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Oscillation Criteria for First Order Non-Linear Delay Differential Equations Caused by Non-Monotone Arguments

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Abstract:
This paper is concerned with the oscillatory behaviour of first – order nonlinear delay differential equation with variable non-monotone deviating arguments and nonnegative coefficients of the form, $z'(t) + p(t)f(z(\tau(t))) = 0$, $\tau(t) < t$, $t \geq t_0 > 0$, where p and τ are continuous on $[t_0, \infty)$, $p(t) \geq 0$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$ and $f \in C(R,R)$ and $zf(z) > 0$ for $z \neq 0$.Sufficient oscillation conditions involving limit superior and limit inferior are obtained . An example, numerically solved, is also given to illustrate the applicability and strength of the obtained results over known ones.

Keywords: Differential equation, non-monotone argument, oscillatory solution, non–oscillatory solution.

1. Introduction

Delay differential equations are the type of differential equations where the derivative of the unknown function, at a certain time, is given in terms of the values of the function, at previous times. Delay differential equations are also referred in the literature as time – delay systems, systems with after effect or dead - time, hereditary systems or equations with delay arguments.

Mathematical modelling involving delay differential equations are widely used for analysis and predictions in various areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, neural networks, see, [1,2,3,8,10,11,18,19,20,22] and the references cited therein. Time delays add to these models memory effects, taking into account the dependence of the model's present state on its past history [20]. Time delay can be related to the duration of certain hidden processes, like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, and so on.

Consider the first order non – linear delay differential equation

$$z'(t) + p(t)f(z(\tau(t))) = 0, \quad t \geq t_0 > 0 \qquad (1.1)$$

where p and τ are continuous on $[t_0, \infty)$, $p(t) \geq 0$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$ and $f \in C(R, R)$ and $zf(z) > 0$ for $z \neq 0$. By a solution of (1.1) we mean a function which is continuous on $[\overline{t_*}, \infty]$ for some $t_* \geq t_0$, where $\overline{t_*} = \inf\{\tau(t): t \geq t_*\}$ and satisfies (1.1) for all $t \geq t_*$.

A solution of (1.1) is said to be oscillatory, if it has arbitrarily large zeroes. Otherwise, it is called non oscillatory.

2. Preliminary Results

The study of the oscillatory behaviour of solutions to linear delay differential equation

$$z'(t) + p(t)(z(\tau(t))) = 0, \quad t \geq t_0 > 0 \qquad (2.1)$$

was initiated by Myshkis [16] in the mid-twentieth century, and proved that all solutions of (2.1) are oscillatory if

$$\lim_{t\to\infty} sup[t - \tau(t)] < \infty \text{ and } \lim_{t\to\infty} inf[t - \tau(t)] \lim_{t\to\infty} inf\, p(t) > \frac{1}{e} \qquad (2.2)$$

In 1972, Ladas, Lakshmikantham and Papadakis [15] proved that the same conclusion holds if τ(t) is nondecreasing and

$$\lim_{t\to\infty} sup \int_{\tau(t)}^{t} p(s)\, ds > 1. \qquad (2.3)$$

In 1982, Koplatadze and Chanturija [13] improved the conditions (2.2) to

$\lim_{t\to\infty} inf \int_{\tau(t)}^{t} p(s)$ ds > $\frac{1}{e}$. (2.4)

It is pointed out in [13] that if $\int_{\tau(t)}^{t} p(s) \leq \frac{1}{e}$, (2.1) has a nonoscillatory solution. Obviously, when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s) \text{ ds}$$

does not exist, a gap appears between the conditions (2.3) and (2.4). How to fill this gap is an interesting problem which has attracted the attention of several authors.

In particular, in 2000, Jaros and Stavroulakis [12] developed new oscillation criteria by sharpening the upper bound for z(τ(t))/z(t) for possible non oscillatory solutions of (2.1), as used in [4, 5, 7, 14, 24]. Their result, formulated in terms of the numbers α and M_α defined by

$\alpha = \lim_{t\to\infty} inf \int_{\tau(t)}^{t} p(s)$ ds and $M_\alpha = \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}$ (2.5)

says that all solutions of (2.1) are oscillatory if, τ(t) is nondecreasing, α ∈ (0,1/e] and

$\lim_{t\to\infty} sup \int_{\tau(t)}^{t} p(s)$ ds > $\frac{1+ln\,\mu_0}{\mu_0}$ - M_α,

where μ_0 is the smaller root of the transcendental equation $\mu = e^{\alpha\mu}$.

