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# On Some Ways for Application of Hybrid Methods to Solve Odes of the First Order

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**Abstract**
With the investigation of the numerical solution of the ODEs, scientists began studying from Newton. Consequently, there were constructed many methods for solving the above –mentioned problem. Have suggested conceptions for the comparison of these methods. And in the results of which are constructed class numerical methods for solving initial-value problems for the ODEs. Among of them, the hybrid methods are better, by taking into receiveaccount that these methods are more exact than the others. It is noted that these methods have some disadvantages, which relate with the calculation of the values at the hybrid points of the sought solution of the problems to be solved. Here, have given some ways for solving these problems. The receiving results are compared with the known by using some simple examples.

**Keywords**: Hybrid methods, initial value problem for ODEs of the first order, bilateral methods.

## 1. Introduction

As was noted the numerical solution of the initial-value problem for ODEs studies beginning from the Newton. That's why constructed methods with the different properties have. Naturally there exist some classes of methods for solving named problems. Among of them one can be noted the Multistep methods with constant coefficients (see for example [1]-[7]). By development of these class methods, scientists have constructed the new class of methods, which have some advantages (see for example [8]-[16]).

Let us consider the following problem:

$$Z'(t) = F(t,z), \quad Z(t_0) = Z_0, \quad t_0 \le t \le T. \tag{1}$$

Assume that the problem (1) has the unique solution defined on the segment $[t_0,T]$. And, also assume that function $f(x,y)$ has defined in some closed set, where has continuous partial derivative to some p, inclusively. The aim of our investigation is the construction of the numerical methods for solving problem (1). Therefore let us the segment of $[t_0,T]$ divide in to N equal parts. And denote by $Z(t_i)$ $(i = 0,1,..,N)$ the exact values of the solution of the problem (1) at the point $t_i\,(i = 0,1,..,N)$, but by the $Z_i\,(i = 0,1,..,N)$ lets denote, respectively, the approximate value at the points $t_i\,(i = 0,1,..,N)$. But mesh-point $t_i$, let's to define as: $t_{i+1} = t_i + h\ (i = 0,1,\ldots,N-1)$ .As it is known, the first direct numerical method for solving problem (1), has constructed by Euler, which can be presented as follows:

$$Z_{i+1} = Z_i + hF(t_i, Z_i). \tag{2}$$

This method has developed by some specialists and in the results of which in the middle of the 20 th century the following method appeared:

$$\sum_{j=0}^{m} \alpha_j Z_{n+j} = h \sum_{j=0}^{m} \beta_i F(t_{n+j}, Z_{n+j}), \quad n = 0,1,\ldots, N-m. \tag{3}$$

It is not difficult to verify, that Adams methods, Stermer and Simpson methods, also some known methods can be received from the method (3) as the partial case. By taking into account Dahlquist's rule, receive that if method (3) is stable and $\alpha_m \ne 0$ , then $p \le 2[m/2]+2$, but if $\alpha_m \ne 0$ and $\beta_m = 0$ , then $p \le m$. Here p-is the degree for the method (3), but m-is the order. Here, have used the conception stability and degree for the method (3) (see for example [4]-[10]).

Definition1. Integer values variable p, is called as the degree for the method (3), if the following takes place:

$$\sum_{i=0}^{m}(\alpha_i Z_{n+j} - h\beta_i Z'_{n+i}) = O(h^p),\ h \to 0,$$

(4)

Definition 2. Method (3) called as the stable, if the roots of the polynomial

$$\rho(\lambda) = \alpha_m \lambda^m + \alpha_{m-1}\lambda^{m-1} + \ldots + \alpha_1 \lambda + \alpha_0$$

located in the unite circle, on the boundary of which there is not multiply roots.

For the construction more exact stable methods, Ibrahimov has investigated the following method:

$$\sum_{j-0}^{m-\nu} \alpha_i Z_{n+i} = h\sum_{j=0}^{m} \beta_i F(t_{n+j}, Z_{n+j}),\ n = 0,1,\ldots, N-k,\ \nu > 0,$$

(5)

which in usually called as the advanced method. He, proved that, if method (5) is stable and has the degree of p, then

$$p \le m + \nu + 1\ (m \ge 3\nu).$$

If method (5) in instable, then $p \le 2m - \nu$.

