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Connectedness in R-Cubic Topological Spaces with Several Types of e-Connectedness

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Abstract

This paper aims to present various forms of e-connectedness within R-cubic topological spaces. The conceptual foundation for these constructions is derived from cubic sets, a framework developed by Y.B. Jun. Furthermore, we explore the interconnections among these different forms of connectedness, providing illustrative examples for clarity.

Keywords and phrases: CS_Re -connected, CS_ReC_5 -connected, CS_Re -strongly connected, CS_ReC_M -disconnected,CS_ReC_s-connected,CS_Re-extremally disconnected.

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1 Introduction

Zadeh introduced the concepts of fuzzy set and interval-valued fuzzy set (IVFS) in [13, 14]. The notion of a fuzzy topological space was first presented by C. L. Chang in 1968 [3]. Building upon the ideas of fuzzy sets and interval-valued fuzzy sets, Y. B. Jun introduced the novel concept of a cubic set in 2012 [9]. Akhtar, in 2016, extended this theory to create the framework of cubic topological space, exploring P-cubic and R-cubic topological spaces [1]. In 2019, Loganayaki and Jayanthi further developed the theory by introducing interior, closure, various types of open sets, and continuous mappings in P-cubic and R-cubic topological spaces [11].

The exploration of e-open sets in topological spaces was initiated by Ekici [4, 5, 6, 7]. This paper's objective is to introduce several forms of e-connectedness within R-cubic topological spaces, extending the research initiated by Ekici. Additionally, we investigate the relationships between these various types of connectedness, providing illustrative examples for better understanding.

Preliminaries

This section covers the fundamental definitions and properties of R-cubic topological spaces.

Definition 2.1 [14] An interval number, denoted by $b=[b^-, b^+]$, is defined as a closed sub-interval of $I=[0,1]$, where $0\leq b^-\leq b^+\leq 1$. The set of all interval numbers is represented by [I].

Definition 2.2 [14] Consider a non-empty set Z. A function $U: Z\to[I]$, mapping elements of Z to interval numbers, is termed an Interval Valued Fuzzy Set (IVFS) in Z. The set of all IVFS in Z is denoted by $[I]^Z$. For any $U\in[I]^Z$ and $z\in Z$, the degree of membership of z in U is represented as $U(z)=[U^-(z), U^+(z)]$. Here, $U^-: Z\to I$ and $U^+: Z\to I$ individually denote fuzzy sets in Z, where U^- is referred to as the lower fuzzy set, and U^+ is termed the upper fuzzy set.

Definition 2.3 [9] Consider a non-empty set Z. A structure $U=\langle z, \mu(z), \lambda(z)\rangle/z\in Z$ is defined as a cubic set($\mathcal{CS}$) in Z, where μ is an interval-valued fuzzy set (IVFS) in Z, and λ is a fuzzy set in Z. A $\mathcal{CS}$ can be succinctly denoted as $U=\langle\mu,\lambda\rangle$, and the collection of all $\mathcal{CS}$'s in Z is represented by C^Z.

Definition 2.4 [9] For a non-empty set Z $(Z\neq\phi)$, a $\mathcal{CS}$ $U=\langle\mu,\lambda\rangle$ is termed an Internal Cubic Set (ICS) if, for all $z\in Z$, the condition $\mu^-(z)\leq\lambda(z)\leq\mu^+(z)$ holds.

Definition 2.5 [9] For a non-empty set Z $(Z\neq\phi)$, a $\mathcal{CS}$ $U=\langle\mu,\lambda\rangle$ is considered an External Cubic Set (ECS) if, for all $z\in Z$, the condition $\lambda(z)\notin(\mu^-(z), \mu^+(z))$ is satisfied.

1. A $\mathcal{CS}$ $U = \langle \mu, \lambda \rangle$ where $\mu(z) = 0$ and $\lambda(z) = 1$ (or, respectively, $\mu(z) = 1$ and $\lambda(z) = 0$) for all $z \in Z$ is denoted as $\ddot{0}$ (or $\ddot{1}$).
2. A $\mathcal{CS}$ $U = \langle \mu, \lambda \rangle$ where $\mu(z) = 0$ and $\lambda(z) = 0$ (or, respectively, $\mu(z) = 1$ and $\lambda(z) = 1$) for all $z \in Z$ is denoted as $\hat{0}$ (or $\hat{1}$).

Let $U = \langle \mu, \lambda \rangle$ and $V = \langle \beta, \eta \rangle$ be two $\mathcal{CS}$'s in Z, Then we define;

1. $U = V \Leftrightarrow \mu = \beta$ and $\lambda = \eta$
2. $U \subseteq_R V \Leftrightarrow \mu \subseteq \beta$ and $\lambda \geq \eta$
3. $U^c = \overline{U} =< \mu^c,\ 1 - \lambda >= \{\langle z,\ \mu^c(z),\ 1 - \lambda(z) \rangle / z \in Z\}$
4. $(U^c)^c = \overline{(\overline{U})} = U$
5. $\ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$
6. $\hat{0}^c = \hat{1}$ and $\hat{1}^c = \hat{0}$
7. R-Union $\bigcup_{i \in \mathbb{N}} U = \{\langle z,\ (\bigcup_{i \in \mathbb{N}} \mu_i)(z), (\wedge \lambda_i i \in \mathbb{N}(z)/z \in Z \rangle\}$
8. R-Intersection $\bigcap_{i \in \mathbb{N}} U = \{\langle z,\ (\bigcap_{i \in \mathbb{N}} \mu_i)(z), (\vee \lambda_i i \in \mathbb{N}(z)/z \in Z \rangle\}$

Definition 2.6 [1] A R-cubic topology is the family $\mathcal{F}_R$ of $\mathcal{CS}$'s in Z which satises the following conditions;

1. $\ddot{0}, \hat{0}, \ddot{1}, \hat{1} \in \mathcal{F}_R$.
2. Let $A_i \in \mathcal{F}_R$, Then $\bigcup_R A_i \in \mathcal{F}_R . i \in \mathbb{N}$
3. Let $A, B \in \mathcal{F}_R$, Then $A \bigcap_R B \in \mathcal{F}_R$.