Define h(t) = $\sup_{s\leq t} \tau(s)$, t ≥ t_0. (2.6)

Clearly, the function h(t) is non decreasing and τ(t) ≤ h(t) < t for all t ≥ t_0.

To start with, we state the lemmas and theorems which we will use in the proof of our main result.

Lemma 2.1: (See [7, Lemma 2.1.1])

If h (t) is defined by (2.6), then $\alpha = \lim_{t\to\infty} inf \int_{\tau(t)}^{t} p(s)$ ds = $\lim_{t\to\infty} inf \int_{h(t)}^{t} p(s)\, ds.$

Lemma 2.2: (See [7, Lemma 2.1.3])

If α ϵ (0, $\frac{1}{e}$] for α defined in Lemma 2.1, and z (t) is an eventually non- negative solution of (1.1) , then

$$\lim_{t\to\infty} \inf \frac{z(t)}{z(h(t))} \geq M_\alpha \qquad (2.7)$$

and

$$\lim_{t\to\infty} \inf \frac{z(h(t))}{z(t)} \geq \mu_0 \qquad (2.8)$$

where μ_0 is the root of the transcendental equation $e^{\alpha\lambda} = \lambda$ which is smaller in value. □

Theorem 2.1: (See [17], Theorem 2.1)

Suppose τ(t) is non – monotone or non – decreasing function. Set h (t) = $\sup_{s\leq t} \tau(s)$, t ≥ 0. Obviously, h(t) is non decreasing, and τ(t) ≤ h (t) for all t ≥ 0.

Suppose that f in equation (1.1) satisfies the following condition

$\lim_{z\to 0} \sup \frac{z}{f(z)}$ = M, 0 ≤ M < ∞ and (2.9)

if $\lim_{t\to\infty} \inf \int_{\tau(t)}^{t} p(s)ds > \frac{M}{e}$ holds, (2.10)

then all solutions of (1.1) oscillate.

Theorem 2.2: (See [9], Theorem 2.1)

Consider the delay linear differential equation z' (t) + p(t) (z (τ(t))) = 0, t ≥ t_0.

Suppose

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} d_\ell(\varepsilon)\, d\varepsilon\right) du\right) ds > 1,$ (2.11)

where

$d_\ell(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) exp\left(\int_{\tau(u)}^{u} d_{\ell-1}(\varepsilon) d\varepsilon\right) du\right) ds\right],$

with

$d_0(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(w) exp\left(\lambda_0 \int_{\tau(\omega)}^{\omega} p(u)\, du\right) d\omega\right) ds\right].$

Then all solutions of (2.1) are oscillatory.

The aim of the present paper is to establish new iterative conditions for the oscillation of all solutions of (1.1), when the arguments are not necessarily monotone. The results obtained essentially improve known results in the literature.

3. Main Results

Theorem 3.1:

Assume that h(t) is defined by (2.6) and for some $\ell \in N$

$$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell (\xi)\, d\xi\right) du\right) ds > 2M, \tag{3.1}$$

where

$$d_\ell (t) = \frac{p(t)}{2M} \left[1 + \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_{\ell-1} (\xi)\, d\xi\right) du\right) ds \right], \tag{3.2}$$

with

$$d_0 (t) = \frac{p(t)}{2M} \left[1 + \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} \exp\left(\mu_0 \int_{\tau(w)}^{w} \frac{p(u)}{2M} du\right) dw\right) ds \right].$$

Then all solutions of (1.1) oscillate.

Proof:

Assume for the sake of contradiction, that there exists a nonoscillatory solution z(t) of (1.1). Since -z(t) is also a solution of (1.1), we can confine our discussion only to the case where the solution is eventually positive. Then, there exists a $t_1 > t_0$ such that z(t) > 0, z (τ (t)) > 0 for all $t \geq t_1$. Thus from (1.1), we have $z'(t) + p(t) f(z(\tau(t))) \leq 0$ for all $t \geq t_1$. It means that z(t) is positive, nonincreasing and has a limit $\ell \geq 0$ as $t \to \infty$.

Now we claim that $\ell = 0$. Condition (2.10) implies that

$$\int_{a}^{\infty} p(t)\, dt = \infty. \tag{3.3}$$

In view of (3.3) and by [17, Theorem 3.1.5], we obtain $\lim_{t\to\infty} z(t) = 0$.

