \alpha(\beta\alpha) - (\beta\alpha)\alpha \in Z(R)$…………………….2.3

Now replacing α by α + 1 in 2.3, and using 2.3., we achieve, $2\alpha\beta - 2\beta\alpha \in Z(R)$.

i.e., $2(\alpha\beta - \beta\alpha) \in Z(R)$.

Since R is a 2-torsion free ring, $\alpha\beta - \beta\alpha \in Z(R)$. We conclude that R is commutative.

Now we present, some examples to see that the unity and 2- torsion free are essential in theorems 2.2 and 2.3

Example 2.1 : The restriction on R, being 2 - torsion free in theorem 2.1 is essential one. For if we consider the ring R of quaternion's over the field of order 4 namely splitting field of α2 + α + 1 over Z2, then it is not of 2-torsion free but satisfies the identity of theorem 2.1. Yet it is non-commutative.

Example 2.2: Theorem 2.2 is false for rings without unity. In fact any nilpotent ring of index ≤ 4 and any nil ring of index 2 will trivially satisfy (αβ)2 = (βα)2 , but such rings may not be commutative. As an example let F be any field define an algebra A over F with basis { α, β, c} , where αβ = c, all other products zero. A is nilpotent of index 3, A is not commutative.

It is well known that a Boolean ring satisfies α2 = α , for all $\alpha \epsilon$ R and this implies commutativity. Similarly we can see the properties of rings in which (αβ)2 = αβ for each pair of elements a, $\beta \epsilon$ R . In [3] Quadri and others proved that an associative semi prime ring in which (αβ)2 - αβ ϵ Z(R) is commutative . In this direction we prove that a 2 – torsion free non – associative ring with unity satisfying $(\alpha\beta)^2$= αβ ϵ Z(R) is commutative. We give an example to show that the unity is essential in the hypothesis. Also, We prove that a non – associative primitive ring (not necessarily having unity) satisfying (αβ)2 – αβ (or) (αβ)2 – βα is central for all α , $\beta \epsilon$ R is commutative.

First we prove the following theorem:

Theorem 2.3 : Let R be a 2 – torsion free non – associative ring with unity satisfying $(\alpha\beta)^2$ - αβ ∈ Z (R) for all α, β in R. then R is commutative.

Proof : By hypothesis $(\alpha\beta)^2$ - αβ ∈ Z (R) 2.4.

Replacing α by α + 1 in 2.4. and using 2.4., we get

(αβ) β + β(αβ) + β^2 - β ∈ Z (R) 2.5.

Again replacing α by α + 1 in 2.5. and using it, we obtaing2β^2 ∈ Z (R)

Since R is a 2 – torsion free , β^2 ∈ Z (R) 2.6.

Replacing β by αβ in. 2.6.

we get $(\alpha\beta)^2$ ∈ Z (R) 2.7.

But by hypothesis $(\alpha\beta)^2$ - αβ ∈ Z (R),

hence we get αβ ∈ Z (R) 2.8.

Now again replacing α by α + 1 in 2.8.,

we get αβ + βα ∈ Z (R) 2.9.

From the equations 2.8. and 2.9. we obtain β ∈ Z (R) for allβ ∈ R .Hence R is commutative.

Theorem 2.4. : Let R be a 2 – torsion free non-associative ring with unity satisfying (αβ)2 - βα ∈ Z (R) for all α, β in R. then R us commutative.

Proof : Given $(\alpha\beta)^2$ - βα ∈ Z (R) 2.10

Replacing α by z + 1 in 2.3.10. and using 2.10., we get (αβ)β + β(αβ) + β^2 –β ∈ Z (R) …………………2.11

Again replacing α by α + 1 in 2.11. and using 2.11., we obtain 2β^2 ∈ Z (R)

Since R is a 2 torsion free, then β^2 ∈ Z (R)………………………2.12.

