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Monte Carlo Method for Financial Option Pricing

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Abstract

Monte Carlo methods allow for the introduction of a statistical approach to risk in financial decision. These methods utilize probabilistic simulations and pseudo random numbers, the precision of these methods is assessed by the size of confidence interval of the estimator's variance. We explore variance reduction methods such as the control variables method and the antithetic variables method, highlighting their significance in enhancing the accuracy of Monte Carlo methods through simulation in financial option pricing through specific examples, with a focus on the European option and the Asian option.

Keywords: Antithetic Variables, Control Variables, Financial Option, Monte Carlo Methods

1. Introduction

Option pricing is a fundamental concept in financial mathematics that plays a crucial role in the world of finance. It involves determining the fair value of an option, which is a financial derivative that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (strike price) within a specified time period (time of option expiration).

Within the realm of options, there exist various types, among which Asian options and European options hold significant importance as distinct categories, each possessing its distinctive attributes and considerations when it comes to pricing.

The key concept that holds utmost importance for our purpose is the representation of derivative prices as expected values, as this forms the underlying principle for the implementation of Monte Carlo simulations. We assume that there is no arbitrage opportunity.

2. Monte Carlo Method

Let π be a probability distribution on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ and f a measurable function on this space, such that

$\int_{\mathbb{R}^d} |f(x)|\, d\pi(x) < \infty$.

Denote,

$I_\pi(f) := \int_{\mathbb{R}^d} f(x) d\pi(x)$.

Let $(X_n)_{n\geq 1}$ be a sequence of d-dimensional random variables in $(\Omega, \mathcal{T}, P)$ with probability distribution π . We have

$\mathrm{E}[f\mathrm{o}X_n] = \int_\Omega f\mathrm{o}X_n(\omega)\, dP(\omega) = \int_{\mathbb{R}^d} f(x) d\pi(x) = I_\pi(f)$.

Thus, the sample mean $f\mathrm{o}X_n$ is an unbiased estimator of

$I_\pi(f)$,

$$\tilde{I}_{\pi,N}(f) = f(X_1) + f(X_2) + \cdots + f(X_N). \qquad (\mathbf{1})$$

Convergence is ensured by the theorem of Strrong law of large numbers (SLLN):

Theorem 1.[2] Let $X_1 + X_2 + \cdots + X_N$ be independent and uniformly distributed (i.i.d) integrable random variables with mean μ, then

$$\tilde{I}_{\pi,N}(f) \rightarrow I_\pi(f).$$

Under stronger assumptions, the Monte Carlo method provides an interval that contains the approximation of the integral with a given probability.

It is the confidence interval. In order to define this interval, assume

$$\int_{\mathbb{R}^d} |f(x)|^2 d\pi(x) < \infty,$$

denote

$$\sigma_\pi^2(f) := \int_{\mathbb{R}^d} |f(x)|^2 d\pi(x) - \left(\int_{\mathbb{R}^d} f(x) d\pi(x)\right)^2$$

Then each random real variable $f\mathrm{o}X_n$ is square integrable and

$$E[(f\mathrm{o}X_n)^2] = \int_\Omega \left(f\mathrm{o}X_n(\omega)\right)^2 dP(\omega) = \int_{\mathbb{R}^d} |f(x)|^2 d\pi(x),$$

hence

$$Var(f\text{o}X_n) = \sigma_\pi^2(f).$$

The sample variance of $f\text{o}X_n$ is an unbiased estimator of $\sigma_\pi^2(f)$. We set

$$\tilde{\sigma}_{\pi,N}^2(f) := \frac{1}{N-1}\left(\sum_{n=1}^{N}(f\text{o}X_n)^2 - N\tilde{I}_{\pi,N}{}^2(f)\right).$$

The convergence rate of Monte Carlo method can be assessed by the central limit theorem (CLT):

Theorem 2. Let $X_1 + X_2 + \cdots + X_N$ be i.i.d integrable random variables with mean μ and variance σ^2, then

$$\frac{\sum_{i=1}^{N} X_i - N_\mu}{\sqrt{N}\sigma} \xrightarrow{D} \mathcal{N}(0,1).$$

Then we have,

$$\frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}(f\text{o}X_n) - E[\sum_{n=1}^{N}(f\text{o}X_n)] \xrightarrow{D} \mathcal{N}\left(0, \sigma_\pi^2(f)\right)\right),$$

$$\frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}(f\text{o}X_n) - NI_\pi(f) \xrightarrow{D} \mathcal{N}\left(0, \sigma_\pi^2(f)\right)\right),$$

$$\sqrt{N}\left(\tilde{I}_{\pi,N}(f) - I_\pi(f)\right) \xrightarrow{D} \mathcal{N}\left(0, \sigma_\pi^2(f)\right),$$

then

$$\lim_{N\to\infty}\left(\left|\tilde{I}_{\pi,N}(f) - I_\pi(f)\right| \le \frac{\sigma_\pi(f).a}{\sqrt{N}}\right) = 2\phi(a) - 1, \quad \text{for } a > 0,$$

the interval

$$\left[\tilde{I}_{\pi,N}(f) - \frac{\sigma_\pi(f).a}{\sqrt{N}}, \tilde{I}_{\pi,N}(f) + \frac{\sigma_\pi(f).a}{\sqrt{N}}\right]$$

