Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 44 No. 12
December 2024

The Fluid Dynamics of Compositional Plumes

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Abstract
The stability of a fully developed plume of compositionally buoyant fluid, of finite width, enclosed between two vertical boundaries and rising in a less buoyant infinite fluid is investigated. The plume and the outside fluid have the same thermal diffusivity, κ, material diffusivity, κ_m, and kinematic viscosity, ν. The linear stability problem is governed by four dimensionless parameters : the (Grashoff) Reynolds number , R $(=UL/\nu$, where U and L are typical velocity and characteristic length , respectively, the Prandtl number , σ $(=\nu/k)$, the Schmidt number , $S_c\,(=\nu/\kappa_m)$, and the dimensionless thickness of the Cartesian plume, x_0. The stability analysis is examined in the case $R \gg 1$. It is found that the presence of diffusion of material introduces an extra term, Ω_{2d}, in the expression for the growth rate. The term Ω_{2d} is found to be negative except in a small region of the wavenumber space where it is positive. The influence of Ω_{2d} on the stability is examined to find that although material diffusion decreases the values of growth rate, the plume remains unstable for all values of the parameters of the dynamical system. Material diffusion is also found to affect the wavenumbers of the unstable mode; 3-dimensional modes can be transformed into 2-dimensional modes or vice versa by the presence of material diffusion.

Keywords: Compositional plume; material diffusion; stability.

1. Introduction

The mathematical modeling of the dynamics of fluid alloys in the presence of pressure and heat is relevant to geophysical [1-5, 10-17, 21-22,26-32,39,43], industrial [9, 18,20,24,34-38,40,42] and environmental [19,23,25,33] applications. Consequently, these varied interests have led to theoretical and experimental studies on various aspects of the fluid dynamics of fluid alloys. Here we will study the linear stability of a compositional plume, which is a directional flow of a column of fluid through a surrounding infinite fluid of different composition.

Iron casting is one of the important industrial applications where compositional plumes may appear [2-5,10-17,22, 34-38]. In iron castings, plumes appear in the form of thin black markings that represent imperfections in the cast because they are devoid of solute. Their appearance signifies a weakness in the iron bars. The need to investigate the nature of these plumes motivated a number of theoretical and experimental studies starting with Copley *et al.* [9]. Copely *et al.* (1970) designed an experiment using a mixed solution of ammonium chloride in water (30-wt% NH_4CL-H_2O). The solution was placed on a cooled surface. The cool surface lowered the temperature of the solution at the bottom of the container until it reached the melting temperature of NH_4CL, which is more than that of water, and ammonium chloride crystals appeared at the bottom of the container. This process continued, producing more crystals, and a layer of mixed solid and fluid, which is mainly water, formed. This layer is known as a mushy layer [2-5, 10-17] . The crystals in the mushy layer formed dendrites. The mushy layer thickened as the process continued further and the dendrites at the bottom of the mushy layer joined to form a solid layer. When the layer reached a certain thickness, the light fluid in the mushy layer was seen to escape through channels called chimneys and through the overlying melt in thin filaments, now referred to as compositional plumes.

The dynamics of the mushy layer has been studied experimentally and theoretically [1-3, 10-17,6-9]. Worster [41-43] made a detailed study on the stability of the mushy layer and found that instability exists at the mushy layer-melt interface and can take one of two forms. One form has small directions of the finger type and occurs in a boundary layer on the melt-mush interface while the other form has larger dimensions and extends to both mushy layer and melt. It is this latter form of instability that is believed to lead to compositional plumes. Moffatt [30] proposed that the mushy layer become unstable, and the rising plumes can break up into blobs of light fluid.

The first comprehensive study of the stability of the compositional plume was carried out by Eltayeb and Loper [15] to show that it is unstable. This study dealt with the simple case of a single vertical interface, across which the concentration of light component is discontinuous. This case represents a plume of infinite thickness. They extended their study to include the finite width of the plume, in which the plume takes the form of a channel between two vertical planes, the so-called Cartesian plume (Eltayeb and Loper [16]) and the more realistic plume form of a circular cylinder (Eltayeb and Loper [17]). All these studies assumed that the compositional interfaces are sharp because they neglected material diffusion, κ_m. These studies showed that a single plume is unstable for all values of the (Grashoff) Reynolds number R and Prandtl number σ defined by

$$R = \frac{UL}{\nu}, \qquad \sigma = \frac{\nu}{\kappa}, \tag{1}$$

where U and L are characteristic velocity and length-scale, respectively, and κ and ν are thermal diffusivity and kinematic viscosity, respectively.