The pair $(Z, \mathcal{F}_R)$ is referred to as an R-cubic topological space, abbreviated as Rcts.

Definition 2.7 [11] A set U is said to be a R-order Cubic set (in brief, CS_R)

1. regular open set (briefly, $CS_R\delta ros$) if $U = CS_R int(CS_R cl(U))$.
2. regular closed set (briefly, $CS_R\delta rcs$) if $U = CS_R cl(CS_R int(U))$.

Definition 2.8 [11] A set U is said to be a CS_R

1. interior(resp. δ interior) of U (briefly, $CS_R int(U)$ (resp. $CS_R\delta int(U)$)) is defined by $CS_R int(U)$ (resp. $CS_R\delta int(U)$) = $\bigcup$ $\{\widetilde{G}: \widetilde{G} \subseteq U$ & $\widetilde{G}$ is a $CS_R os$ (resp. $CS_R\delta os$) in $Z\}$.
2. closure(resp. δ closure) of U (briefly, $CS_R cl(U)$ (resp. $CS_R\delta cl(U)$)) is defined by $CS_R cl(U)$ (resp. $CS_R\delta cl(U)$) = $\bigcap$ $\{\widetilde{G}: \widetilde{G} \supseteq U$ & $\widetilde{G}$ is a $CS_R cs$ (resp. $CS_R\delta cs$) in $Z\}$.

Definition 2.9 [11] A set U is said to

1. β open set (briefly, $CS_R\beta os$) if $U \subseteq CS_R cl(CS_R int(CS_R cl(U)))$.

Definition 2.10 [12] A set U is said to be a CS_R

1. δ-pre open set (briefly, $CS_R\delta\mathcal{P}os$) if $U \subseteq CS_R int(CS_R\delta cl(U))$.
2. δ-semi open set (briefly, $CS_R\delta\mathcal{S}os$) if $U \subseteq CS_R cl(CS_R\delta int(U))$.
3. e-open set (briefly, $CS_R eos$) if $U \subseteq CS_R cl(CS_R\delta int(U)) \cup CS_R int\ (CS_R\delta cl(U))$.
4. e^*-open set (briefly, $CS_R e^* os$) if $U \subseteq CS_R cl(CS_R int(CS_R\delta cl(U)))$.
5. a-open set (briefly, $CS_R aos$) if $U \subseteq CS_R int(CS_R cl(CS_R\delta int(U)))$.

The complement of a CS_Re-open set (resp. $CS_R\delta os$, $CS_R\delta\mathcal{P}os$, $CS_R\delta\mathcal{S}os$ & $CS_R e^* os$) is called CS_Re- (resp. δ, δ-pre, δ-semi & e^*) closed set (briefly, $CS_R ecs$ (resp. $CS_R\delta cs$ $CS_R\delta\mathcal{P}cs$, $CS_R\delta\mathcal{S}cs$ & $CS_R e^* cs$)) in Z.
The family of all $CS_R\delta\mathcal{P}os$ (resp. $CS_R\delta\mathcal{P}cs$, $CS_R\delta\mathcal{S}os$, $CS_R\delta\mathcal{S}cs$, $CS_R eos$, $CS_R ecs$ $CS_R e^* os$ & $CS_R e^* cs$) of Z is denoted by $CS_R\delta\mathcal{P}OS(Z)$ (resp. $CS_R\delta\mathcal{P}CS_R(Z)$, $CS_R\delta\mathcal{S}OS(Z)$, $CS_R\delta\mathcal{S}CS_R(Z)$, $CS_R eOS(Z)$, $CS_R eCS_R(Z)$ $CS_R e^* OS(Z)$ & $CS_R e^* CS_R(Z)$).

Definition 2.11 [12] A set U is said to be a C

1. e interior(resp. δ pre interior & δ semi interior) of U (briefly, $CS_R eint(U)$ (resp. $CS_R\delta\mathcal{P}int(U)$ & $CS_R\delta\mathcal{S}int(U)$)) is defined by $CS_R eint(U)$ (resp. $CS_R\delta\mathcal{P}int(U)$ & $CS_R\delta\mathcal{S}int(U)$) = $\bigcup$ $\{\widetilde{G}: \widetilde{G} \subseteq U$ & $\widetilde{G}$ is a $CS_R eos$ (resp. $CS_R\delta\mathcal{P}os$ & $CS_R\delta\mathcal{S}os$) in $Z\}$.
2. e closure(resp. δ pre closure & δ semi closure) of U (briefly, $CS_R ecl(U)$ (resp. $CS_R\delta\mathcal{P}cl(U)$ & $CS_R\delta\mathcal{S}cl(U)$)) is defined by $CS_R ecl(U)$ (resp. $CS_R\delta\mathcal{P}cl(U)$ & $CS_R\delta\mathcal{S}cl(U)$) = $\bigcap$ $\{\widetilde{G}: \widetilde{G} \subseteq U$ & $\widetilde{G}$ is a $CS_R ecs$ (resp. $CS_R\delta\mathcal{P}cs$ & $CS_R\delta\mathcal{S}cs$) in $Z\}$.