Is named the confidence interval at level $2\phi(a) - 1$. In fact, the variance $\sigma_\pi^2(f)$ is as unknown as $I_\pi(f)$, that's why the following proposition which is a consequence of CLT is practically useful.

3. Variance Reduction Techniques

We explore techniques that aim to enhance the effectiveness of Monte Carlo simulation through the reduction of variance of simulation estimates. There are several techniques available for variance reduction, including importance sampling [4], control variates, partition of the region [5], stratified sampling, Latin hypercube sampling [4], moment matching methods, antithetic variates and quasi-Monte Carlo methods [1]. Each technique has its strengths and weaknesses, and the choice of which technique to use depends on the specific problem and the characteristics of the data. We discuss control variates and antithetic variates methods, and we illustrate them with examples.

Let π be a probability distribution on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ and f a measurable function on this space, such that

$\int_{\mathbb{R}^d}(f(x))^2$ d π(x) < ∞.

Denote

$I_\pi(f) := \int_{\mathbb{R}^d} f(x)d\,\pi(x)$ and $\sigma_\pi^2(f) := \int_{\mathbb{R}^d}\left(f(x)\right)^2 d\pi(x) - \left(\int_{\mathbb{R}^d} f(x)d\pi(x)\right)^2$.

Let $(X_n)_{n\ge1}$ be a sequence of d-dimensional random variables, distributed with respect to π.

We have shown in last section that, for every $n \ge 1$ we have

$E[f\text{o}X_n] = I_\pi(f)$ and $Var(f\text{o}X_n) = \sigma_\pi^2(f)$,

And for all $a > 0$,

$$\lim_{N\to\infty}\left(\left|\frac{1}{N}\sum_{n=1}^{N} f\text{o}X_n - I_\pi(f)\right| \le \frac{\sigma_\pi(f).a}{\sqrt{N}}\right) = 2\phi(a) - 1.$$

In the purpose of reducing the length of the confidence interval

$$\left[\frac{1}{N}\sum_{n=1}^{N} f\text{o}X_n - \frac{\sigma_\pi(f).a}{\sqrt{N}}, \frac{1}{N}\sum_{n=1}^{N} f\text{o}X_n + \frac{\sigma_\pi(f).a}{\sqrt{N}}\right],$$

we reduce the variance of the random variable whose expected value is being estimated using the Monte Carlo method.

- Antithetic Variates

The basic idea behind this technique is to take advantage of the fact that certain pairs of random variables have negative correlations with each other [5]. which means, when one variable is high the other tends to be law. By generating pairs of random variables that are negatively correlated , we can use them to estimate the expected value of a function with less variance than we would get from using the simple Monte Carlo method.

Assume that the probability distribution π is symmetric about $x_0 \in \mathbb{R}^d$, then

$$\forall x \in \mathbb{R}^d;\ \pi(x) = \pi(2x_0 - x).$$

And let X be a d-dimensional random variable, distributed with respect to π.

Denote

$V := f(X)$ and $V_a := \frac{1}{2}(f(X) + f(2x_0 - X))$,

then we have

$E[V] = I_\pi(f)$ and $var(V) = \sigma_\pi^2(f)$.

And

$$E[V] = \frac{1}{2}E[f(X)] + \frac{1}{2}E[f(2x_0 - X)]$$
$$= \frac{1}{2}E[f(X)] + \frac{1}{2}\int_{\mathbb{R}^d} f(2x_0 - x)d\pi(x)$$
$$= \frac{1}{2}E[f(X)] + \frac{1}{2}\int_{\mathbb{R}^d} f(y)d\pi(2x_0 - y)$$
$$= \frac{1}{2}E[f(X)] + \frac{1}{2}\int_{\mathbb{R}^d} f(y)d\pi(y)$$
$$= E[f(X)] = E[V].$$

Similarly,

$var(f(2x_0 - X)) = var(f(X))$, hence

$$var(V_a) = \frac{1}{4}var(f(X)) + \frac{1}{2}Cov(f(X), f(2x_0 - X)) + \frac{1}{4}var(f(X))$$
$$= \frac{1}{2}var(f(X)) + \frac{1}{2}Cov(f(X), f(2x_0 - X)). \quad (\mathbf{2})$$

It follows from the Cauchy-Shwarz's inequality that

$$Cov(f(X), f(2x_0 - X)) \leq \sqrt{var\big(f(X)\big)}\sqrt{var\big(f(2x_0 - X)\big)} = var\big(f(X)\big),$$

Thus

$$var(V_a) \leq var\big(f(X)\big) = var(V) = \sigma_\pi^2(f).$$

Proposition 1. Let $A_1, A_2, \ldots, A_d$ be subsets of $\mathbb{R}$, $X_1, X_2, \ldots, X_d$ independent random real variables, such that each random variable X_i takes values in A_i.