Suppose that M > 0. Then, by virtue of (2.9) we can choose $t_2 > t_1$ so large that

$$f(z(t)) \geq \frac{1}{2M} z(t) \text{ for all } t \geq t_2. \tag{3.4}$$

Since τ(t) ≤ h(t) and z(t) is nonincreasing, by (1.1) and (3.4) we have

$$z'(t) + p(t) \left(\frac{1}{2M} z(\tau(t))\right) \leq 0 \quad , \ t \geq t_3. \tag{3.5}$$

Dividing (3.5) by z(t) and integrating on [τ(t), t], we get

$\int_{\tau(t)}^{t} \frac{z'(u)}{z(u)} du + \int_{\tau(t)}^{t} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du \leq 0$. That is,

$\ln \frac{z(t)}{z(\tau(t))} + \int_{\tau(t)}^{t} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du \leq 0.$

Thus,

$$z(\tau(t)) \geq z(t) \exp\left(\int_{\tau(t)}^{t} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right). \tag{3.6}$$

Combining (3.5) and (3.6),

$$z'(t) + \frac{1}{2M} p(t)\, z(t) \exp\left(\int_{\tau(t)}^{t} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) \leq 0. \tag{3.7}$$

Again dividing (3.7) by z(t) and integrating on [τ(s), t], we get

$\int_{\tau(s)}^{t} \frac{z'(w)}{z(w)} dw \leq - \int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw,$

$\ln \frac{z(\tau(s))}{z(t)} \geq \int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw.$ That is,

$$z(\tau(s)) \geq z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw\right). \tag{3.8}$$

Similarly integrating (3.5) on [τ(t), t], we have

$$z(t) - z(\tau(t)) \leq -\int_{\tau(t)}^{t} \frac{p(s)}{2M} z(\tau(s))\, ds. \tag{3.9}$$

Combining (3.8) and (3.9), gives

$z(\tau(t)) - z(t) \geq z(t) \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw \right) ds .$

Multiplying the last inequality by p(t), gives

$z(t)\, p(t) - z(\tau(t))\, p(t) \leq - p(t)\, z(t) \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw \right) ds,$

$2Mz'(t) + p(t) \left(1 + \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(\tau(u))}{z(u)} du\right) dw\right) ds\right) z(t) \leq 0.$

Since τ(u) ≤ h(u), and z(t) is non-increasing, we have

$2Mz'(t) + p(t) \left(1 + \int_{\tau(t)}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp\left(\int_{\tau(w)}^{w} \frac{p(u)}{2M} \frac{z(h(u))}{z(u)} du\right) dw\right) ds\right) z(t) \leq 0.$

By Lemma 2.2, we have

$\frac{z(h(u))}{z(u)} > \mu_0 - \varepsilon \ for\ some\ \varepsilon > 0\ , so$ the last inequality becomes

2Mz$'$(t) + p(t) [1 + $\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp((\mu_0 - \varepsilon) \int_{\tau(w)}^{w} \frac{p(u)}{2M}$ du) dw) ds] z(t) ≤ 0

or

z$'$(t) + $\frac{p(t)}{2M}$ [1 + $\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp((\mu_0 - \varepsilon) \int_{\tau(w)}^{w} \frac{p(u)}{2M}$ du) dw) ds] z(t) ≤ 0

or

z$'$(t) + d_0(t, ε)z(t) ≤ 0, (3.10)

where

d_0(t, ε) = $\frac{p(t)}{2M}$ [1 + $\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{t} \frac{p(w)}{2M} exp((\mu_0 - \varepsilon) \int_{\tau(w)}^{w} \frac{p(u)}{2M}$ du) dw) ds].

Integrating (3.10) on [s , t]

z(s) ≥ z(t) exp($\int_{s}^{t} d_0(\xi, \varepsilon)$ d ξ) , t ≥ s

Thus, z(τ(u)) ≥ z(u) exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) d ξ), since τ(u) ≤ u. (3.11)

Therefore

$\frac{z(\tau(u))}{z(u)}$ ≥ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) d ξ).

Dividing (3.5) by z(t) > 0 and integrating on [s,t], gives

$\int_{s}^{t} \frac{z'(u)}{z(u)}$ du + $\int_{s}^{t} \frac{p(u)}{2M} \frac{(z(\tau(u)))}{z(u)}$ du ≤ 0

or

ln $\frac{z(s)}{z(t)}$ ≥ $\int_{s}^{t} \frac{p(u)}{2M} \frac{(z(\tau(u)))}{z(u)}$ du. (3.12)

Combining (3.11) and (3.12) , we obtain

ln $\frac{z(s)}{z(t)}$ ≥ $\int_{s}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ)du,

or

z(s) ≥ z(t) exp($\int_{s}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du). (3.13)

since τ(s) < s ≤ t, (3.13) guarantees that

z(τ(s)) ≥ z(t) exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du) . (3.14)

Combining (3.9) and (3.14), we obtain

z(t) – z(τ(t)) + $z(t) \int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du) ds ≤ 0.