3. Various types of R-order e-connectedness in cubic topological spaces

Definition 3.1 In an Rcts $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$, it is considered $\mathrm{CS_R}$e-disconnected if there exist $\mathrm{CS_R}$eos L and J in Z, where $\mathrm{L} \neq \ddot{0}$, $\mathrm{J} \neq \ddot{0}$, $\mathrm{L} \cup \mathrm{J} = \ddot{1}$, and $\mathrm{L} \cap \mathrm{J} = \ddot{0}$. If Z is not $\mathrm{CS_R}$e-disconnected, then it is termed $\mathrm{CS_R}$e-connected.

Example 3.1 Let Z be a non-empty set and $\mathcal{F}_{\mathrm{R}}$ be the collection of $\mathcal{CS}$'s in Z then $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ be Rcts $\{\ddot{0}, \hat{0}, \ddot{1}, \hat{1}, \mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4, \mathrm{H}_5, \mathrm{H}_6, \mathrm{H}_7, \mathrm{H}_8, \mathrm{H}_9, \mathrm{H}_{10}, \mathrm{H}_{11}, \mathrm{H}_{12}, \mathrm{H}_{13}, \mathrm{H}_{14}, \mathrm{H}_{15}\}$ where $\mathrm{H}_1 = \langle[0,0],0.3\rangle, \mathrm{H}_2 = \langle[0,0],0.4\rangle, \mathrm{H}_3 = \langle[0,0],0.6\rangle, \mathrm{H}_4 = \langle[0.2,0.4],0\rangle, \mathrm{H}_5 = \langle[0.2,0.4],0.6\rangle, \mathrm{H}_6 = \langle[0.2,0.4],1\rangle, \mathrm{H}_7 = \langle[0.5,0.7],0\rangle, \mathrm{H}_8 = \langle[0.5,0.7],0.3\rangle, \mathrm{H}_9 = \langle[0.5,0.7],1\rangle, \mathrm{H}_{10} = \langle[0.6,0.8],0\rangle, \mathrm{H}_{11} = \langle[0.6,0.8],0.4\rangle, \mathrm{H}_{12} = \langle[0.6,0.8],1\rangle, \mathrm{H}_{13} = \langle[1,1],0.3\rangle, \mathrm{H}_{14} = \langle[1,1],0.4\rangle, \mathrm{H}_{15} = \langle[1,1],0.6\rangle$ H_{16} and H_{17} are $\mathrm{CS_R}$eos in $\mathrm{Z}, \mathrm{H}_{16} \neq 0 \neq 0, \mathrm{H}_{17} \neq 0 \neq 0$ and $\mathrm{H}_{16} \cup \mathrm{H}_{17} = \mathrm{H}_{17} \neq \hat{1}$ and $\mathrm{H}_{16} \cap \mathrm{H}_{17} = \mathrm{H}_{16} \neq \hat{0}$. Hence Z is $\mathrm{CS_R}$e-connected.

Example 3.2 In Example 3.1 Consider the $\mathcal{CS}$ $\mathrm{H}_{16} = \langle[0,0],1\rangle$, $\mathrm{H}_{17} = \langle[1,1],0\rangle$ are $\mathrm{CS_R}$eos in Z, $\mathrm{H}_{16} \neq \hat{0}$ and $\mathrm{H}_{17} \neq \hat{0}$ and $\mathrm{H}_{16} \cup \mathrm{H}_{17} = \hat{1}, \mathrm{H}_{16} \cap \mathrm{H}_{17} = \hat{0}$. Hence Z is $\mathrm{CS_R}$e-connected

Definition 3.2 In an Rcts $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$, it is termed $\mathrm{CS_R}$eC5-disconnected if there exists a $\mathcal{CS}$ L in Z, which is both a $\mathrm{CS_R}$eos and a $\mathrm{CS_R}$ecs , such that $\mathrm{L} \neq \ddot{0}$ and $\mathrm{L} \neq \ddot{1}$. If Z is not $\mathrm{CS_R}\mathrm{eC}_5$-disconnected, then it is considered $\mathrm{CS_R}\mathrm{eC}_5$-connected.

Example 3.3 In Example 3.1 Consider the $\mathcal{CS}$ $\mathrm{H}_{16} = \langle[0.5,0.7],0.3\rangle$ is a $\mathrm{CS_R}$eos but not $\mathrm{CS_R}$ecs

Example 3.4 In Example 3.1 Consider the $\mathcal{CS}$ $\mathrm{H}_{16} = \langle[0.6,0.8],0.4\rangle$ which is a $\mathrm{CS_R}$eos also $\mathrm{CS_R}$ecs such that $\hat{1} \neq \mathrm{H}_{16} \neq \hat{0}$. Thus Z $\mathrm{CS_R}\mathrm{eC}_5$-connected.

Proposition 3.1 If an Rcts $(\mathrm{Z}, \mathcal{F}\mathrm{R})$ is $\mathrm{CS_R}$eC5-connected, then it is also $\mathrm{CS_R}$e-connected.

Proof. If there exist nonempty $\mathrm{CS_R}$eos sets L and J in an Rcts $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ such that $\mathrm{L} \cup \mathrm{J} = \ddot{1}$ and $\mathrm{L} \cap \mathrm{J} = \ddot{0}$ (making it $\mathrm{CS_R}$e-disconnected), and if $\bar{\mathrm{J}} = \mathrm{L}$ (closure of J is L), then L is $\mathrm{CS_R}$e-clopen, which implies that Z is $\mathrm{CS_R}$eC5-disconnected.