Let $B = A_1 \times A_2 \times \ldots \times A_d$ and $h: B \to \mathbb{R}, k: B \to \mathbb{R}$.

Assume that it exists $I \subset \{1,2,\ldots,d\}$ such that

- h and k are increasing functions with respect to each variable X_i, where $i \in I$,
- h and k are decreasing functions with respect to each variable X_i, where $i \in I^c$.

If $h(X_1, X_2, \ldots, X_d)$ and $\mathrm{k}(X_1, X_2, \ldots, X_d)$ are square integrable functions, then

$$Cov\big(h(X_1, X_2, \ldots, X_d), \mathrm{k}(X_1, X_2, \ldots, X_d)\big) \geq 0.$$

Proof: The proof follows the same pattern as the previous one; we just need to multiply the functions by -1 when $i \in I^c$.

Proposition 2. Let g be a probability density function over $\mathbb{R}^d$, equals to zero except for $A \in \mathfrak{B}(\mathbb{R}^d)$, symmetric about $x_0 \in \mathbb{R}^d$. Let $f: A \to \mathbb{R}$, monotone with respect to each variable and let X be a d-dimensional random variable of pdf g.

Denote

$V := f(X)$ and $W := f(2x_0 - X)$,

then

$$cov(V, W) \leq 0.$$

Corollary 1. Let $f: [0,1]^d \to \mathbb{R}$, be a monotone function with respect to each variable such that

$$\int_{[0,1]^d} |f(x)|^2 < \infty,$$

Let $U \sim \mathcal{U}[0,1]^d$. Denote

$V := f(U)$ and $V_a := \frac{1}{2}\big(f(U) + f(1 - U)\big)$, then

$$var(V_a) \leq \frac{1}{2}var(V).$$

Proof: Since the uniform distribution $\mathcal{U}[0,1]^d$ is symmetric about $x_0 = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, the proof is straightforward from the proposition 3 and the identity (2).

Corollary 2. Let $f: \mathbb{R}^d \to \mathbb{R}$, be a monotone function with respect to each variable such that

$$\int_{\mathbb{R}^d} |f(x)|^2 e^{-\|x\|_2^2} dx < \infty,$$

Let $X \sim \mathcal{N}(0, I^d)$, denote

$V := f(X)$ and $V_a := \frac{1}{2}\big(f(X) + f(-X)\big)$, then

$$var(V_a) \leq \frac{1}{2}var(V).$$

Proof: Since the normal distribution $\mathcal{N}(0, I^d)$ is symmetric about $x_0 = (0,0,\ldots,0)$, the proof is straightforward from the proposition 3 and the identity **(2)**.

- Control Variates

In this technique, instead of estimating a parameter directly, the difference between the problem of interest and some analytical models is concerned.

The underlying principle of this method is to introduce a correlated secondary variable called the control variate, which is associated with the main variable of interest, in order to reduce the variance and enhance the accuracy of the simulation results.

A random variable C is a control variate for V, if it is correlated with V and if its expectation μ_c is known.

The control variate C is used to construct an estimator for $\mu = E[V]$ that has a smaller variance than V.

For any α∈ℝ, define

$V_a := V - \alpha(C - \mu_C)$.

Then

$$E[V_a] = E[V] - \alpha(C - \mu_C) = E[V],$$

hence $V_\alpha$is an unbiased estimator for μ.

We have

$$\begin{aligned} var(V_\alpha) &= var(V) + \alpha^2 var(C - \mu_C) \\ &\quad - 2\alpha cov(V, C - \mu_C) \\ &= \alpha^2 var(C) - 2\alpha cov(V, C) \\ &\quad + var(V), \end{aligned}$$

Which is a second-degree polynomial about α, with a positive leading coefficient $var(C) > 0$, then the parabola opens upwards and it has a unique minimum for

$$\alpha = \alpha^* := \frac{cov(V, C)}{var(C)}.$$

Then

$$\begin{aligned} var(V_{\alpha^*}) &= var\left(V - \frac{cov(V, C)}{var(C)}(C - \mu_C\right) \\ &= var(V) \\ &\quad + \left(\frac{cov(V, C)}{var(C)}\right)^2 var(C - \mu_C) \\ &\quad - 2\frac{cov(V, C)}{var(C)} cov(V, C - \mu_C) \\ &= var(V) - \frac{cov(V, C)^2}{var(C)} \\ &= var(V)\left(1 - \rho_{V,C}^2\right) \leq var(V), \end{aligned}$$

Where $\rho_{V,C}$ is the correlation coefficient between V

4. European Option

A European option is a specific type of options contract (either a call or put option) that imposes limitations on its exercise until the predetermined expiration date[3]. In simpler terms, once an investor acquires a European option, even if the price of the underlying security moves favorably (such as a rise in stock price for call options or a decline in stock price for put options), the investor is unable to capitalize on this movement by exercising the option before the expiration date. However, European options remain highly relevant and widely traded in financial markets.