Multiplying the last inequality by p(t) we find

z(t)p(t) – z(τ(t)) p(t) + $z(t)p(t) \int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ)du) ds ≤ 0,

p(t) [1 + $\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du) ds] z(t) ≤ z(τ(t)) p(t),

which in view of (3.5), becomes

p(t) [1 + $\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du) ds] z(t) ≤ -2M z' (t).

2M z' (t) + p(t) [1 +$\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε)dξ) du) ds] z(t) ≤ 0,

or

z' (t) + $\frac{p(t)}{2M}$ [1 +$\int_{\tau(t)}^{t} \frac{p(s)}{2M}$ exp($\int_{\tau(s)}^{t} \frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_0$(ξ, ε) dξ) du) ds] z(t) ≤ 0,

z' (t) + d_1(ξ, ε) z(t) ≤ 0, by the definition of d_ℓ(ξ, ε) (3.15)

Now it becomes apparent that by repeating the above steps, we can get

z' (t) + d_ℓ(ξ, ε) z(t) ≤ 0, (ℓ ϵ N),

and

z(τ(s)) ≥ z(h(t)) exp($\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$(ξ, ε) d ξ) du) (3.16)

Integrating (3.5) from h(t) to t, and using (3.16), we have z(t) – z(h(t))) + $\frac{1}{2M}\int_{h(t)}^{t} p(s)\ (\ z(\tau(s\ ))ds \quad \leq\ 0.$

z(t) − z(h(t)) +$\frac{1}{2M}$z(h(t) $\int_{h(t)}^{t} p(s)\exp(\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\ \ \exp(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\ d\xi\ )du\ )ds\ \leq 0.$ (3.17) The inequality is valid even if we omit z(t) > 0 in the left – hand side. Therefore,

$z(h(t)\left[\frac{1}{2M}\int_{h(t)}^{t} p(s)\exp(\ \int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\ \ \exp(\ \int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\ \ d\xi\ )du\ )ds\ -\ 1\right] \leq 0,$

which means that

$\lim_{t\to\infty} sup \int_{\tau(t)}^{t} p(s)$ exp ($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_\ell$ (ξ, ε) dξ) du) ds ≤ 2M.

Since ε may be taken arbitrarily small, this inequality contradicts (3.1).

The proof of the theorem is complete. □

Theorem 3.2:

Assume that h(t) is defined by (2.6) and let α ϵ (0,$\frac{1}{e}$].If for some ℓ ϵ N,

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)$ exp($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ dξ)du) ds > 2M (1 - M_α) (3.18)

where d_ℓ is defined by (3.2), then all solutions of (1.1) are oscillatory.

Proof:

Let z(t) be an eventually positive solution of (1.1). Then as in the proof of Theorem 3.1, we obtain (3.17) That is for sufficiently large t, we have

z(t) – z(h(t)) + z(h(t)) $\int_{h(t)}^{t}\frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ) du) ds ≤ 0

z(h(t)) [1 − $\int_{h(t)}^{t}\frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ)du) ds] ≥ z(t)

or

1 - $\int_{h(t)}^{t}\frac{p(s)}{2M}$ exp ($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ)du) ds ≥ $\frac{z(t)}{z(h(t))}$

or

$\int_{h(t)}^{t} p(s)$ exp ($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ)du) ds ≤ 2M (1 - $\frac{z(t)}{z(h(t))}$).

Now taking limit supremum as t → ∞ , we get

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)$ exp($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ)du) ds ≤ 2M [1 – $\lim_{t\to\infty} inf\ \frac{z(t)}{z(h(t))}$]

By Lemma 2.2, the last inequality becomes

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)$ exp($\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ dξ)du) ds ≤ 2M (1 – M_α)

which contradicts (3.18).

Hence the proof of the Theorem is complete. □

Theorem 3.3:

Assume that h(t) is defined by (2.6) and let α ϵ (0,$\frac{1}{e}$].If for some ℓ ϵ N

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)$ exp($\int_{\tau(s)}^{t}\frac{p(u)}{2M}$ exp($\int_{\tau(u)}^{u} d_\ell$ d ξ)du) ds > 2M ($\frac{1}{M\alpha}$ - 1) (3.19)

where d_ℓ is defined by (3.2), then all the solutions of (1.1) are oscillatory.