The converse need not be true, and we provide an example to illustrate that.

Example 3.5 In Example 3.1 Consider the $\mathcal{CS}$ $\mathrm{H}_{16} = \langle[0.6,0.8],0.4\rangle$,$\mathrm{H}_{17} = \langle[0.3,0.5],0.7\rangle$, H_{16} and H_{17} are $\mathrm{CS_R}$eos in Z. Also $\mathrm{H}_{16} \cup \mathrm{H}_{17} = \mathrm{H}_{16} \neq \hat{1}, \mathrm{H}_{16} \cap \mathrm{H}_{17} = \mathrm{H}_{17} \neq \hat{0}$.Hence Z is $\mathrm{CS_R}$e-connected. Since H_{16} is both $\mathrm{CS_R}$eos and $\mathrm{CS_R}$ecs in Z, Z is $\mathrm{CS_R}\mathrm{eC}_5$-disconnected.

Proposition 3.2 Let $\mathrm{f}: (\mathrm{Z}, \mathcal{F}_{\mathrm{R}}) \rightarrow (\mathrm{Y}, \mathcal{T}_{\mathrm{R}})$ be a $\mathrm{CS_R}$e-irresolute surjection. If $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ is $\mathrm{CS_R}$e-connected, then $(\mathrm{Y}, \mathcal{T}_{\mathrm{R}})$ is also $\mathrm{CS_R}$e-connected.

Proof. Assume, for a contradiction, that $(\mathrm{Y}, \mathcal{T}_{\mathrm{R}})$ is not $\mathrm{CS_R}$e-connected. Consequently, there exist nonempty $\mathrm{CS_R}$eos sets L and J in $(\mathrm{Y}, \mathcal{T}_{\mathrm{T}})$ such that $\mathrm{L} \cup \mathrm{J} = \ddot{1}$ and $\mathrm{L} \cap \mathrm{J} = \ddot{0}$.

Given that f is $\mathrm{CS_R}$e-irresolute, consider the preimages of L and J in $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ as $\mathrm{R} = \mathrm{f}^{-1}(\mathrm{L})$ and $\mathrm{U} = \mathrm{f}^{-1}(\mathrm{J})$. Since $\mathrm{L} \cup \mathrm{J} = \ddot{1}$, it implies $\mathrm{R} \cup \mathrm{U} = \ddot{1}$, and since $\mathrm{L} \cap \mathrm{J} = \ddot{0}$, it implies $\mathrm{R} \cap \mathrm{U} = \ddot{0}$.

This suggests that Z is $\mathrm{CS_R}$e-disconnected, contradicting our assumption that $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ is $\mathrm{CS_R}$e-connected. Therefore, the initial assumption that $(\mathrm{Y}, \mathcal{T}_{\mathrm{R}})$ is not $\mathrm{CS_R}$e-connected is invalid, leading to the conclusion that Y is indeed $\mathrm{CS_R}$e-connected

Proposition 3.3 $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ is $\mathrm{CS_R}$eC5-connected if and only if there are no nonempty $\mathrm{CS_R}$eos sets L and J in Z such that $\mathrm{L} = \bar{\mathrm{J}}$.

Proof. Suppose there exist $\mathrm{CS_R}$eos sets L and J in Z such that $\mathrm{L} \neq \ddot{0} \neq \mathrm{J}$ and $\mathrm{L} = \bar{\mathrm{J}}$. As $\mathrm{L} = \bar{\mathrm{J}}$, $\bar{\mathrm{J}}$ is a $\mathrm{CS_R}$eos and J is a $\mathrm{CS_R}$ecs. The condition $\mathrm{L} \neq \ddot{0}$ implies $\mathrm{J} \neq \ddot{1}$. However, this contradicts the assumption that Z is $\mathrm{CS_R}\mathrm{eC}_5$-connected.

Conversely, let L be both $\mathrm{CS_R}$eos and $\mathrm{CS_R}$ecs in Z such that $\ddot{0} \neq \mathrm{L} \neq \ddot{1}$. Take $\mathrm{J} = \bar{\mathrm{L}}$. Since J is a $\mathrm{CS_R}$eos and $\mathrm{L} \neq \ddot{1}$, it implies $\mathrm{J} = \bar{\mathrm{L}} \neq \ddot{0}$, which leads to a contradiction.

Definition 3.3 An Rcts $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ is considered $\mathrm{CS_R}$e-strongly connected if there are no nonempty $\mathrm{CS_R}$ecs sets L and J in Z such that $\mathrm{L} \cap \mathrm{J} = \ddot{0}$.

Proposition 3.4 An Rcts $(\mathrm{Z}, \mathcal{F}_{\mathrm{R}})$ is defined as $\mathrm{CS_R}$e-strongly connected if there are no $\mathrm{CS_R}$eos sets L and J in Z (where $\mathrm{L} \neq \ddot{1} \neq \mathrm{J}$) such that $\mathrm{L} \cap \mathrm{J} = \ddot{0}$.

Example 3.6 In Example 3.1 Consider the $\mathcal{CS}$ $\mathrm{H}_{16} = \langle[0.3,0.5],0.7\rangle$, $\mathrm{H}_{17} = \langle[0.6,0.8],0.4\rangle$, H_{16} is a $\mathrm{CS_R}$eos. $\mathrm{H}_{16}, \mathrm{H}_{17}$ is a $\mathrm{CS_R}$eos. Hence Z is $\mathrm{CS_R}$e-strongly connected.