Now replacing β by αβ in 2.12.. we get $(\alpha\beta)^2$ ∈ Z (R)………………………………………………………2.13

But by hypothesis $(\alpha\beta)^2$ - βα ∈ Z (R).

Hence we have βα ∈ Z (R)…………………………………………2.14

Now again replacing α by α + 1 in 2.14 , we get βα+ β ∈ Z (R) ……………………………………………………2.15

Using 2.14 and 2.15 , we obtain β ∈ Z (R) for all β ∈ R ,then R is commutative.

Theorem 2.5 : If R is a 2 – torsion free primitive ringwhich satisfy $(\alpha\beta)^2$ - αβ ∈ Z (R) for all α, β in R, then R is commutative.

Proof : By hypothesis , $(\alpha\beta)^2$ - αβ ϵ Z(R) 2.16

Replacing a by α + β in 2.16 and using 2.16,

we obtain(αβ) β^2 + β^2 (αβ) + β^4 – β^2 ϵ Z(R) 2.17

Now replacing α by β in $(\alpha\beta)^2$ - αβ ϵ Z(R), we get

β^4 - β^2 ϵ Z(R) 2.18

Using 2.3.17 and 2.3.18, we obtain

(αβ) β^2 + β^2 (αβ) ϵ Z(R) 2.19

We replacing α by α + β in 2.19, then (αβ) β^2 + β^4 + β^2 (αβ)
+β^4 ϵ Z(R).

By . 2.12 β^4 + β^4 ϵ Z(R)., i.e., 2β^4 ϵ Z(R).

Since R is a 2 – torsion free ring, β^4 ϵ Z(R) 2.20

Using 2.18 and 2.20, we obtain

β^2 ϵ Z(R)2.21

Taking β by αβ in 2.21, we get $(\alpha\beta)^2$ ϵ Z(R).But by hypothesis $(\alpha\beta)^2$ - αβ ϵ Z(R).

Hence , αβ ϵ Z(R)2.22

Replacing β by α + β in 2.3.21, we get α^2 + β^2 + αβ + βα ϵ
Z(R).

Since α^2 , β^2 ϵ Z(R)., we get

αβ + βα ϵ Z(R) 2.23

From 2.22 and 2.23, βα ϵ Z(R). Hence αβ - βα ϵ Z(R).

If R is a primitive ring, ten R has a maximal right ideal which contains no non – zero ideal of R .Consequently, we obtain(αβ - βα) R = 0,

which further yields αβ - βα = 0

Due to primitivity of R. Hence R is commutative.

Theorem 2.6 : Let R be a 2 – torsion free primitive ring which satisfy the identity $(\alpha\beta)^2$ – βα ϵ Z(R). for all α, β in R . Then R is commutative.

Proof : Given (αβ)2 - βα ϵ Z(R)……………………………………………………………..2.24

Replacing α by α + β in 2.24, and using 2.3.24, we obtain(αβ)β^2 + β2 (αβ) + β4 – β2 ϵ Z(R)…….2.25

Replacing α by y in 2.24, we get

β^4 - β^2 ϵ Z(R)…………………………………………………………………………
………………2.26

Using 2.25 and 2.26, we get

(αβ) β^2 + β^2 (αβ) ϵ Z(R)……………………………………………………………………………
………2.27

Now we replacing α by (α + β) in 2.3.27 , then (αβ)β^2 + β^4 +β^2 (αβ) + β^4 ϵ Z(R).

But by 2.27, β^4 + β^4 ϵ Z(R)., i.e., 2β^4 ϵ Z(R).

Since R is a 2 – torsion free ring, then β^4 ϵ Z(R) 2.28

Using 2.26 and 2.28, we get β^2 ϵ Z(R) 2.29

Now replacing β by αβ in 2.29, $(\alpha\beta)^2 \in Z(R)$.
By assumption, $(\alpha\beta)^2 - \beta\alpha \in Z(R)$. Hence, $\beta\alpha \in Z(R)$ 2.30
Replacing β by (α + β) in 2.30 ,
we get, $\alpha\beta \in Z(R)$ 2.31
Hence, $\alpha\beta - \beta\alpha \in Z(R)$.
Now using the same argument as in the proof of theorem 2.5 we conclude that R is commutative.Now we give examples showing that unity in the statement of the theorems is essential.