There are two types of European options;

1. Call option: Holders of such contracts have the ability to buy a predetermined quantity of the underlying asset at the expiration date at a predetermined price.
   - If at the expiration T, the price S_T of the asset is lower than the strike price K, the option holder has no incentive to exercise it.
   - If the price S_T is higher than K, exercising the option allows the holder to realize a profit equal to $S_T - K$: they buy the asset at the strike price K and sell it on the market at the price S_T. Therefore, at expiration T, the value of the call option is given by

     $$(S_T - K)_+ = max(S_T - K, 0).$$

2. Put option: Investors have the option to sell a predetermined quantity of the underlying asset at the strike price on the expiration date.
   - If at the expiration date T, the price S_T of the asset is higher than the strike price$K$, the option holder has no incentive to exercise it.
   - If the price S_T is lower than$K$, exercising the option allows the holder to realize a profit equal to $S_T - K$: they buy the asset on the market at the price S_T and sell it at the strike price$K$. Therefore, at expiration, the value of the put option is given by

     $$(K - S_T)_+ = max(K - S_T, 0).$$

We assume that it is possible to borrow or invest money in the market at a constant interest rate r, known as the risk-free interest rate.

Let S_t^0 be the price of the risk-free asset at time t, its price at a later time $t' > t$ is given by

$$S_t^0 e^{r(t'-t)}.$$

Let C_t denote the price of the call option and P_t denote the price of the put option at time $t < T$.

- Suppose

$$C_t - P_t > S_t - Ke^{-r(T-t)},$$

at time t, we can buy the asset at the market price S_t and a put P_t and sell a call option C_t: the profit is

$$C_t - S_t - P_t,$$

- If this amount is positive, we invest it at the rate r;
- if it is negative, we borrow it at the rate r.
- At time T, this amount becomes $(C_t - S_t - P_t)e^{r(T-t)}$.
- If $S_T > K$, the call option is exercised by the holder, we sell the asset and obtain K, resulting in a wealth equal to

$$K + (C_t - S_t - P_t)e^{r(T-t)} > 0.$$

- If $S_T \leq K$, we exercise the put option and sell the asset, obtaining K, resulting in a wealth equal to

$$K + (C_t - S_t - P_t)e^{r(T-t)} > 0.$$

In both cases, there exists an arbitrage opportunity, which is

excluded by assumption.

- Suppose $C_t - P_t < S_t - Ke^{-r(T-t)}$,

At time t, we can buy a call option C_t and sell the asset at the market price S_t and a put option P_t; the profit is

$$S_t + P_t - C_t,$$

- If this amount is positive, we invest it at the rate r;
- If it is negative, we borrow it at the rate r.

At time T, this amount becomes

$(S_t + P_t - C_t)e^{r(T-t)}$.

- If $S_T \geq K$, we exercise the call option and buy the asset at price , resulting in a wealth equal to

$(S_t + P_t - C_t)e^{r(T-t)} - K > 0$;

- If $S_T < K$, the put option is exercised by the holder, we buy the asset at price K, resulting in a wealth equal to

$$(S_t + P_t - C_t)e^{r(T-t)} - K > 0$$

In both cases, there exists an arbitrage opportunity, which is excluded by assumption.

Consequently, we have the parity relationship;

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

In the classical modeling of financial markets, the price S_t of a stock at time t is given by:

$$S_t = S_0 exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right),$$

Where S_0: the initial stock price, μ: the drift rate, σ: the volatility and $\{B_t : t \geq 0\}$is a standard Brownian motion.

Thus, the price S_t is a log-normal random variable.

$\mathfrak{L}\mathcal{N}\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$.

Its expectation is:

$$E[S_t] = exp\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2}{2}t\right) = S_0 e^{\mu t}.$$

Its variance is

$$Var(S_t) = \left(e^{\sigma^2} - 1\right)(E[S_t])^2 = S_0^2 e^{2\mu t\left(e^{\sigma^2 t} - 1\right)}.$$

We consider a risk-neutral world where $\mu = r$. The black-Scholes formula is commonly used to calculate the theoretical price of a European option.

