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A Study on the Classes of Additively Regular Semiring and Ordered Semiring

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Abstract:

In this paper, we study the conditions under which the classes of Additively Regular semiring is additively and/or multiplicative idempotent. We also study the classes of totally ordered additively Regular semiring. We prove that the additive and multiplicative structures are positively totally ordered.

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Keywords: Almost idempotent, Idempotent, Multiplicatively sub idempotent, Periodic, Rectangular band, singular semigroup.

1. INTRODUCTION

Various concepts of regularity on semigroups have been investigated by R.Croisot. His studies have been presented in the book of Clifford A.H. and G.B.Preston as R.Croisot theory one of the central places in the theory is held by the left regularity.
K.S.S. Nambooripad studied on the structures of regular semigroups. We used the notion of Clifford semiring. Sen, Ghosh & Mukhopadhyay studied the congruences on inverse semirings with the commutative additive reduct and Maity improved this to the regular semirings with the set of all additive idempotents a bi semilattice. The study of regular semigroups has yielded many interesting results. These results have applications in other branches of algebra and analysis. Some other applications of semiring areas are cryptography, optimization theory, graph theory, dynamical systems, and automata theory. The paper is organized as follows: Section one deals with introduction. Section two contains definitions. In third section we study the classes of Additively Regular semiring. In section 4 we focused on the classes of totally ordered Additively Regular semiring and the last section is the conclusion.

2 PRELIMINARIES:

Definition 2.1:

An algebraic structure (S, +, •) is termed as semiring if the additive and multiplicative reducts are semigroups and u(x + y) = ux + uy and (x + y)u = xu + yu for every u, x, y in S.

Definition 2.2:

An additive semigroup is said to be additively idempotent if u + u = u for all u in S.
A multiplicative semigroup is multiplicatively idempotent or band if $u^2 = u$ for all u in S.If

both (S, +) and (S, •) are idempotents then S is known as an idempotent semiring.

Definition 2.3:

A semiring is termed as mono-semiring if u + x = ux for all u, x in S.

Definition 2.4:

A multiplicative semigroup is assumed to be left (right) singular if ux = u (xu = x) for allu, x in S.

An additive semigroup is said to be left (right) singular if u + x = u (u + x = x) for all u, xin S.

Definition 2.5:

An element u is periodic if $u^m = u^n$, where m and n are positive integers.

A multiplicative semigroup is said to be periodic if every one of their elements isperiodic.

An element u is periodic if mu = nu, where m and n are positive integers.

An additive semigroup is said to be periodic if every one of their elements is periodic.

Definition 2.6:

An additive semigroup (multiplicative semigroup) is rectangular band if u = u + x + u(u = uxu) for all u, x in S.

Definition 2.7:

A semiring is said to be zerosumfree if u + u = 0 for all u in S.A semiring is said to be zero square if $u^2 = 0$ for all u in S

Definition 2.8:

In a semiring S, the semigroup (S, •) is right square regular (left square regular) $ax^2=x(a^2 x = a)$ for some x in S.

Definition 2.9:

An additive semigroup (multiplicative semigroup) is commutative if u + x = x + u(ux = xu) for all u, x in S.

Definition 2.10:

A semiring is almost idempotent if $u + u^2 = u^2$ for all u, x in S.

Definition 2.11:

A semigroup (S, •) is said to be lateral zero if abc = b for some elements a, b, c in S.

Definition 2.12:

In a semiring S, an element u is Multiplicatively Subidempotentif $u + u^2 = u$ for all u inS.

Definition 2.13:

In a totally ordered semiring (S, +, •,≤) (i) (S, +, ≤) is p.t.o, if u + x ≥ u, x for all u, x in S. (ii) (S, •, ≤) is p.t.o, if ux≥ u, x for all u, x in S.

Definition 2.14:

A totally ordered semigroup (S, +, ≤) is assumed to be non-negatively (non-positively) ordered if every element of S is non-negative/non-positive.

Definition 2.15:

An element u in a totally ordered semiring is said to be a minimal/maximal if u ≤ x(u ≥ x) for every u ∈S.

Definition 2.16:

In a semiring if for an element a there exists an element b such that a = a + b + a anda (a + b) = a + b.

If all the elements satisfy the above conditions. Then S is said to be Clifford Semiring.