Proof:

Assume for the sake of contradiction, that there exists a non – oscillatory solution z(t) of (1.1) and that z(t) is eventually positive. Then as in the proof of Theorem 3.1, for sufficiently large t, we have (3.16).That is,

z(τ(s)) ≥ z(t) exp($\int_{\tau(s)}^{t}\frac{p(u)}{2M}$ exp ($\int_{\tau(u)}^{u} d_\ell$ dξ) du) (3.20)

Integrating (3.5) from h(t) to t gives

$\int_{h(t)}^{t} z'$ (s) ds + $\int_{h(t)}^{t}\frac{p(s)}{2M}$ z(τ(s)) ds ≤ 0

or

z(t) – z(h(t)) + $\int_{h(t)}^{t}\frac{p(s)}{2M}$ z(τ(s)) ds ≤ 0,

which in view of (3.20) gives

$z(t) - z(h(t)) + \int_{h(t)}^{t} \frac{p(s)}{2M} z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi \right) du\right) ds \le 0$. That is

$\frac{1}{2M} \int_{h(t)}^{t} p(s)\, z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds \le \frac{z(h(t))}{z(t)} - 1.$

Now taking limit superemum on both sides, we get

$$\frac{1}{2M} \lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)\, z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi \right) du\right) ds \le \lim_{t\to\infty} sup \frac{z(h(t))}{z(t)} - 1 \qquad (3.21)$$

By Lemma 2.2, (3.21) leads to

$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)\, z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi \right) du\right) ds \le 2M \left(\frac{1}{M_\alpha} - 1\right).$

This inequality contradicts (3.19).

The proof of the theorem is complete. □

Theorem 3.4:

Assume that h(t) is defined by (2.6) and let $\alpha \in (0, \frac{1}{e}]$. If for some $\ell \in N$,

$$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)\, z(t) \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds > 2M \left(\frac{1+ ln\, \mu_0}{\mu_0} - M_\alpha\right) \qquad (3.22)$$

where d_ℓ is defined by (3.2), then all solutions of (1.1) are oscillatory.

Proof:

By taking z(t) to be an eventually positive solution of (1.1) we obtain (3.20) as in Theorem 3.3. That is

$z(\tau(s)) \ge z(t) \exp\left(\int_{\tau(s)}^{t} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right).$

Since $\tau(s) \le h(s)$, the above inequality gives

$$z(\tau(s)) \ge z(h(s)) \exp\left(\int_{\tau(s)}^{h(s)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi \right) du \right). \qquad (3.23)$$

Observe that (2.8) implies that for each $\varepsilon > 0$ there exists t_ϵ such that

$$\frac{z(h(t))}{z(t)} > \mu_0 - \varepsilon \text{ for } t \ge t_\epsilon \ge t_1 \qquad (3.24)$$

Noting that by nondecreasing nature of the function $\frac{z(h(t))}{z(s)}$ in s, it holds

$1 = \frac{z(h(t))}{z(h(t))} \le \frac{z(h(t))}{z(s)} \le \frac{z(h(t))}{z(t)}$, $t_\varepsilon \le h(t) \le s \le t$,

In particular for $\varepsilon \in (0, \mu_0 - 1)$, by continuity we see that there exists a $t^* \in (h(t), t]$ such that

$$1 < \mu_0 - \varepsilon = \frac{z(h(t))}{z(t^*)}. \qquad (3.25)$$

Integrating (3.5) from t^* to t , we have

$\int_{t^*}^{t} z'(s)\, ds + \int_{t^*}^{t} \frac{p(s)}{2M} z(\tau(s))\, ds \le 0.$

$z(t) - z(t^*) + \int_{t^*}^{t} \frac{p(s)}{2M} z(\tau(s))\, ds \le 0.$

Using (3.23) along with $h(s) \le h(t)$ in combination with the nonincreasing nature of z,
we have

$z(t) - z(t^*) + z(h(t)) \left(\int_{t^*}^{t} \frac{p(s)}{2M} \exp\left(\int_{\tau(s)}^{h(s)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds \right) \le 0.$

$\frac{1}{2M} \int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds \le \frac{z(t^*)}{z(h(t))} - \frac{z(t)}{z(h(t))}.$

In view of (3.25) and Lemma 2.2, for the ε considered, there exists a $t_\varepsilon' \ge t_\varepsilon$ such that

$$\frac{1}{2M} \int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds < \frac{1}{\mu_0 - \varepsilon} - M_\alpha + \varepsilon \text{ for } t \ge t_\varepsilon' \qquad (3.26)$$