The price of the call option at the initial time is given by

$$E[C_0] = e^{-rT}E[(S_T - K)_+]$$

$$= e^{-rT}E\left[(S_T 1_{S_T > K})\right] - Ke^{-rT}E\left[1_{S_T > K}\right] \quad \textbf{(3)}$$

$$= e^{-rT}E\left[(S_T 1_{S_T > K})\right] - Ke^{-rT}P(S_T > K). \; (\mathbf{4})$$

But we have

$$P(S_T > K) = P\left(S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T\right) > K\right)$$

$$= P\left(\frac{S_0}{K} > exp\left(-\left(r - \frac{\sigma^2}{2}\right)T - \sigma B_T\right)\right)$$

$$= P\left(ln\left(\frac{S_0}{K}\right) >\right) - \left(r - \frac{\sigma^2}{2}\right)T - \sigma B_T$$

$$= P\left(\frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma} > -B_T\right)$$

$$= P\left(\frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T - \sigma^2 T}{\sigma\sqrt{T}} > -\frac{B_T}{\sqrt{T}}\right)$$

$$= P\left(-W < d_1 - \sigma\sqrt{T}\right)$$

Where $W := \frac{B_T}{\sqrt{T}} \sim \mathcal{N}(0,1)$ and $d_1 := \frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T-\sigma^2 T}{\sigma\sqrt{T}}$.

Since W and $-W$ have the same distribution, then

$$P(S_T > K) = \phi(d_2). \quad (\mathbf{5})$$

Furthermore,

$$e^{-rT}E\left[(S_T 1_{S_T>K})\right]$$
$$= E\left[exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}W\right)1_{W>-d_2}\right]$$
$$= e^{-rT}S_0\int_{-d_2}^{\infty}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\,dx$$
$$= S_0\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}\,dx.$$

By changing variable $u = \mathrm{x} - \sigma\sqrt{T}$,

$$e^{-rT}E\left[(S_T 1_{S_T>K})\right] = S_0\int_{-d_2-\sigma\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du$$
$$= S_0\int_{-\infty}^{d_2+\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du$$
$$= S_0\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du = S_0\phi(d_1). \quad (\mathbf{6})$$

By substituting **(6)** and **(5)** in **(3)**, we get

$$E[C_0] = S_0\phi(d_1) - Ke^{-rT}\phi(d_2), \quad (\mathbf{7})$$

where

$$d_1 := \frac{1}{\sigma\sqrt{T}}\left(ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right), d_2 := d_1 - \sigma\sqrt{T}$$

And φ is the cdf of the standard normal distribution.

The price of the put option at the initial time is given by

$$\begin{aligned} E[P_0] = e^{-rT}E[(K-S_T)_+] &= Ke^{-rT}E\left[1_{K>S_T}\right] \\ &- e^{-rT}E\left[S_T 1_{K>S_T}\right] \\ &= Ke^{-rT}P(K > S_T) \\ &- e^{-rT}E\left[S_T 1_{K>S_T}\right]. \end{aligned}$$

We have,

$$P(K > S_T) = \mathrm{P}\left(K > S_0 exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma B_T\right)\right)$$
$$= P\left(\frac{B_T}{\sqrt{T}} < \frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{-\sigma\sqrt{T}}\right)$$
$$= P\left(\frac{B_T}{\sqrt{T}} < \frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T-\sigma^2 T}{-\sigma\sqrt{T}}\right)$$
$$= P\left(W < -d_1 + \sigma\sqrt{T}\right) = \phi(-d_2).$$

Moreover,

$$e^{-rT}E\left[S_T 1_{K>S_T}\right]$$
$$= E\left[exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}W\right)1_{W<-d_2}\right]$$
$$= e^{-rT}S_0\int_{-\infty}^{-d_2}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\,dx$$
$$S_0\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}\,dx,$$

we set $u = x - \sigma\sqrt{T}$,

$e^{-rT}E\left[S_T 1_{K>S_T}\right] =$
$S_0\int_{-\infty}^{-d_2-\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du = S_0\phi(-d_1)$.

Then, we obtain

$$E[P_0] = Ke^{-rT}\phi(-d_2) - S_0\phi(-d_1). \quad (\mathbf{8})$$

We chek the parity relationship:

$$E[C_0] - E[P_0] = S_0 - Ke^{-rT}.$$

Example 1. Consider

S_0	r	σ	T	K
5	0.06	0.3	1	10

These parameters represent an option on a financial asset where the current price is 5 units, the risk-free interest rate is 6% per year, the volatility of the asset's returns is 30% per year, the option has a maturity of 1 year, and the strike price is 10 units. From **(7)** we get $E[C_0] \approx 0.0128194$ and from **(8),** $\mathrm{E}[\mathrm{P}_0] \approx 4.4304648$.