Proposition 3.5 If $\mathrm{f}: (\mathrm{Z}, \mathcal{F}_{\mathrm{R}}) \rightarrow (\mathrm{Y}, \mathcal{T}_{\mathrm{R}})$ is a $\mathrm{CS_R}$e-irresolute surjection and Z is $\mathrm{CS_R}$e-strongly connected, then Y is also CS_Re-strongly connected.

Proof. Suppose Y is not CS_Re-strongly connected; then, there exist CS_Recs sets C and D in Y such that $C \neq \ddot{0}$, $D \neq \ddot{0}$, and $C \cap D = \ddot{0}$. As f is CS_Re-irresolute, $f^{-1}(C)$ and $f^{-1}(D)$ are CS_Recs in Z, and $f^{-1}(C) \cap f^{-1}(D) = \ddot{0}$, $f^{-1}(C) \neq \ddot{0}$, and $f^{-1}(D) \neq \ddot{0}$.

(If $f^{-1}(C) = \ddot{0}$, then $f(f^{-1}(C)) = C$ implies $f(\ddot{0}) = C$, leading to $C = \ddot{0}$, a contradiction.)

Therefore, Z is CS_Re-strongly disconnected, contradicting the assumption. Thus, $(Y, \mathcal{T}_R)$ is CS_Re-strongly connected.

It's also noted that CS_Re-strongly connected does not imply CS_ReC_5-connected, and vice versa. For this purpose we see the following examples:

Example 3.7 In Example 3.1 Consider the $\mathcal{CS}$ $H_{16} = \langle[0.6,0.8],0.4\rangle. H_{17} = \langle[0.2,0.4],0.6\rangle$, Z is CS_ReC_5-connected. Hence Z is CS_Re-strongly connected. But Z is not CS_ReC_5-connected since H_{16} is CS_Reos and CS_Recs in Z.

Example 3.8 In Example 3.1 Consider the $\mathcal{CS}$ $H_{16} = \langle[0.2,0.4],0.6\rangle, H_{17} = \langle[0.3,0.4],0.2\rangle$, Z is CS_ReC_5-connected. But Z is not CS_Re-strongly connected since H_{16}, H_{17} are CS_Reos in Z.

Definition 3.4 L and J are non-zero $\mathcal{CS}$ in $(Z, \mathcal{F}_R)$. Then L and J are said to be

1. CS_Re-weakly separated if $CS_R\mathrm{ecl}(L) \subseteq \bar{J}$ and $CS_R\mathrm{ecl}(J) \subseteq \bar{L}$.
2. CS_Re-q-separated if $(CS_R\mathrm{ecl}(L)) \cap J = \ddot{0} = L \cap (CS_R\mathrm{ecl}(J))$.

Definition 3.5 An Rcts $(Z, \mathcal{F}_R)$ is considered CS_ReC5-disconnected if there exist CS_Re-weakly separated non-zero $\mathcal{CS}$'s L and J in $(Z, \mathcal{F}_R)$ such that $L \cup J = \ddot{1}$.

Example 3.9 In Example 3.1 Consider the $\mathcal{CS}$ $H_{16} = \langle[0,0],1\rangle, H_{17} = \langle[1,1],0\rangle$ are CS_Reos in Z. $CS_R\mathrm{ecl}(H_{16}) \subseteq H_{17}$ and $CS_P\mathrm{ecl}(H_{17}) \subseteq H_{16}$. Hence H_{16} and H_{17} are CS_Re-weakly separated and so Z is CS_ReC_S-disconnected

Definition 3.6 An Rcts $(Z, \mathcal{F}_R)$ is termed CS_ReCM-disconnected if there exist CS_Re-q-separated non-zero cubic sets L and J in $(Z, \mathcal{F}_R)$ such that $L \cup J = \ddot{1}$.

Example 3.10 In Example 3.1 Consider the $\mathcal{CS}$ $H_{16} = \langle[0,0],1\rangle, H_{17} = \langle[1,1],0\rangle$ are CS_Reos in Z. $CS_R\mathrm{ecl}(H_{16}) \cap H_{17} = 0$ and $CS_P\mathrm{ecl}(H_{17}) \cap H_{16} = 0$ which implies H_{16} and H_{17} are CS_Re-q-separated. Hence Z is CS_ReC_M-disconnected.

Remark 3.1 An Rcts $(Z, \mathcal{F}_R)$ is CS_ReCS-connected if and only if it is CS_ReC_M-connected.

Definition 3.7 An Rcts $(Z, \mathcal{F}_R)$ is considered CS_Re-super disconnected if there exists a CS_Reros set L in Z such that $\ddot{0} \neq L \neq \ddot{1}$. Z is referred to as CS_Re-super connected if it is not CS_Re-super disconnected.

Example 3.11 In Example 3.1 Consider the $\mathcal{CS}$'s $H_{16} = \langle[0.2,0.4],0.6\rangle, H_{17} = \langle[0.6,0.8],0.4\rangle$, H_{16}, H_{17} are CS_Reos's in Z and $CS_R\mathrm{ecl}(CS_R\mathrm{eint}(H_{16})) = H_{16}$. This implies H_{16} is a CS_Reros in Z. Hence Z is a CS_Re-super disconnected

Proposition 3.6 The equivalence of the given statements in the context of an Rcts $(Z, \mathcal{F}_R)$ is as follows:

1. Z is CS_Re-super connected.
2. For each CS_Reos $L \neq \ddot{0}$ in Z, we have $CS_R\mathrm{ecl}(L) = \ddot{1}$.
3. For each CS_Recs $L \neq \ddot{1}$ in Z, we have $CS_R\mathrm{eint}(L) = \ddot{0}$.
4. There exists no CS_Reos L and J in Z such that $L \neq \ddot{0} \neq J$ and $L \subseteq \bar{J}$.
5. There exists no CS_Reos L and J in Z such that $L \neq \ddot{0} \neq J$, $J = \overline{CS_R\mathrm{ecl}(L)}$, and $L = \overline{CS_R\mathrm{ecl}(J)}$.
6. There exists no CS_Recs L and J in Z such that $L \neq \ddot{1} \neq J$, $J = \overline{CS_R\mathrm{eint}(L)}$, and $L = \overline{CS_R\mathrm{eint}(J)}$.