Example : Let $R = \left\{\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} / \ \alpha, \beta, c \in Z \right\}$
α, β, c ϵ Z

. Clearly, R is not commutative though it satisfy the relations $(\alpha\beta)^2 - \alpha\beta \in Z(R)$ or $(\alpha\beta)^2 - \beta\alpha \in Z(R)$, for all α, β in R.
Ram Awatar [4] generalized Gupta`s [5] result and proved that if R is an associative semi prime ring in which $\alpha\beta^2 \alpha - \beta\alpha^2 \beta$ is central, then R is commutative. In this section we show that if R is an alternative prime ring in which $(\alpha beeta\alpha^2)\alpha - (\beta\alpha^2)\beta$ is central, then R is commutative.
Now we prove the following theorem.
Theorem 2.7 : Let R be a non – associative primitive ring with unity satisfying $(\alpha\beta)^2 - \beta(\alpha^2\beta) \in Z(R)$
for all α, β in R. then R is commutative
Proof: By hypothesis $(ab)^2 - b(a^2b) \in Z(R)$ 2.32
For all a, b in R
Replacing α by α+1 in 2.32, we get $((\alpha+1)\beta) - \beta((\alpha+1)^2\beta) \in {}^2Z(R)$.
i.e., $(\alpha\beta + \beta)^2 - \beta(\alpha^2\beta + 2\alpha\beta + \beta) \in Z(R)$.
Using 2.32 , we obtain $(\alpha\beta)\beta - \beta(\alpha\beta) \in Z(R)$……………2.33
Now replacing β by (β+1) in 2.3.33 and using 2.33 ,we get $\alpha\beta - \beta\alpha \in Z(R)$.
If R is a primitive ring then R has a maximal right ideal which contains no non-zero ideal of R. Consequently, we obtain $(\alpha\beta - \beta\alpha) R = 0$. This further yields $\alpha\beta - \beta\alpha = 0$ due to primitivity of R. Hence R is commutative.
Theorem 2.8: Let R be an alternative prime ring with $(\alpha\beta^2)\alpha - (\beta\alpha^2)\beta \in Z(R)$ for all α, β in R. then R is commutative.
Proof : First we shall prove that $Z(R) \neq (0)$ Let us suppose that Z(R) = (0)
Hence by hypothesis, $(\alpha\beta^2)\alpha = (\beta\alpha^2)\beta$,…………………………………2.34
for all α, β in R.
Replacing β by $\beta+\beta^2$ in 2.3.34. we obtain $(\alpha(\beta^2 + \beta^4 + 2\beta^3))\alpha$
$= (\beta\alpha^2 + \beta^2\alpha^2)(\beta+\beta^2)$
i.e., $(\alpha\beta^2)\alpha + (\alpha\beta^4)\alpha + 2(\alpha\beta^3)\alpha = (\beta\alpha^2)\beta + (\beta\alpha^2)\beta^2 + (\beta^2\alpha^2)\beta + (\beta^2\alpha^2)\beta^2$
i.e., $2(\alpha\beta^3)\alpha = (\beta^2\alpha^2)\beta + (\beta\alpha^2)\beta^2$……………………………………………2.35
Since $(\beta^2\alpha^2)\beta = (\beta(\beta\alpha^2))\beta = \beta(\beta\alpha^2)\beta) = \beta((\alpha\beta^2)\alpha = ((\beta\alpha)\beta^2)\alpha$
$= (\beta\alpha)(\beta^2\alpha)$
and $(\beta\alpha^2)\beta^2 = ((\beta\alpha)\alpha)\beta^2 = (\beta\alpha)(\alpha\beta^2)$
Hence 2.35 reduced to, $2(\alpha\beta^3)\alpha = (\beta\alpha)(\beta^2\alpha + \alpha\beta^2)$…………………………2.36