It is easy to understand that, method (5) cannot be receive from the method (4) as the partial case, because $\nu > 0$. And if take into account that $\alpha_m \ne 0$, then receive that the class of method (5) is the independent object of research. Considering what has been said, that class method (5) is separate, Ibrahimov has fundamentally investigated of method (5). And constructed concrete methods with the degree $p = 3$ and $p = 5$. For the study advanced methods let us to consider the following method (see for example [17]):

$$Z_{n+1} = Z_n + h(8F(t_{n+1}, Z_{n+1}) + 5F(t_n, Z_n)$$
$$- F(t_{n+2}, Z_{n+2}))/12,$$

(6)

which is stably and has the degree p=3.

In the application of this method to solving problem (1) arises nessarty for the calculation of the value $Z_{n+1}$ and $Z_{n+2}$. For the finding the value $Z_{n+1}$ one can be used the following predictor-corrector method:

$$\bar{Z}_{n+1} = Z_n + hF(t_n, Z_n);$$
$$Z_{n+1} = Z_n + h(F(t_n, Z_n) + F(t_{n+1}, \bar{Z}_{n+1}))/2,$$

which is known, as the trapezoidal rule. For the calculation of the value $Z_{n+2}$ one can be suggest some methods. For example:

$$Z_{n+2} = Z_{n+1} + h(3F_{n+1} - F_n)/2,$$

here, $F_l = F(t_l, Z_l)\ l(0,1,2,\ldots)$. In this case receive the following:

$$Z_{n+1} = Z_n + (8F_{n+1} + 5F_n)/12$$
$$- hF(t_{n+2}, Z_{n+1} + h(3F_{n+1} - F_n)/2)/12. \qquad (7)$$

Let us to consider the case $F(t,Z) \equiv -\lambda Z\ (\lambda > 0)$. In this case receive the following:

$$(12 + 7h\lambda + 3h^2\lambda^2/2)Z_{n+1} = (12 - 5h\lambda + h^2\lambda^2)/Z_n$$

.

As follows from here method (7) is A-stable. But if method (6) presented as follows:

$$Z_{n+1} = Z_n + h(8F_{n+1} + 5F_n)/12$$
$$- hF(t_{n+2}, 3Z_{n+1} - 2Z_n + hF_n)/12,$$

(8)

then receive that method (8) is not A-stable.

It follows from here that the properties of the predictor-corrector method depends from the choosing of the predictor formula.Noted that in usually methods of type (7) or (8) applied to solving of complex tasks by taking into account of the properties of A-stability. It is known that if method (3) is A-stable, then $p \le 2$. Method (7) is A-stable and has the degree $p = 3$. It follows from here that method (5) can be taking as the perspective. Let us to consider to construction of A-stable methods.

**§1. Construction A-stable methods with the high degree.**

By using Dahlquist's results, many authors for the construction A-stable methods, investigated the following method (see for example [17]-[19]):

$$\sum_{i=0}^{m} \alpha_i Z_{n+i} = h\sum_{i=0}^{m} \beta_i F(t_{n+i}, Z_{n+i}) + h^2 \sum_{i=0}^{m} \gamma_i G(t_{n+i}, Z_{n+i}),$$

(9)

here the function $G(t,Z)$ has defined as the

$$G(t,Z) = F'_t(t,Z) + F'_Z(t,Z)F(t,Z)$$

.

Method (9) has investigated by many authors (see for example [20]-[35]).