Proof. (i) $\Rightarrow$ (ii): Assume there exists a $L \neq \ddot{0}$ such that $CS_R\mathrm{ecl}(L) \neq \ddot{1}$. Choose $L = CS_R\mathrm{eint}(CS_R\mathrm{ecl}(L))$. This creates a proper CS_Reros in Z, contradicting the assumption that Z is CS_Re-super connected.

(ii) $\Rightarrow$ (iii): Consider $L \neq \ddot{1}$ as a CS_Recs in Z. If we take $J = \bar{L}$, then J is a CS_Reos in Z, and $J \neq \ddot{0}$. Using (ii), we get $CS_R\mathrm{ecl}(J) = \ddot{1}$, which implies $\overline{CS_R\mathrm{ecl}(J)} = \ddot{0}$. Therefore, $CS_R\mathrm{eint}(\bar{J}) = \ddot{0}$, leading to $CS_R\mathrm{eint}A = \ddot{0}$.

(iii) $\Rightarrow$ (iv): Suppose L and J are CS_Reos in Z such that $L \neq \ddot{0} \neq J$ and $L \subseteq \bar{J}$. As $\bar{J}$ is a CS_Recs in Z, and $\bar{J} \neq \ddot{1}$, by (iii) we have $CS_R\mathrm{eint}\bar{J} = \ddot{0}$. However, $L \subseteq \bar{J}$ implies $L = CS_R\mathrm{eint}(L) \subseteq CS_R\mathrm{eint}(\bar{J}) = \ddot{0}$, leading to a contradiction.

(iv) $\Rightarrow$ (i): Let $\ddot{0} \neq L \neq \ddot{1}$ be a CS_Reros in Z. If we take $J = \overline{CS_R ecl(L)}$, then $J \neq \ddot{0}$, and $L \subseteq \bar{J}$, which is a contradiction. Therefore, Z is CS_Re-super connected.

(i) $\Rightarrow$ (v): Suppose L and J are two CS_Reos in $(Z, \mathcal{F}_R)$ such that $L \neq \ddot{0} \neq J$, $J = \overline{CS_R ecl(L)}$, and $L = \overline{CS_R ecl(J)}$. Now, we have $CS_R eint(CS_R ecl(L)) = CS_R eint(\bar{J}) = \overline{CS_R ecl(J)} = L$, and $L \neq \ddot{0}$ and $L \neq \ddot{1}$. Since $L = CS_R eint(L)$, it contradicts the assumption (i). Thus, (v) is true.

(v) $\Rightarrow$ (i): Let L be a CS_Reos in Z such that $L = CS_R eint(CS_R ecl(L))$, $0_\sim \neq L \neq \ddot{1}$. Now, take $J = \overline{CS_R ecl(L)}$. In this case, we have $J \neq \ddot{0}$, and J is a CS_Reos in Z, $J = \overline{CS_R ecl(L)}$, and $\overline{CS_R ecl(J)} = L$. However, this contradicts the assumption (v). Therefore, $(Z, \mathcal{F}_R)$ is CS_Re-super connected.

(v) $\Rightarrow$ (vi): Suppose L and J are CS_Recs in $(Z, \mathcal{F}R)$ such that $L \neq \ddot{1} \neq J$, $J = \overline{CS_R eint(L)}$, and $L = \overline{CS_R eint(J)}$. Taking $C = \bar{L}$ and $D = \bar{J}$, C and D become CS_Reos in $(Z, \mathcal{F}R)$, $C \neq \ddot{0} \neq D$, $\overline{CS_R eclC} = D$, and $\overline{CS_R eclD} = C$. This contradicts (v). Hence, (vi) is true.

(vi) $\Rightarrow$ (i): The proof is similar to the case of (v) $\Rightarrow$ (vi).

Proposition 3.7 Consider a surjective mapping $f: (Z, \mathcal{F}_R) \to (Y, \mathcal{J}_R)$ that is CS_Re-irresolute. If Z is CS_Re-super connected, then it follows that Y is also CS_Re-super connected.

Proof. Assume that Y is CS_Re-super disconnected. This implies the existence of CS_Reos C and D in Y, with $C \neq \ddot{0} \neq D$ and $C \subseteq \overline{D}$. Given that f is CS_Re-irresolute, it follows that $f^{-1}(C)$ and $f^{-1}(D)$ are CS_Reos in Z. Moreover, $C \subseteq \overline{D}$ implies $f^{-1}(C) \subseteq \overline{f^{-1}(D)}$. Therefore, $f^{-1}(C) \neq \ddot{0} \neq f^{-1}(\overline{D})$, indicating that Z is CS_Re-super disconnected. However, this contradicts the assumption that Z is CS_Re-super connected, leading to a contradiction

Definition 3.8 An Rcts $(Z, \mathcal{F}_R)$ is termed CS_ReC5-connected between two $\mathcal{CS}$'s L and J if there is no $\mathcal{CS}$ E in $(Z, \mathcal{F}_R)$ such that $L \subseteq E$ and $E\bar{q}J$.