If R is not 2 –torsion free, 2.36, becomes $(\beta\alpha)(\beta^2\alpha + \alpha\beta^2) = 0$With α = (α+β), this gives $(\beta\alpha + \beta^2)(\beta^2\alpha + \beta^3 + \alpha\beta^2 + \beta^3) = 0$ i.e., $\beta^2(\beta^2\alpha + \alpha\beta^2) = 0$………………………………………………2.37
put α = rα in 2.37, then we get
$\beta^2(\beta^2(r\alpha) + (r\alpha)\beta^2) = 0$………………………………………………………2.38
since $\beta^2(\beta^2 r) = \beta^2(r\beta^2)$
From 2.37 and 2.38, we have $\beta^2(r(\beta^2\alpha + \alpha\beta^2)) = 0$.
We write this as $\beta^2 r (\beta^2\alpha + \alpha\beta^2) = 0$
Since R is prime, either $\beta^2 = 0$ or $\beta^2\alpha + \alpha\beta^2 = 0$. i.e.., $\beta^2 \in$
Z(R) = 0.
Thus in either case $\beta^2 = 0$ for every β in R.
If R is 2- torsion free, we replace β by $\beta + \beta^3$ in 2.33, andget
$2(\alpha\beta^4)\alpha = (\beta^3\alpha^2)\beta + (\beta\alpha^2)\beta^3$ or
$2(\beta^2\alpha^2)\beta^2 = \beta^2((\beta\alpha^2)\beta) + ((\beta\alpha^2)\beta)\beta^2 = \beta^2((\alpha\beta^2)\alpha) + ((\alpha\beta^2)\alpha)\beta^2$
We write this as $(\beta^2\alpha^2)\beta^2 - \beta^2((\alpha\beta^2)\alpha) = ((\alpha\beta^2)\alpha)\beta^2 - (\beta^2\alpha^2)\beta^2$or
$(\beta^2\alpha)(\alpha\beta^2 - \beta^2\alpha) = (\alpha\beta^2 - \beta^2\alpha)\beta^2$
We replacing α by α + β : Then we get
$\beta^3(\alpha\beta^2 - \beta^2\alpha) = (\alpha\beta^2 - \beta^2\alpha)\beta^3$…………………………………………………………2.39
For all α, β in R
Let $I\beta^2$ be the inner derivation by β^2 i.e., α ⃞ $\alpha\beta^2 - \beta^2\alpha$ and $I\beta^3$ be the inner derivation by β^3. Then 2.39 becomes $1\beta^3 1\beta^2(\alpha) = 0$
Thus the product of these derivation is again a derivation. wecan conclude
that either β^2 or β^3 in Z(R), i.e., β^2 or β^3 is zero.
If $\beta^3 = 0$, then 2.35, becomes $(\beta^2\alpha^2)\beta + (\beta\alpha^2)\beta^2 = 0$
Substituting α +β for α, we get
$(\beta^2\alpha^2 + \beta^3 + 2\beta^2(\alpha\beta))\beta + (\beta\alpha^2 + \beta^3 + 2\beta(\alpha\beta))\beta^2 = 0$i.e., $2(\beta^2\alpha)\beta^2 + 2(\beta(\alpha\beta^3)\beta^2) = 0$
Then we get $2(\beta^2\alpha)\beta^2 = 0$ or $(\beta^2\alpha)\beta^2 = 0$, Then $\beta^2 = 0$
Thus if Z (R) = (0), then $\beta^2 = 0$ for every β in R. Then $0 = (\alpha+\beta) = (\alpha\beta)\alpha$ or $\alpha R \alpha = 0$
Then α = 0 or R = 0, a contradiction. Therefore $Z(R) \neq (0)$
Taking $\lambda \neq 0$ in Z(R) and let $\alpha = \alpha + \lambda$ in $(\alpha\beta^2)\alpha - (\beta\alpha^2)\beta$ inZ(R), we get
$\lambda(\alpha\beta^2 - 2(\beta\alpha)\beta + \beta^2(\alpha))$ in Z(R).
Since R is prime, we must have
$\alpha\beta^2 - 2(\beta\alpha)\beta + \beta^2\alpha$ in Z(R)…………………………………………2.40
if $\lambda\alpha$ is in Z(R), then $\lambda\alpha\beta - \beta\lambda\alpha = 0 = \lambda(\alpha\beta - \beta\alpha)$