3.CLASSES OF ADDITIVELY REGULAR SEMIRING

THEOREM 3.1: If S be an additively regular semiring, (S, +) is idempotentand commutative, then $a^n = a^n + a^n a'$.

Proof: Given that S is an additively regular semiring $a = a+a^1+a$ $\rightarrow$**(1)**
Given (S,+) is idempotent a + a = a for all a in SFrom equation (1) $a = a + a' + a \Rightarrow a = a + a'$
Multiplying 'a' on both sides $a^2 = a^2 + aa'$ Adding 'a^2' on both sides we get $a^2 + a^2 = a^2 + a^2 + aa' \Rightarrow a = a + aa'$Multiplying 'a' on both sides we get $a^2 = a^2 + a^2 a'$
Again multiplying 'a' on both sides we get $a^3 = a^3 + a^3 a'$Continue this process we get $a^n = a^n + a^n a'$

REMARK 3.2: Let S be an additively regular semiring with zerosumfree semiring. Then we can prove that a + a'= 0 and a'+a = 0.
Proof: Firstly Let 'S' be an additive regular semiring then a= a+a'+a
Adding 'a' on both sides and using the definition of zerosumfree then above equationtakes the form a+a'=0 and a'+a = 0 Hence the theorem

Proposition 3.3: Let S be an additively regular semiringand absorption semiring. Then $a=a+(a'+a)x^n$.
Proof: Given that 'S' is additively regular semiring then a=a+ a'+a$\rightarrow$ **(1)**and we knowthat S is also a absorption semiringa+ax=a$\rightarrow$ **(2)**
By considering first equation and multiplying with an element 'x' and by the addition of element 'a' we get a+ax=a+ax+a'x+ax
Using second equation in above we obtain a=a+ a'x+ax
By continuing this process, finally we obtain the equation $a = a + (a' + a) x^n$

Proposition 3.4: Let S be an additively regular semiringand multiplicatively sub idempotent with (S, +) commutative. Then $a=a+a^na'$.
Proof: By hypothesis 'S' is additively regular semiringthen a=a+ a'+ a $\rightarrow$ **(1)**
And also S is multiplicative sub idempotent $a+a^2= a$$\rightarrow$ **(2)**
For the first equation if we multiply and add the element 'a' on both sidesThen we get $a+a^2 = a+a^2+a\,a'+a^2$
By using multiplicatively sub idempotent law and (S, +) commutativewe get a=a+a a'
On generalizationthe above steps we obtain the conclusion as $a=a+a^n a'$

THEOREM 3.5: Let S be an additively regular semiring and if (S, •) isregular. Then $a^nx^na' = a^nx^na' + a + a^nx^na'$.
Proof: Given that S is additively regular semiringthen a'= a'+ a+ a' $\rightarrow$ **(1)**

First if we multiply first equation with ‘ax’ on both sideswe getaxa' = axa' + axa+axa'
Using (S, •) regular in above it reduces to axa'=axa'+a+axa'
Again continuing same process ‘n’ times we get $a^n x^n a' = a^n x^n a' + a + a^n x^n a'$

THEOREM 3.6: If S is an additively regular and (S,+) left singular semigroup ,then(S, •) is periodic.
Proof: Given that S is an additively regular a = a + a ' + a → (1)
We know that (S,+) is left singular semigroup a' + a = a' for all a , a' in SFrom equation (1) a = a + a'
Multiplying ‘a’ on both sides $a^2 = a^2 + aa' \Rightarrow a^2 = a(a + a')$ → **(2)**
By using Clifford semiringa(a + a') = a + a'
Then equation (2) becomes as $a^2 = a + a'$ → **(3)**
$\Rightarrow a^2 = a \Rightarrow a^3 = a^2 \Rightarrow a^m = a^n$ Thus (S, •) is periodic

PROPOSITION 3.7:If S is a semiring in which (S,+) is a left singular semigroup, (S, •) is lateral zero and band, then for every three element a,b,c in S then $(ac)^n + b = (ac)^n$ for n ≥ 1.
Proof : Given that (S,+) is an left singular semigroup then a+b=a ∀ a,b in SMultiplying 'c' on both sides ac + bc = ac
Since (S, •)is lateral zero abc=b then ac+bc=ac Multiplying ‘a’ on both sides, we get $a^2c + abc = a^2c$
(S, •) is band we have $a^2=a$ then above equation becomes ac+ b = acMultiplying ‘c’ on both sides, we get $ac^2+bc^2=ac^2$
Multiplying ‘a’ on both sides we get $a^2c^2+abc=a^2c^2 \Rightarrow ac^2+b=ac^2 \Rightarrow ac+b=ac$ $\Rightarrow aacc+abc=aacc \Rightarrow (ac)^2 +b =(ac)^2$
By generalizing the above concept we obtain $(ac)^n+ b =(ac)^n$ for n ≥ 1