We estimate $E[C_0]$ and $\mathrm{E}[\mathrm{P}_0]$ using a simple Monte Carlo method, taking $N = 10$ to $N = 100000$ points, and calculating the confidence interval limits at a 95% confidence level.

The results are shown in Figure.1 for the call option and Figure.2 for the put option (the x-axis represents the number of the random variables being generated in logarithmic scale).

If we use the technique of antithetic variables, we obtain the results shown in Figure.3 for the call option and Figure.4 for the put option (the x-axis represents the number of the random variables being generated in logarithmic scale).

Comparing the results obtained by the simple Monte Carlo method and the one using the antithetic variables technique, we observe the advantage of this technique for option pricing.

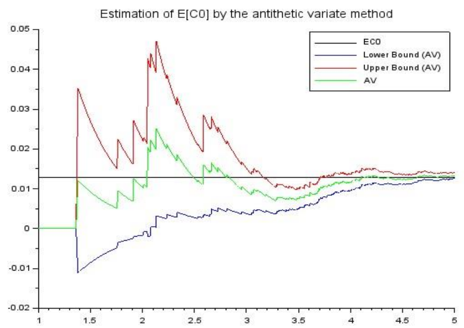

Figure 1:European call option by the simple MC method.

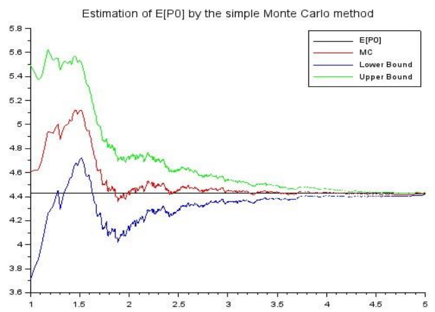

Figure 2; European put option by the simple MC method.

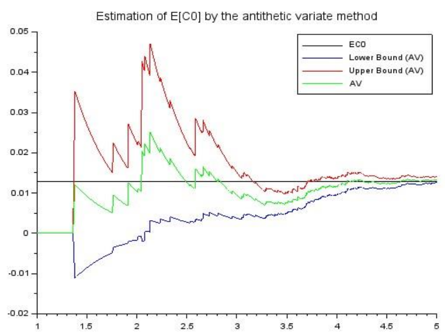

Figure 3: European call option by the antithetic variates technique.

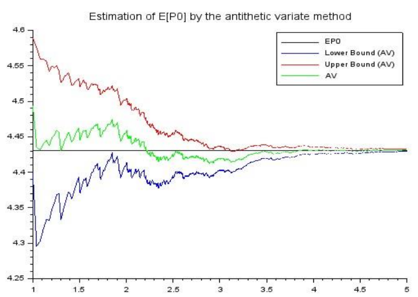

Figure 4: European put option by the antithetic variates technique.

5. Asian Option

Asian options derive their name from their pricing approach, which relies on the average value of the underlying asset over a predetermined duration \$n\$. The length of this period varies depending on the specific terms of the option agreement, spanning from a few days to several months. The determination of the average price involves employing diverse techniques, including arithmetic or geometric averaging.

We denote S_t as the price of an asset at time t. For an Asian option, we compare the exercise price K to an average of the asset prices before the expiration T.

Geometric averaging: Let $t_i = i\Delta t$, where

$1 \leq i \leq n$, where $\Delta t = \frac{T}{n}$. The geometric mean $\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}$ is a random variable following a log normal distribution.

The price of the Asian call option is

$$E[C_g] = e^{-rT} E\left[\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} - K\right)_+\right].$$

We have

$$\prod_{i=1}^{n} S_{t_i} = \left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)\left(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right)^2 \ldots \left(\frac{S_{t_3}}{S_{t_2}}\right)^{n-2} \left(\frac{S_{t_2}}{S_{t_1}}\right)^{n-1} S_{t_1}^n.$$

And

$$\frac{S_{t_n}}{S_{t_{n-1}}} = \frac{S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_{t_n}\right)}{S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)(n-1)T/n + \sigma B_{t_{n-1}}\right)}$$

$$= exp\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}X_1\right),$$

Where X_1 is a $\mathcal{N}(0,1)$ distributed random variable.

Similarly

$$\frac{S_{t_{n-1}}}{S_{t_{n-2}}} = exp\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}X_2\right),$$

$$\frac{S_{t_2}}{S_{t_1}} = exp\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}X_{n-1}\right),$$

$S_{t_1} = S_0\, exp\left(\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}+\sigma\sqrt{\frac{T}{n}}X_1\right),$

where $\{X_i\}_{1\le i\le n}$ are independent standard normal distributed random variables.