Then, R λ (αβ– βα) = 0 = λ R(αβ – βα) and sing λ ≠ 0, wehave
αβ – βα = 0, i.e., is in Z(R).
In 2.3.40., let α = αβ and get
$\alpha\beta^2$ - 2(βα)β + $(\beta^2\alpha)\beta$ in Z(R), then β is in Z(R), unless $\alpha\beta^2 - 2(\beta\alpha)\beta + \beta^2\alpha = 0$. So if β is not in Z(R),
$\alpha\beta^2 - 2(\beta\alpha)\beta + \beta^2\alpha = 0$, for every β α in R, and
β is in Z(R) , then $\alpha\beta^2 - 2(\beta\alpha)\beta + \beta^2\alpha$ is still zero.
Therefore, $\alpha\beta^2 + \beta^2\alpha = 2(\beta\alpha)\beta$,………………………………………2.41
for every α, β in R
If R is 2 –torsion free, then R is commutative.
If R is not 2 –torsion free, then 2.3.41 becomes $\alpha\beta^2 + \beta^2 z = 0$ or β^2 is in Z(R) for every
y in R. Then $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + \alpha\beta + \beta\alpha$ is in Z(R)
i.e. , αβ + βα is in Z(R)
Let α = αβ and get (αβ + βα)β is in Z(R)
Then β is in Z(R) , unless αβ + βα = 0, which also means β isin Z(R).
Thus Z(R) = R and R is commutative
We give an example showing that the unity in the statement ofthe theorem 2.7. is essential.

Example: Let R ={$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ / α, β ∈ Z}

α, β ∈ Z, We can easily verify the identity of theorem 2.7. i.e., $(\alpha\beta)^2 - \beta(\alpha^2\beta) \in Z(R)$. But Ris not commutative.

References

[1] Gupta.R.N: A note on commutativity of Rings. The Math. Stu., 39(1971), 184 – 186.
[2] Giri. R.D. Rakhunde. R.R. and Dhoble A.R: On commutativity of non associative Rings, the Math. Stu, 61 (1-4), (1992), 149 – 152
[3] Quadri. M.A., Ashraf. M. and Khan M.A: A commutativity condition for semiprime rings H. Bull Austral Math., Soc., 33(1986), 71-73.
[4] Ram Awtar : A remark on the commutativity of certain Rings,Proc. Amar. Math. Soc., 41 (1973), 370 – 372
[5] Herstein.I.N: A generalization of a theorem of Jacobson, Amar J.Math,73(1951),756-762.
[6] Soundararajan, R., Stanislaus, P.M., Ramasamy, S.G., Dhabliya, D., Deshpande, V., Sehar, S., Bavirisetti, D. P. Multi-Channel Assessment Policies for Energy-Efficient Data Transmission in Wireless Underground Sensor Networks (2023) Energies, 16 (5), art. no. 2285,
[7] Anand, R., Khan, B., Nassa, V.K., Pandey, D., Dhabliya, D., Pandey, B.K., Dadheech, P. Hybrid convolutional neural network (CNN) for Kennedy Space Center hyperspectral image (2023) Aerospace Systems, 6 (1), pp. 71-78.