THEOREM 3.8: If S is an additively regular semiring with multiplicative identity‘1’, and (S, •) is idempotent, then (S, +) is periodic.
Proof: Given that S is an additively regular semiringthen a = a+a'+a
Case -1: let us take a'=1then a +1+a = a
Multiplying ‘a’ on both sides then we get$a^2+a+ a^2 = a^2$
By using (S, •) idempotent in above equation we get 3a = a Thus (S, +) is periodic
Case-2: Again let us consider a + a' + a = a
Multiplying ‘a’ on both sides then, we get$a^2 + aa' + a^2= a^2$
By using (S, •) idempotent in above equation we get a + aa' + a= aMultiplying ‘a'’ on both sides then we get a'a+ a'aa'+a'a=a'a Multiplying ‘a'’ on both sides then we get3a'aa' = a'aa'
Thus an element a'aa' is periodic
Hence S contains additively periodic elements

PROPOSITION 3.9 :If S is an additively regular semiring and (S, +) be absorbing .Then $a^n + a'a^n + a^{n-1} a' + a^n = a^n$ for n ≥ 1.

Proof: Given that (S, +) is absorbingthena +1 = 1
Multiplying ‘a'’ on both sidesto above equation we get a'a + a'= a'Adding ‘a’ on both sides we geta + a'a + a'= a + a'
Again adding ‘a’ on both sidesa + a'a + a' +a= a + a' +a
$\Rightarrow$a + a'a + a' + a = a
Multiplying ‘a’ on both sides we get $a^2 + a'a^2 + aa' + a^2 = a^2$
$\Rightarrow a^n + a'a^n + a^{n-1} a' + a^n = a^n$ for $n \geq 1$

THEOREM 3.10: If S is an additively regular semiring with (S, +) commutative, then na + a' = (n -1) a and a + n (a' + a) = a where ‘a'’ depends on ‘a’ and n>1.
Proof: Given that S is an additively regularsemiring a + a'+ a = aBy using (S, +) commutative in above equation we get2a + a'= a Again Adding ‘a’on both sideswe get 3a + a' = 2a
Proceeding in a similar manner we get na + a'= (n -1)a for n >1Again let us consider a + a' + a = a
Adding ‘a'’ on both sides then we get a + a' + a +a' = a + a'
Adding ‘a’ on both sides then we get a + (a' + a) + (a' + a) = a + a' + a $\Rightarrow$ a + 2 (a' + a) = a
Adding ‘a' + a’ on both sides to above equation we get
a + 2 (a' + a) + (a' + a) = a + a'+ a $\Rightarrow$a + 3 (a' + a) = aContinuing like this we get a + n (a' + a) = a for n >1

PROPOSITION 3.11: If S is an additively regular and Clifford semiring, then(a + a') + a^n =a^n for n >1**.**
Proof :Given that S is Clifford semiring then a (a + a') = a + a' Also Given that S is an additively regular semiring a = a + a'+ aa (a + a') + a.a = a.a
By using Clifford semiring then above equation reduces to $a + a' + a^2 = a^2$Again on multiplication of an element ‘a’ on both sides we get
$a(a + a') + a^3 = a^3 \Rightarrow a + a' + a^3 = a^3$
On generalizing the above concept we obtain $(a + a') + a^n = a^n$ for n >1

THEOREM 3.12: If S is an additively regular semiring and absorption semiring, then a = a + (a' + a) x^n
Proof : Given that S is an additively regular semiring a = a + a' + a $\rightarrow$**(1)**Multiplying ‘x’ on both sides to above equation we get ax = ax + a'x+ ax Adding ‘a’ on both sides then a + ax = a + ax + a'x+ ax
Using absorption semiring in above we get a = a + a'x+ ax$\rightarrow$**(2)**Again multiplying ‘x’ on both sides$\Rightarrow$ ax = ax + a'xx+ axx
Again adding ‘a’ on both sidesand using absorption semiring we get $a = a + a'x^2 + ax^2$From above equations we can conclude that $a = a + a'x^n + ax^n \Rightarrow a = a + (a' + a)x^n$

THEOREM 3.13: If S is an additively regular semiring and also multiplicativelysub idempotent, then $a + a^n a' = a^n a'$.