Example 3.12 In Example 3.1 Consider the $\mathcal{CS}$'s $H_{16} = \langle[0.5,0.7],0.3\rangle, H_{17} = \langle[0.6,0.8],0.2\rangle$ $H_{18} = \langle[0.8,0.9],0.1\rangle$, H_{16} is CS_Reos in Z. Then Z is $CS_R eC_5$-connected between H_{16} and H_{17}.

Theorem 3.1 If an Rcts $(Z, \mathcal{F}_R)$ is $CS_R eC_5$-connected between two $\mathcal{CS}$'s L and J, then it is also $CS_R C_5$-connected between the same sets L and J.

Proof. Assume that $(Z, \mathcal{F}_R)$ is not $CS_R C_5$-connected between two $\mathcal{CS}$'s L and J. This implies the existence of CS_Ros E in $(Z, \mathcal{F}_R)$ such that $L \subseteq E$ and $E\bar{q}J$. Since every CS_Ros is also a CS_ReOS, there exists a CS_ReOS E in $(Z, \mathcal{F}_R)$ satisfying $L \subseteq E$ and $E\bar{q}J$. However, this contradicts the assumption that $(Z, \mathcal{F}_R)$ is not CS_Re-connected between L and J. Therefore, $(Z, \mathcal{F}_R)$ is $CS_R C_5$-connected between L and J.

The converse of the above theorem may not hold true, as illustrated by the following example.

Example 3.13 In Example 3.1 Consider the $\mathcal{CS}$'s $H_{16} = \langle[0.2,0.4],0.6\rangle$ $H_{17} = \langle[0.1,0.2],0.8\rangle$ $H_{18} = \langle[0.4,0.6],0.3\rangle$, H_{16} is CS_Ros in Z. Then $(Z, \mathcal{F}_R)$ $CS_R C_5$-connected between H_{17} and H_{18}. Consider $H_{19}\langle[0.2,0.4],0.6$ is a CS_Reos such that $H_{17} \leq H_[19]$ and $H_{19} \leq \overline{H_{18}}$ which implies $(Z, \mathcal{F}_R)$ is $CS_P C_5$-disconnected between H_5 and H_6.

Theorem 3.2 Let $(Z, \mathcal{F}_R)$ be an Rcts, and let L and J belong to $(Z, \mathcal{F}_R)$. If $L\bar{q}J$, then $(Z, \mathcal{F}_R)$ is $CS_R eC_5$-connected between L and J.

Proof. Assume that $(Z, \mathcal{F}_R)$ is not $CS_R eC_5$-connected between L and J. Consequently, there exists a CS_ReOS E in $(Z, \mathcal{F}_R)$ such that $L \subseteq E$ and $E \subseteq \bar{J}$. This implies $L \subseteq \bar{J}$. However, this contradicts the assumption that $(Z, \mathcal{F}_R)$ is not $CS_R eC_5$-connected between L and J. Therefore, $(Z, \mathcal{F}_R)$ is $CS_R eC_5$-connected between L and J.

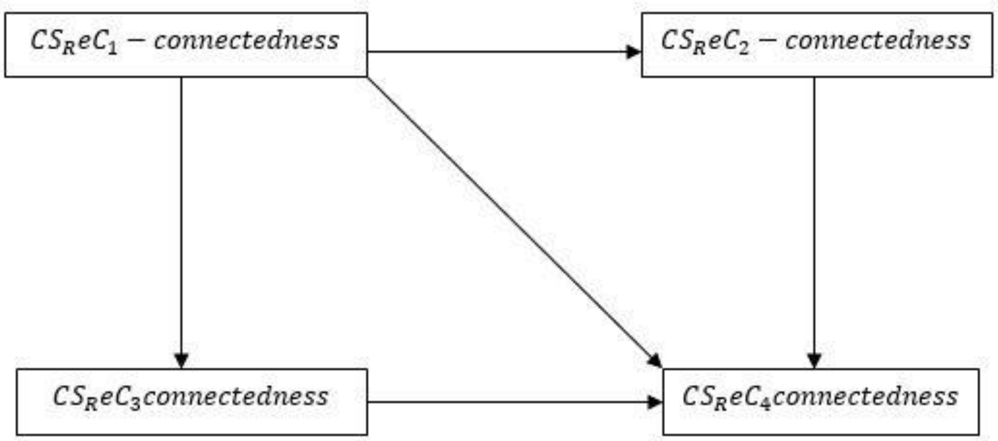
The converse of the above theorem may not hold true, as demonstrated by the following example.

Example 3.14 In Example 3.1 Consider the CS_Ros's $H_{16} = \langle[0.5,0.7],0.3\rangle$ $H_{17} = \langle[0.6,0.8],0.2\rangle$ $H_{18} = \langle[0.8,0.9],0.1\rangle$, H_{16} is CS_Reos in Z. Then $(Z, \mathcal{F}_R)$ is $CS_R eC_5$-connected between H_{16} and H_{17}. But H_{16} is not q-coincident with H_{16}.