Then

$$ln\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} = \frac{1}{n}\left[\ln\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)+2ln\left(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right)+\cdots + nln\, S_{t_1}\right]$$

$= \ln S_0 + \left(r-\frac{\sigma^2}{2}\right)\frac{(n+1)}{2n}T+\frac{\sigma}{n}\sqrt{\frac{T}{n}}\sum_{i=1}^{n} iX_i.$

Since $X_i \sim \mathcal{N}(0,1)$, then $\frac{\sigma}{n}\sqrt{\frac{T}{n}}\sum_{i=1}^{n} iX_i \sim \mathcal{N}\left(0, \sigma^2\frac{(n+1)(2n+1)}{6n^2}T\right),$

Hence

$$\frac{\sigma}{n}\sqrt{\frac{T}{n}}\sum_{i=1}^{n} iX_i = \sigma\sqrt{\frac{T(n+1)(2n+1)}{6n^2}}Y,$$

With $Y \sim \mathcal{N}(0,1)$.

Furthermore

$$ln\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} = \ln S_0 + \left(r-\frac{\sigma^2}{2}\right)\frac{(n+1)}{2n}T + \sigma\sqrt{\frac{T(n+1)(2n+1)}{6n^2}}Y$$

$$= \ln S_0 + \left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)T+\hat{\sigma}\sqrt{T}Y,$$

where

$$\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right) = \left(r-\frac{\sigma^2}{2}\right)\frac{(n+1)}{2n}$$

and $\hat{\sigma} = \sigma\sqrt{\frac{(n+1)(2n+1)}{6n^2}}$.

$$E\left[C_{g,0}\right] = e^{-rT}E\left[\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} - K\right)_+\right]$$

$$= e^{-rT}E\left[S_0 exp\left(\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)T+\hat{\sigma}\sqrt{T}Y-K\right)\right].$$

From equality **(7)** it follows that

$E\left[C_{g,0}\right] = e^{-rT}\left(e^{\hat{\mu}T}S_0\phi(\widehat{d_1}) - K\phi(\widehat{d_2})\right),$ **(9)**

where $\hat{\mu}$ and $\hat{\sigma}$ are defined as

$$\hat{\sigma} = \frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}},\quad \hat{\mu} = \left(\mu-\frac{\sigma^2}{2}\right)\frac{n+1}{2n}+\frac{\hat{\sigma}^2}{2},$$

and $\hat{d}_1 := \frac{1}{\hat{\sigma}\sqrt{T}}\left(ln\left(\frac{S_0}{K}\right)+\left(\hat{\mu}+\frac{\hat{\sigma}^2}{2}\right)T\right),\ \hat{d}_2 := \hat{d}_1 - \hat{\sigma}\sqrt{T}$

and φ is the cdf of the standard normal distribution.

The price of the Asian put option is

$$E\left[P_{g,0}\right] = e^{-rT}E\left[\left(K-\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}\right)_+\right]$$

$$= e^{-rT}E\left[K - S_0 exp\left(\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)T+\hat{\sigma}\sqrt{T}Y\right)\right]$$

From **(8) it follows that**

$$E\left[P_{g,0}\right] = e^{-rT}\left(K\phi(-\widehat{d}_2) - e^{\widehat{\mu}T}S_0\phi(-\widehat{d_1})\right).\quad \mathbf{(10)}$$

Arithmetic averaging: Let $0 \le t_1 < t_2 < \cdots < t_n \le T$.

The price of the call option is given by $E\left[C_{a,0}\right] = e^{-rT}E\left[\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)_+\right],$

the price of the put option is given by $E\left[P_{a,0}\right] = e^{-rT}E\left[\left(K-\frac{1}{n}\sum_{i=1}^{n} S_{t_i}\right)_+\right].$

We do not have the exact value of the Asian option prices based on the arithmetic averaging, we can estimate it using the simple Monte Carlo method or the one using the control variates technique (taking the geometric mean as the control variable for the arithmetic mean).

Example 2. Consider

S_0	r	σ	T	K	n
100	0.04	0.4	1	120	52

These parameters represent an option on a financial asset where the current price is 100 units, the risk-free interest rate is 4% per year, the volatility of the asset's returns is 40% per year, the option has a maturity of 1 year, the strike price is 120 units, and the option is based on an average of the asset price over 52 periods (e.g., weeks).\

Equation **(9)** gives $E\left[C_{g,0}\right] \approx 3.2431371$ and **(10)** gives $E\left[P_{g,0}\right] \approx 21.778575$.

For the geometric averaging, we estimate $E\left[C_{g,0}\right]$ and $E\left[P_{g,0}\right]$ using the Monte Carlo method, taking $N = 1000$ to $N = 1000000$points, and calculate the confidence interval limits at a 95% confidence level. The results are shown in Figure 5 the call option and Figure 6 for the put option (the x-axis represents the number of the random variables being generated in logarithmic scale).