Dividing (3.6) by z(t) and integrating from h(t) to t^*, we find

$\int_{h(t)}^{t^*} \frac{z'(s)}{z(s)}\, ds + \int_{h(t)}^{t^*} \frac{p(s)}{2M} \frac{z(\tau(s))}{z(s)}\, ds \le 0$

or

$\int_{h(t)}^{t^*} \frac{p(s)}{2M} \frac{z(\tau(s))}{z(s)}\, ds \le - \int_{h(t)}^{t^*} \frac{z'(s)}{z(s)}\, ds.$

Using (3.23) in the last inequality ,

$$\int_{h(t)}^{t^*} \frac{p(s)}{2M} \frac{z(h(s))}{z(s)} \exp\left(\int_{\tau(s)}^{h(s)} \frac{p(u)}{2M} \exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi \right) du\right) ds \le - \int_{h(t)}^{t^*} \frac{z'(s)}{z(s)}\, ds \qquad (3.27)$$

By (2.8) for $s \ge h(s) \ge t_\varepsilon'$, we have $\frac{z(h(s))}{z(s)} > \mu_0 - \varepsilon$.

From (3.27), we get

$$\frac{(\mu_0 - \varepsilon)}{2M}\int_{h(t)}^{t^*} p(s)\exp\left(\int_{\tau(s)}^{h(s)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds \le -\int_{h(t)}^{t^*}\frac{z'(s)}{z(s)} ds.$$

Hence for sufficiently large t, we have

$$\frac{1}{2M}\int_{h(t)}^{t^*} p(s)\exp\left(\int_{\tau(s)}^{h(s)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds < -\frac{1}{(\mu_0 - \varepsilon)}\int_{h(t)}^{t^*}\frac{z'(s)}{z(s)} ds$$

$$= -\frac{1}{(\mu_0 - \varepsilon)}\left[\ln \frac{z(h(t))}{z(t^*)}\right]$$

$$< \frac{1}{(\mu_0 - \varepsilon)}\left[\ln ((\mu_0 - \varepsilon)\right] .$$

That is,

$$\frac{1}{2M}\int_{h(t)}^{t^*} p(s)\exp\left(\int_{\tau(s)}^{h(s)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds < \frac{1}{(\mu_0 - \varepsilon)}\left[\ln ((\mu_0 - \varepsilon)\right]. \tag{3.28}$$

Adding (3.26) and (3.28) and then taking the limit as $t \to \infty$, we have

$$\lim_{t\to\infty} sup \int_{h(t)}^{t} p(s)\, z(t) \exp\left(\int_{\tau(s)}^{h(s)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds \le 2M\left[\frac{1+\ln ((\mu_0 - \varepsilon)}{(\mu_0 - \varepsilon)} - M_\alpha + \varepsilon\right]$$

Since ε may be taken arbitrarily small, this inequality contradicts (3.19).

The proof of the theorem is complete.

Next, let us proceed to an oscillation condition involving limit infimum.

Theorem 3.5:

Assume that h(t) is defined by (2.6) and for some $\ell \in N$,

$$\lim_{t\to\infty} inf \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell \, d\xi\right) du\right) ds > \frac{1}{e} \tag{3.29}$$

where d_ℓ is defined by (3.2), then all the solutions of (1.1) are oscillatory.

Proof:

Assume for the sake of contradiction that, there exists a nonoscillatory solution z(t) of (1.1).

Since -z(t) is also a solution of (1.1), we can confine our discussion only to the case where the solution z(t) is eventually positive.

Then, there exists a $t_1 > t_0$ such that z(t) and z(τ(t)) > 0 for all $t \ge t_1$.

Thus from (1.1), we have

$z'(t) \le -\frac{1}{2M} p(t)z(\tau(t))$ for all $t \ge t_1$

which means that z(t) is an eventually nonincreasing function of positive numbers.

We note that we may obtain (3.16) as in the proof of Theorem 3.1.

Dividing (3.5) by z(t) and integrating from h(t) to t, we have

$\int_{h(t)}^{t}\frac{z'(s)}{z(s)} ds + \int_{h(t)}^{t^*}\frac{p(s)}{2M}\frac{z(\tau(s))}{z(s)} ds \le 0$. That is,

$$\ln\frac{z(h(t))}{z(t)} \ge \frac{1}{2M}\int_{h(t)}^{t} p(s)\,\frac{z(\tau(s))}{z(s)} ds.$$

From which, in view of τ(s) ≤ h(s) and by (3.16), we obtain

$$\ln\frac{z(h(t))}{z(t)} \ge \frac{1}{2M}\int_{h(t)}^{t} p(s)\,\frac{z(h(s))}{z(s)}\exp\left(\int_{\tau(s)}^{h(s)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\, d\xi\right) du\right) ds.$$

Using monotonocity of z(t) and h(s) < s in the last inequality leads to

$$\ln\frac{z(h(t))}{z(t)} \ge \frac{1}{2M}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\, d\xi\right) du\right) ds. \tag{3.30}$$

From (3.29), it follows that there exists a constant c > 0 such that for a sufficiently large t, the following inequality holds,

$$\frac{1}{2M}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\, d\xi\right) du\right) ds \ge c > \frac{1}{e} .$$

Choose c' such that $c > c' > \frac{1}{e}$.