Definition 3.9 Let N be an $\mathcal{CS}$ in Rcts $(Z, \mathcal{F}_R)$. (a) N is called $CS_R eC_i$-disconnected if there exist CS_Reos

M and W in Z satisfying the following properties. (i=1,2,3,4):
$C_1: N \subseteq M \cup W, M \cap W \subseteq \overline{N}, N \cap M \neq \ddot{0}, N \cap W \neq \ddot{0}$,
$C_2: N \subseteq M \cup W, N \cap M \cap W = \ddot{0}, N \cap M \neq \ddot{0}, N \cap W \neq \ddot{0}$,
$C_3: N \subseteq M \cup W, M \cap W \subseteq \overline{N}, M \nsubseteq \overline{N}, W \nsubseteq \overline{N}$,
$C_4: N \subseteq M \cup W, N \cap M \cap W = \ddot{0}, M \nsubseteq \overline{N}, W \nsubseteq \overline{N}$,
(b) N is said to be CS_ReC_i-connected $(i = 1,2,3,4)$ if N is not CS_ReC_i-disconnected $(i = 1,2,3,4)$.

Certainly, we can derive the following implications among various types of CS_ReC_i-connected sets, where i takes values from 1 to 4:



Example 3.15 In Example 3.1 $M = \langle[0.2,0.4],0.6\rangle, W = \langle[0.6,0.8],0.4\rangle$ be CS_Peos. Consider the $\mathcal{CS}$ $N = \langle[0.5,0.6],1\rangle$, N is $CS_ReC_2, CS_ReC_3, CS_ReC_4$-connected but CS_ReC_1-disconnected.
Example 3.16 In Example 3.1 $M = \langle[0,0],1\rangle, W = \langle[1,1],0\rangle$ be CS_Reos. Consider the $\mathcal{CS}$ $N = \langle[0.2,0.4],0.1\rangle$, N is CS_ReC_2-disconnected but CS_ReC_4-connected.
Example 3.17 In Example 3.1 $M = \langle[0.6,0.7],0.4\rangle, W = \langle[0,0.8],0.9\rangle$ be CS_Reos. Consider the $\mathcal{CS}$ $N = \langle[0.3,0.6],0.7\rangle$, N is CS_ReC_3-disconnected but CS_ReC_4-connected.

4. R-order e-Extremally Disconnectedness in Cubic Topological Spaces

Definition 4.1 Consider any Rcts $(Z, \mathcal{F}_R)$. Z is termed CS_Re-extremally disconnected if the e-closure of every CS_Reos in Z is itself a CS_Reos.
Theorem 4.1 For an Rcts $(Z, \mathcal{F}_R)$, the following statements are equivalent:

1. $(Z, \mathcal{F}_R)$ is a CS_Re-extremally disconnected space.
2. For each CS_Recs L, $CS_Reint(L)$ is a CS_Recs.
3. For each CS_Reos L, $CS_Recl(L) = \overline{CS_Recl(\overline{CS_Recl(L)})}$ is a CS_Recs.
4. For each CS_Reos L and J with $CS_Recl(L) = \overline{J}$, $CS_Recl(L) = \overline{CS_Recl(J)}$.

Proof. (i) $\Rightarrow$ (ii): Consider any CS_Recs denoted as L. This implies that the closure of L, denoted as $\overline{L}$, is a CS_Reos. Consequently, the closure of the interior of $\overline{L}$, i.e., $CS_Recl(\overline{L})$, becomes the closure of the interior of $CS_Reint(L)$, denoted as $\overline{CS_Reint(L)}$. As a result, $CS_Reint(L)$ is recognized as a CS_Recs within the context of $(Z, \mathcal{F}_R)$.
(ii) $\Rightarrow$ (iii): Let L be a CS_Reos. In this case, $CS_Recl(\overline{CS_Recl(L)})$ is equivalent to $CS_Recl(CS_Reint(\overline{L}))$, and $\overline{CS_Recl(\overline{CS_Recl(L)})}$ becomes $\overline{CS_Recl(CS_Reint(\overline{L}))}$. Given that L is a CS_Reos, implying that $\overline{L}$ is a CS_Recs,

according to (ii), $\mathrm{CS_R eint}(\bar{\mathrm{L}})$ is a $\mathrm{CS_R ecs}$. Consequently, $\mathrm{CS_R ecl}(\mathrm{CS_R eint}(\bar{\mathrm{L}})) = \mathrm{CS_R eint}(\bar{\mathrm{L}})$, leading to $\overline{\mathrm{CS_R ecl}(\mathrm{CS_R eint}(\bar{\mathrm{L}}))} = \overline{\mathrm{CS_R eint}(\bar{\mathrm{L}})} = \mathrm{CS_R ecl}(\mathrm{L})$.

(iii) $\Rightarrow$ (iv): Take any two $\mathrm{CS_R eos}$, L and J, in $(\mathrm{Z}, \mathcal{F}_\mathrm{R})$ such that $\mathrm{CS_R ecl}(\mathrm{L}) = \bar{\mathrm{J}}$. As per (iii), $\mathrm{CS_R ecl}(\mathrm{L}) = \overline{\mathrm{CS_R ecl}(\mathrm{J})}$.

(iv) $\Rightarrow$ (i): Let L be any $\mathrm{CS_R eos}$ in $(\mathrm{Z}, \mathcal{F}_\mathrm{R})$. Define J as $\overline{\mathrm{CS_R ecl}(\mathrm{L})}$. Consequently, $\mathrm{CS_R ecl}(\mathrm{L}) = \bar{\mathrm{J}}$. Applying (iv), it follows that $\mathrm{CS_R ecl}(\mathrm{L}) = \overline{\mathrm{CS_R ecl}(\mathrm{J})}$. This implies that $\mathrm{CS_R ecl}(\mathrm{L})$ is a $\mathrm{CS_R eos}$ in $(\mathrm{Z}, \mathcal{F}_\mathrm{R})$, establishing that $(\mathrm{Z}, \mathcal{F}_\mathrm{R})$ is $\mathrm{CS_R}$e-extremally disconnected.

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