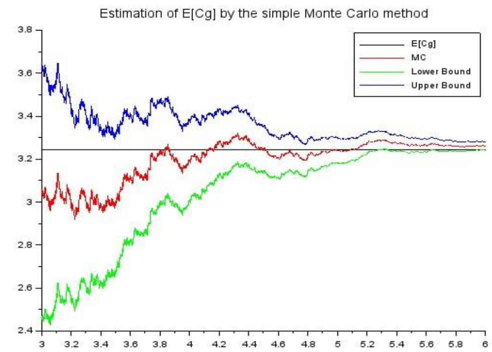

Figure 5: Asian call option by the simple MC method using geometric averaging.

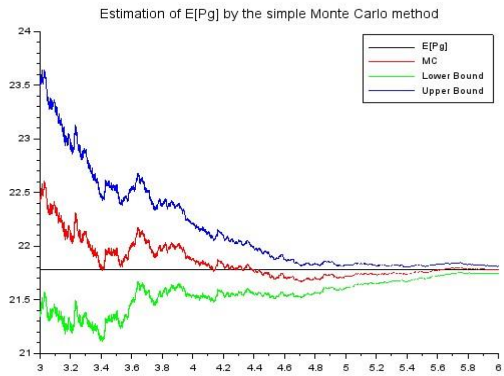

Figure 6: Asian put option by the simple MC method using geometric averaging

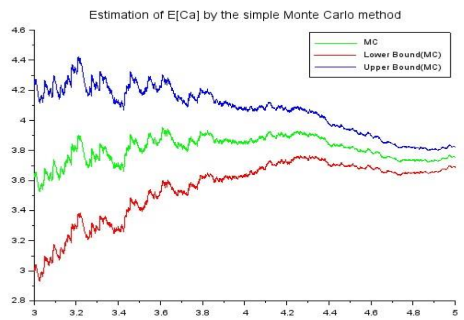

Figure 7: Asian call option by the simple MC method using arithmetic averaging.

For the arithmetic avereging, we estimate $E\left[C_{a,0}\right]$ and $E\left[P_{a,0}\right]$ using the Monte Carlo method, taking $N = 1000$ to $N = 1000000$ points, and calculate the confidence interval limits at a 95% confidence level. The results are shown in Figure 7 for the call option and Figure 8 for the put option (the x-axis represents the number of the random variables being generated in logarithmic scale).

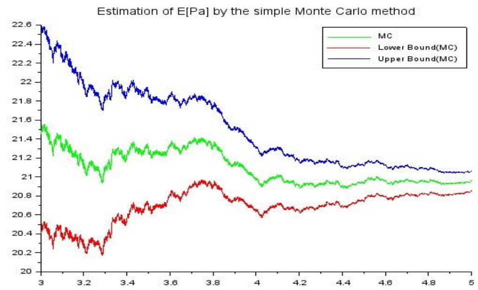

Figure 8: Asian put option by the simple MC method using arithmetic averaging.

We compare the results obtained by the simple Monte Carlo method and the one using the controlc variates technique, we obtain the results shown in Figures 9 (call option) and Figure 10 (put option(the x-axis represents the number of the random variables being generated in logarithmic scale). We observe the advantage of this technique for evaluating both options.

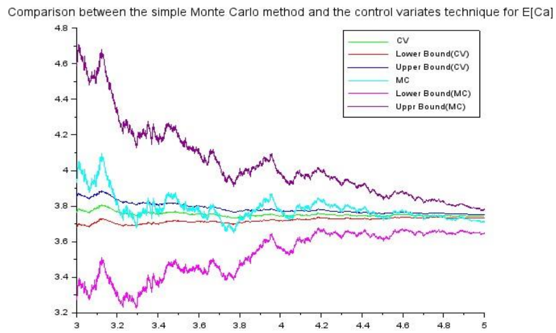

Figure 9: Comparison between the simple MC and the control variates technique for asian arithmetic call option.

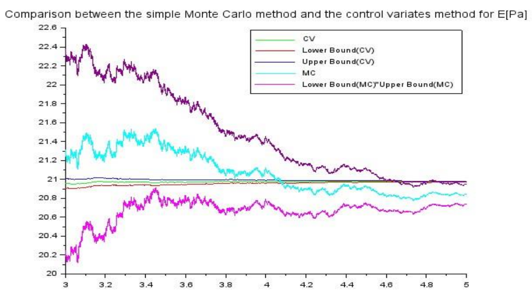

Figure 10: Comparisaon between the simple MC and the control variates technique for asian arithmetic put option.

6. Conclusion

The application of Monte Carlo methods for option pricing was be established, we compared the two variance reduction methods presented in Section 1 with the Monte Carlo method for European and Asian options. For the European option, we compared the Monte Carlo method with the antithetic variable method since an exact value is not available. However, for the Asian option, since exact values are available, we compared it with the control variable method as the antithetic variable method performs well when the integrated function is monotone.

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