As is known many scientists have considered constructing A-stable methods by using multistep second derivative methods as (9). Note that the following method can be received from the method of(9), as the following:

$$Z_{n+1} = Z_n + h(F_{n+1} + F_n)/2 + h^2(-G_{n+1} + G_n)/12,$$

(10)

here reminder term can been presented as:

$$R_n = h^5 Z^{(5)}/720.$$

This method has the degree p=4 and A-stable. If compares method (7) with the method of (10), then receive that method (7) has some advantages, which refers to methods of advanced type. Let us noted that if method (9) is stable and has the degree of p, then there are methods with the degree $p \le 2m + 2$. Method (10) receive from the method (9) for the value $m = 1$, degree for which satisfies $p = 4$. Thus have shown that if method (9) is stable then $p_{\max} = 2m + 2$. For the construction more exact methods, Ibrahimov offered to use the following method:

$$\sum_{i=0}^{k} \alpha_i Z_{n+i} = h \sum_{i=0}^{l} \beta_i F_{n+i} + h^2 \sum_{i=0}^{s} \gamma_i G_{n+i}, \ n = 0,1,.., N-k.$$

(11)

This is more general, than the method (9). Ibrahimov has proved that if method (11) stable and has the degree of p, then there exist methods of type (11), with the degree $p = s + l + m + 2$, here $l - k = m$ or $s - k = m \ (m \ge 0)$. Ibrahimov has constructed more specifically methods for the case k=2, l=s=3 and k=l=s=3. And also have given some comparison of the constructed methods.

To illustrate all the results obtained above, it is proposed in the following methods(see for example [11]):

$$Z_{n+2} = (11Z_n + 8Z_{n+1})/19 + h(10F_n + 57F_{n+1} + 24F_{n+2})/57 - hF_{n+3}/57.$$

(12)

This method stable and has the degree of p=5.

This method is the advanced, have the degree p=5 and stable. Thus receive that method (12) has the maximum degree in the case k=3. But in the application of that the papers some difficulties associated with the calculation of $Z_{n+3}$. For solving this problem one can be use the following method, which is the modification of the method (12):

$$Z_{n+2} = (11Z_n + 8Z_{n+1})/19 + h(10F_n + 57F_{n+1} + 24F_{n+2})/57 - hF(t_{n+3}, Z_{n+2} + (23F_{n+2} - 16F_{n+1} + 5f_n)/12)/57.$$

(13)

This method can be taking as the explicit method of nonlinear type. Noted that this method is A-stable, which can be comparison with the method (9).One can note that method (13) is better than method (9). Usually for the comparison of the numerical methods are used the conception of optimal methods, which has the maximum degree and stability. By using this description receive methods (12) and (13) can be taken as the better. As was noted above method (13) is A-stable, the for method (13) can be taken as the better. And now let us consider construction of stable methods of type (11) and take l=s=k. In the case k=3, one can construct stable methods, which can be presented as follows.

$$Z_{n+3} = (Z_{n+2} + Z_{n+1} + Z_n)/3 + h(10781F_{n+3} + 22707F_{n+2} + 16659F_{n+1} + 4285F_n)/27216 + h^2(-2099G_{n+3} + 7227G_{n+2} + 2853G_{n+1} + 979G_n)/45360 \ (p = 8),$$

.

(14)

$$Z_{n+2} = (416Z_{n+1} - 103Z_n)/313 + h(157F_{n+3} + 11232F_{n+2} + 8451F_{n+1} - 2830F_n)/25353 + h^2(-11G_{n+3} - 630G_{n+2} + 1557G_{n+1} - 92G_n)/8451 \ (p = 9).$$

(15)

$$Z_{n+1} = Z_n + h(1985F_{n+3} + 12015F_{n+2} + 42255F_{n+1} + 34465F_n)/90720 + h^2(-163G_{n+3} - 2421G_{n+2} - 7659G_{n+1} + 1283G_n)/30240,$$

(16)

here $G_s = G(t_s, Z_s) \ (s = 0,1,2,....)$.

As is known, all methods have their own advantages and disadvantages. Above presented methods also have their advantages and disadvantages. For example, implicit methods are more exact than the explicit method, but in the application of the implicit methods there arises some difficulty, which relates with finding the solution of the nonlinear algebraic equation. Note that similar difficulties arise in the case of advanced methods to solve some problems. In usually this difficult relation with the calculation the values of the solution of given problems at the next points. Thus proved that indeed each method has its own disadvantages and advantages. However, there are ways in which one can overcome some of the above disadvantages. For example, to eliminate the shortcomings of implicit methods, predictor-corrector methods can be used. Noted that one of the basic questions is the choosing of the predictor methods. In [8] is given the way for choosing predictor methods, depending from the corrector methods. In some cases it is available to use the corrector method as the predictor. Consequently there are different ways for the construction of the predictor-corrector method. To explain these and in others cases, let us to consider the following paragraph.