For every ε > 0 such that $c - \varepsilon > c'$, we have

$$\frac{1}{2M}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)}\frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\, d\xi\right) du\right) ds \ge c - \varepsilon > c' > \frac{1}{e} . \tag{3.31}$$

Combining the inequalities (3.30) and (3.31),

$\ln \frac{z(h(t))}{z(t)} > c'$

$\frac{z(h(t))}{z(t)} > e^{c'} \geq ec' > 1$, by the choice of c'. That is ,

z(h(t)) ≥ ec′ z(t) .

Repeating the above procedure, it follows by induction that for any positive integer k,

$\frac{z(h(t))}{z(t)} \geq (ec')^{k}$, for sufficiently large t.

Since ec′ > 1 there is a k ε N satisfying

$k > \frac{2(\ln 2 - \ln c')}{1 + \ln c'}$,

such that for t sufficiently large

$\frac{z(h(t))}{z(t)} \geq (ec')^{k} > (\frac{2}{c'})^2$. (3.32)

Taking the integral on [h(t),t] whose length is not less than c′, we split the interval into two parts where their lengths are not less than $\frac{c'}{2}$.

Let t_m ϵ (h(t),t) ,be the splitting point. Then

$$\frac{1}{2M}\int_{h(t)}^{t_m} p(s)\ \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\ \exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\ d\xi\right)du\right) ds \geq \frac{c'}{2} \quad (3.33)$$

$$\frac{1}{2M}\int_{t_m}^{t} p(s)\ \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\ \exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\ d\xi\right)du\right) ds \geq \frac{c'}{2} \quad (3.34)$$

Integrating (3.5) from h(t) to t_m ,and using the fact that z(t_m) > 0 ,we obtain

$\int_{h(t)}^{t_m} z'(s)\,ds + \int_{h(t)}^{t_m} \frac{p(s)}{2M} z(\tau(s))\,ds \leq 0$.That is,

$z(h(t) - z(t_m) \geq \int_{h(t)}^{t_m} \frac{p(s)}{2M} z(\tau(s))\,ds$

or

$z(h(t) \geq \int_{h(t)}^{t_m} \frac{p(s)}{2M} z(\tau(s))\,ds$. Now using (3.16),this inequality becomes

$z(h(t)) > \int_{h(t)}^{t_m} \frac{p(s)}{2M} z(h(t)) \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\ \exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\,d\xi\right) du\right) ds)$.That is

$z(h(t)) > \frac{z(h(t_m)}{2M}\int_{h(t)}^{t_m} p(s)\ \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\ \exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\,d\xi\right) du\right) ds)$,

which in view of (3.33) gives

$z(h(t)) > \frac{c'}{2} z(h(t_m)$. (3.35)

Similarly integrating (3.5) from t_m to t, we get

$z(t_m) > \frac{z(h(t))}{2M}\int_{t_m}^{t} p(s)\ \exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\ \exp\left(\int_{\tau(u)}^{u} d_\ell(\xi,\varepsilon)\,d\xi\right) du\right) ds$,

which in view (3.34) implies

$z(t_m) > \frac{c'}{2} z(h(t))$. (3.36)

Combining (3.35) and (3.36)

$z(h(t_m) < \frac{2}{c'} z(h(t) < (\frac{2}{c'})\ (\frac{2}{c'})\ z(t_m)$ which contradicts (3.32) .

The proof of the theorem is complete.

Example:

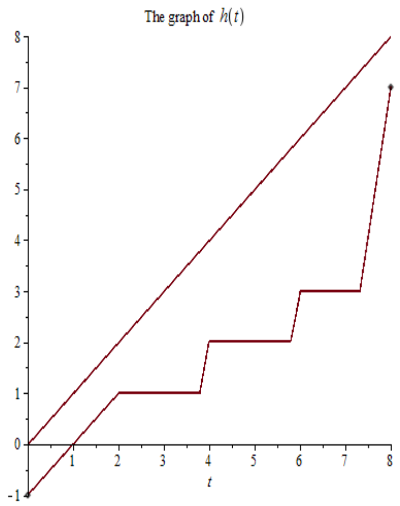
Consider the nonlinear delay differential equation