Proof : Given that S is an additively regular semiring a' = a'+ a+ a'→**(1)**By multiplicatively sub idempotent $a+a^2=a$

Multiplying 'a' on both sides for equation(1)From equation (1) $aa'=aa'+a^2+aa'$

Adding 'a' on both sides ⇒$a + aa'=a+aa'+a^2+aa'$⇒a+aa'=a+aa'+aa'Multiplying 'a' on both sides$a^2+a^2a' =a^2+a^2a'+a^2a'$

Adding 'a' on both sides ⇒$a+a^2+a^2a' =a+a^2+a^2a'+a^2a'$⇒$a+a^2a' = a+a^2a'+a^2a'$ continuing this process we get $a+a^na' = a^na' + a + a^na'$ we conclude $a + a^na'=a^na'$

THEOREM 3.14: If S is an additively regular semiring with multiplicatively sub idempotent, then $a = a + a^n(a'+a)$.

Proof : Given that S is an additively regular semiring a = a+a'+a →**(1)**

Multiplying 'a' on both sides we get $a^2 = a^2+aa'+ a^2$ Adding 'a' on both sides ⇒ $a+a^2 = a+a^2+aa' +a^2$ →**(2)**By using multiplicatively sub idempotent $a + a^2= a$ From equation (2) $a = a + aa'+ a^2$

Again multiplying 'a' on both sides ⇒ $a^2=a^2+a^2a'+a^3$

Adding 'a' on both sides ⇒$a+ a^2=a+a^2+a^2a'+a^3$→**(3)** ⇒$a = a + a^2a'+ a^3$And continue this process then we obtain $a = a + a^na'+a^{n+1}$

Now these process can be generalized then we get $a = a + a^n(a'+a)$

4. CLASSES OF TOTALLY ORDERED ADDITIVELY REGULAR SEMIRING

THEOREM 4.1: If S is an additively regular (S, •) is band and positively totally ordered, then (S, +) is periodic.

Proof: Given that S is an additively regular then a'= a'+a+ a'→ **(1)**

Multiplying 'a'' on both sides we get $(a')^2=(a')^2+aa'+(a)^2$Suppose a'=b then $b^2 = b^2 + ab + b^2$ → **(2)**

Using (S, •) p.t.o in equation (2) we get $b^2 = b^2 + b + b^2$

Using (S, •) is band in above we get b=3b for all b in S Therefore (S, +) is periodic

THEOREM 4.2 : Suppose S is an totally ordered additively regular semiring and(S, +) is positively totally ordered, then (S, +) is idempotent

Proof: By hypothesis S is additively regular semiring element then there exists anelement 'x' such that a + a' + a = a→ **(1)**

Given that (S, +) is positively totally ordered then a + a'≥ a and a + a'≥ a' Adding 'a' on both sides ⇒a + a' + a ≥ a + a and a + a' + a ≥a'+ a → **(2)** Using equation (1) in equation (2) it reduces to a ≥a + a and a ≥ a + a'→ **(3)**

Since (S, +) is p.t.o a + a ≥ a and also we obtained a ≥ a + a(S, +) is idempotent

PROPOSITION 4.3: Suppose S is an totally ordered additively regular semiring and (S, +) is positively totally ordered (negatively totally ordered). Then (S, •) is non-negative(non-positive).

Proof: By additively regular semiring we have $a + a' + a = a \rightarrow$ **(1)**

Since (S, +) is positively totally ordered, $a + a' \geq a, a'$ for all a, x in S Then equation (1) becomes

$a = a + a' + a \geq a' \Rightarrow a \geq a' \Rightarrow a^2 \geq aa'$

$\Rightarrow a + a^2 \geq a + aa' \Rightarrow a + a^2 + a \geq a + aa' + a$

Since (S, +) is positively totally ordered, $\Rightarrow a + a^2 \geq a, a' \Rightarrow a^2 \geq a$ (S, •) is non-negative

In a similar manner if Since (S, +) is negatively totally ordered then (S, •) is non-positive

Conclusion:

In this article we can observe that on application of different constrains to additively regular semiring mostly it results in periodic law either in multiplicative structure or an additive structure.

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