**§2.On the application of the multistep multiderivative methods to solve problem (1).**

Above have, given some information about multistep and multistep second derivative method with constant coefficients and also comparison of these methods. And have shown that stable multistep second derivative methods are more exact than the multistep methods. However, stable advanced methods in usually are more exact than the stable methods of type (3). And stable multistep second derivative more exact than the methods of type (1). Noted that stable methods of type (1) more exact than the others. As was noted above in the application methods of type (9) to solving problem (1) arises some difficult, which are related with calculation the function $G(t,Z)$ at the mesh points. Consequently, stable methods of type (9), can be taking as better in the application of that to solving following problem:

$$Z'' = G(t,Z,Z'),\ Z(0)=Z_0,$$
$$Z'(0)=Z'_0,\ t_0 \le t \le T.$$

(17)

For the sheik of objectivity, noted that in the application of the method (9) to solving of problem (17) it is arises some difficulty related with the finding the values of the function $Z'(t)$ at the mesh points, $t_{i+1}=t_i+h$. For this aim one can be used methods of type (3), but the maximum value of the degree for the stable methods of type (3) equal to m+2 for $m=2\nu$ even and to m+1 for $m=2\nu-1$(odd). But the maximum value for the degree of the stable method of type (9) equals to 2m+2. This combination can be used for the values $m \le 2$. In the case $m \ge 3$, here recommend to use following hybrid method:

$$\sum_{i=0}^{m}\alpha_i Z_{n+i} = h\sum_{i=0}^{m}\beta_i F(t_{n+i+\nu_i}, Z_{n+i+\nu_i}),\ |\nu_i|<1,$$
$$i=0,1,\dots,m.$$

(18)

A well-known representative method of this class in the midpoint, which can written as the follows:

$$Z_{n+1} = Z_n + hF(t_n + h/2, Z_{n+1/2}).$$

(19)

The advantage of this method is known to all specialists. Let us noted that the known representative method for the class method (3) is the trapezoidal rule which can be as following:

$$Z_{m+1} = Z_m + h(F(t_m,Z_m)+F(t_{m+1},Z_{m+1}))/2.$$

(20)

This method is stable and has degree p=2. Method (19) also is stable and has degree p=2. However, method (19) is explicit, but method (20) is implicit. Let us compare these methods. These methods are stable, has the degree of p=2. But the equation (20) is nonlinear algebraic equation to respect $Z_{m+1}$. As is known, to solve the nonlinear algebraic equation is not easy. However, in the calculation of unknown $Z_{n+1}$ by equality of (19), there is any difficulty. It should be noted that many experts claim that implicit methods usually give better results than explicit methods. That's why we mainly used implicit methods here. For the correction of the above shortcomings of implicit methods, it was proposed here to use the predictor-corrector methods. For example, predictor-corrector method to using trapezoid method (20), can been constructed as follows:

$$\hat{Z}_{m+1} = Z_m + hF(t_m, Z_m),$$

(21)

$$Z_{m+1} = Z_m + hF(t_m,Z_m) + F(t_{m+1},\hat{Z}_{m+1}))/2,$$

(22)

Here the value of $\hat{Z}_{m+1}$ has calculated by the formula (21), then that corrected by the method (22) . Note that by the formulas (21) and (22) calculate the approximately value $Z_{m+1}$ of the solution of some problems. As was noted above that method (19) is explicit but in using that it is arises calculation of the value $Z_{n+1/2}$, for which one can be used the method (21). In some cases it is desirable to use simpler structure than the scheme (21) and (22). For example, let us to consider the following predictor-corrector method:

$$\bar{Z}_{m+1} = Z_m + hF(t_m + h, \hat{Z}_{m+1}),$$

(23)

here, $\hat{Z}_{m+1}$ value, which is calculated by the method (21). Let us consider the following half sum:

$$Z_{m+1} = (\bar{Z}_{m+1} + \hat{Z}_{m+1})/2.$$

(24)