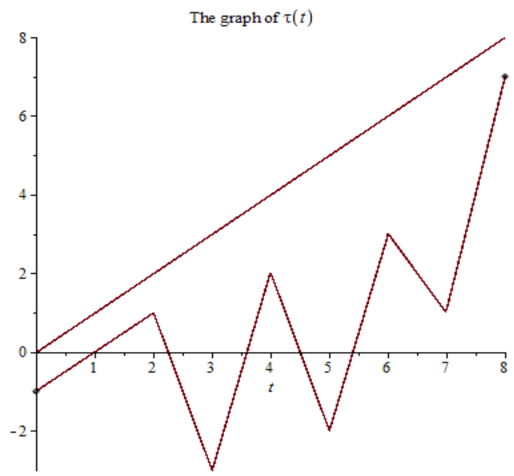
$$z'(t) + \frac{21}{200}\ z(\tau(t))(11 + |z(\tau(t))|) = 0,\ \ t \geq 1 \quad (3.37)$$

with

$$
\tau(t) = \begin{cases} t - 1, & t \in [8k, 8k+2] \\ -4t + 40k + 9, & t \in [8k+2, 8k+3] \\ 5t - 32k - 18, & t \in [8k+3, 8k+4] \\ -4t + 40k + 18, & t \in [8k+4, 8k+5] \\ 5t - 32k - 27, & t \in [8k+5, 8k+6] \\ -2t + 24k + 15, & t = [8k+6, 8k+7] \\ 6t - 40k - 41, & t = [8k+7, 8k+8] \end{cases}
$$

$$
h(t) = \begin{cases} t - 1, & t \in [8k, 8k+2] \\ 8k + 1, & t \in [8k+2, 8k+\frac{19}{5}] \\ 5t - 32k - 18, & t \in [8k+\frac{19}{5}, 8k+4] \\ 8k + 2, & t \in [8k+4, 8k+\frac{29}{5}] \\ 5t - 32k - 27, & t \in [8k+\frac{29}{5}, 8k+6] \\ 8k + 3, & t \in [8k+6, 8k+\frac{44}{6}] \\ 6t - 40k - 41, & t \in [8k+\frac{44}{6}, 8k+8]. \end{cases}
$$





Comparing (3.37) with (3.1), we have

$f(z) = \frac{z}{z(11+|z(\tau(t))|)} = \frac{1}{(11+|z(\tau(t))|)}$ and $p(t) = \frac{21}{200} = 0.105$.

It is easy to see that

$M = \limsup_{z \to 0} \frac{z}{z(11+|z|)} = 0.09$ and

β = $\lim_{t\to\infty}\inf\int_{\tau(t)}^{t} p(s)ds$ = $\lim_{t\to\infty}\inf\int_{8k+2}^{8k+4}(0.105)ds$ = 0.21.

The smaller root of $e^{0.21\mu}$= μ is μ_0 = 1.26

Let ε = 1.25, then μ_0 – ε = 0.01.

Let F(t) = $\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} \frac{p(u)}{2M}\exp\left(\int_{\tau(u)}^{u} d_1(\xi)d\xi\right) du\right) ds$.

with

$d_0(t) = \frac{p(t)}{2M}\left[1+\int_{\tau(t)}^{t}\frac{p(s)}{2M}\exp\left(\int_{\tau(s)}^{t}\frac{p(\omega)}{2M}exp\left((\mu_0-\varepsilon)\int_{\tau(\omega)}^{\omega}\frac{p(u)}{2M}\,du\right)d\omega\right)ds\right]$

$d_1(t) = \frac{p(t)}{2M}\left[1+\int_{\tau(t)}^{t}\frac{p(s)}{2M}\exp\int_{\tau(s)}^{t}\frac{p(u)}{2M}exp\int_{\tau(u)}^{u}(d_0(\varepsilon)\,d\varepsilon)du)\,ds\right]$

We obtain

$d_0(t)$ = 2.74

$d_1(t, €)$ = 4.45 x 10^{120}.

We obtain

$\lim_{t\to\infty} sup$ F(t =8k + 4) ≅ 0.21 > 2M= 0.18 .

That is condition of Theorem 3.1 is satisfied. Therefore, all the solutions of (3.37) are Oscillatory.

4. Conclusion

In this paper, the oscillatory behaviour of first order delay differential equations with non-monotone arguments is discussed. The sufficient conditions namely (3.1), (3.18), (3.19), (3.22) and (3.25) establish the oscillations of the solutions of (1.1). Moreover, an example to substantiate Theorem (3.1) is also given.

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