It is not difficult to prove that methods (22) and (24) are one and the same. However, it is easier to use the method (24), than the method of (22). And now, let us to consider the generalization of the method (19), in one version, than can be presented as follows:

$$\sum_{i=0}^{m} \bar{\alpha}_i Z_{n+i} = h\sum_{i=0}^{m} \bar{\beta}_i F(t_{n+i+\nu_i}, Z_{n+i+\nu_i}),$$

$$(|\nu_i| < 1;\ i = 0,1,\ldots,m).$$

(25)

For the receiving method (19)from the method (25), it is enough to take $\bar{\beta}_0 = 1,\ \beta_1 = 0$ and $\nu_0 = 1/2$. Noted that if method (25) is stable, then in the class of methods (25), there are methods with the degree p=2m+2. For the illustration the receiving results here, let us to consider the following section.

**2. Numerical results.**

As is known one of the popular mathematical model for the study many problems which we encounter in our activity can be presented as follows:

$$Z' = \lambda Z(t),\ Z(0) = 1,\ 0 \le x \le 1.$$

(26)

Exact solution of this example: $Z(t) = \exp(\lambda t)$.

There is an idea that first order ordinary differential equations can be reduced to second order ordinary differential equations, to solve which one can be used the multistep second derivative methods. If this idea applied to example (26), then receive the following:

$$Z''(t) = \lambda^2 Z(t),\ Z(0) = 1,\ Z'(0) = \lambda$$

(27)

Exact solution of this problem can be presented as: $Z(t) = \exp(\lambda t)$. To solving of this problem one can applied the following methods:

$$Z_{n+2} = 2Z_{n+1} - Z_n + h^2(4Z''_{n+2-\beta} + Z''_{n+1} + 4Z''_{n+1+\beta})/9$$

(28)

$$Z_{n+2} = 2Z_{n+1} - Z_n + h^2(Z''_{n+2} + 10Z''_{n+1} + Z''_{n+2})/12.$$

(29)

**Table 1. Results for the $h = 0.01$**

| $\lambda$ | $x_n$ | Error for method (13) | Error for method (12) |
|---|---|---|---|
| $\lambda = 1$ | 0.1<br>0.4<br>0.7<br>1.0 | 4.66E-12<br>2.96E-11<br>7.15E-11<br>1.39E-10 | 1.28E-10<br>7.27E-10<br>1.72E-9<br>3.34E-9 |
| $\lambda = -1$ | 0.1<br>0.4<br>0.7<br>1.0 | 3.86E-12<br>1.34E-11<br>1.78E-11<br>1.90E-11 | 1.04E-10<br>3.24E-10<br>4.23E-10<br>4.49E-10 |

**Table 2.Results for the $h = 0.1$**

| $\lambda$ | $x_n$ | Error for method (13) | Error for method (12) |
|---|---|---|---|
| $\lambda = 1$ | 0.1<br>0.4<br>0.7<br>1.0 | 3.16E-9<br>1.99E-8<br>4.8E-8<br>9.3E-8 | 4.56E-9<br>2.46E-8<br>5.82E-8<br>1.12E-7 |

**Table 3. Results for the methods (28) and (29).**

| $x$ | $\lambda = 1$ | $\lambda = -1$ | $\lambda = 5$ | $\lambda = -5$ |
|---|---|---|---|---|
| 0,2<br>0<br>0,6<br>0<br>1,0<br>0 | 2.04E-12<br>1.7E-11<br>4.1E-11 | 2.16E-12<br>2.05E-11<br>5.53E-11 | 2.53E-08<br>3.17E-08<br>2.86E-06 | 3.26E-08<br>1.17E-08<br>1.97E-07 |

**Table 4. Results for the methods (28) and (29).**

| $x$ | $\lambda = 1$ | $\lambda = -1$ | $\lambda = 5$ | $\lambda = -5$ |
|---|---|---|---|---|
| 0,20 | 5,66E-13 | 4.92E-13 | 1.28E-08 | 6.5E-09 |
| 0,60 | 6,23E-12 | 4.17E-12 | 4.31E-07 | 8.13E-08 |
| 1,00 | 2,12E-11 | 1.10E-11 | 5.76E-06 | 6.11E-07 |

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