Motivated by geophysical applications, Eltayeb and Hamza [12] studied the linear stability of compositional plumes in the presence of rotation to find that rotation destabilizes the plume. Classen *et al*. [8] investigated the dynamics of plumes under the influence of rotation experimentally to find that the plumes are unstable. Eltayeb *et al.* [13-14] examined the stability of Cartesian plumes in the presence of a uniform magnetic field to find that the magnetic field can destabilize the plume slightly. They also found that magnetic diffusion can enhance instability of the plume. The both effect of magnetic field and rotation on compositional plumes was investigated by Eltayeb [10-11] to conclude that the growth rate of the instability was controlled by the action of rotation. However, all these studies on compositional plumes considered that the compositional plume rises vertically in unbounded domains of fluid. The theoretical studies by Al Mashrafi and Eltayeb [2] examined the influence of the two vertical boundaries on the stability of the plumes in the absence of rotation and magnetic fields. They found that the presence of two fixed vertical boundaries affects the stability of the plume, but the plumes remain unstable. They extended their studies by including the vertical rotation (Al Mashrafi and Eltayeb [3]) to find that the plume remains unstable. However, all these studies neglected the material diffusion.

Since diffusion is generally stabilizing, it is of interest to investigate the influence of material diffusion on the stability of the plume in order to see whether the plume can be stabilized. In section 2, we formulate the problem. The system of the model is governed by the same equations used by Eltayeb and Loper [16] except for the presence of material diffusion in the equation of the concentration of light material. This introduces a new dimensionless number, S_c, as a measure of material diffusion. In section 3, we derive the perturbation equations and solve the resulting eigenvalue problem for the growth rate Ω_2 of the system. It is found that material diffusion adds an extra term, $S_c\,\Omega_{2d}$, to the growth rate. In section 4, we investigate the properties of Ω_{2d} and its influence on the growth rate Ω_2 to find that material diffusion tends to stabilize the plume. In section 5, we present some concluding remarks.

2. Formulation of the problem

We assume a two-material incompressible fluid of infinite extent. The concentration of the light material in the fluid is C, and the temperature of the fluid is T. The two fluids have the same thermal diffusivity, κ, kinematic viscosity, ν, and material diffusion, κ_m.

This system is governed by the equations of motion, continuity, heat, concentration, and state. These equations are

$$\rho_0\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}.\nabla)\boldsymbol{u}\right]=-\nabla p+\rho_0\nu\,\nabla^2\boldsymbol{u}-\rho g\,\hat{\boldsymbol{z}}\,,$$

(2)

$$\nabla.\boldsymbol{u}=0\,,$$

(3)

$$\frac{\partial T}{\partial t}+\boldsymbol{u}.\nabla T=\kappa\nabla^2 T\,,$$

(4)

$$\frac{\partial C}{\partial t}+\boldsymbol{u}.\nabla C=\kappa_m\,\nabla^2 C\,,$$

(5)

$$\frac{\rho}{\rho_0}=1-\alpha\left(T-T_0\right)-\beta\left(C-C_0\right),$$

(6)

where $\boldsymbol{u}$ is the velocity vector, g the uniform acceleration of gravity, p the pressure, t the time, $\hat{\boldsymbol{z}}$ is the upward unit vector, (α,β) are the coefficients of thermal and compositional expansions, ρ the density, (ρ_0, T_0, C_0) the constant reference values, where we have considered that the fluid is Boussinesq.

The system (2)-(6) allows a hydrostatic state in which the fluid is stably stratified by a temperature with uniform temperature gradient, γ.

To write the system (2) - (6) in dimensionless form, we take the scale of length as the salt finger length scale, defined by

$$L=\left(\frac{\nu\kappa}{\alpha\gamma g}\right)^{\frac{1}{4}},$$

(7)

and the amplitude of concentration of light material, $\tilde{C}$, as the unit of the concentration. We further take U as a unit of velocity by

$$U=\beta\tilde{C}\left(\frac{g\,\kappa}{\alpha\nu\gamma}\right)^{\frac{1}{2}},$$

(8)

and $\beta\tilde{C}/\alpha$, L/U and $\rho_0\beta\tilde{C}\left(\nu g^3\kappa/\alpha\gamma\right)^{\frac{1}{4}}$ as units of temperature, time and pressure, respectively.

We eliminate ρ from (2) using (6) and write the dimensionless equations as

$$R\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}.\nabla)\boldsymbol{u}\right]=-\nabla\left(p+\frac{z}{\beta\tilde{C}}\right)+\nabla^2\boldsymbol{u}+\left(T-T_0+C-C_0\right)\hat{z}\,,$$

(9)

$$\nabla.\boldsymbol{u}=0\,,$$

(10)

$$R\,\sigma\left[\frac{\partial T}{\partial t}+\boldsymbol{u}.\nabla T\right]=\nabla^2 T\,,$$

(11)

$$R\,S_c\left[\frac{\partial C}{\partial t}+\boldsymbol{u}.\nabla C\right]=\nabla^2 C\,,$$

(12)

where R is Grashoff (Reynolds) number, σ is Prandtl number, and S_c is Schmidt number:

$$R=\beta\tilde{C}\left(\frac{g\,\kappa^3}{\alpha^3\gamma^3\nu^5}\right)^{\frac{1}{4}}\left(=\frac{UL}{\nu}\right),\qquad \sigma=\frac{\nu}{\kappa},\qquad S_c=\frac{\nu}{\kappa_m}\,.$$

(13)

We consider a Cartesian coordinate system $O(x,y,z)$ in which Ox, Oy are horizontal and Oz is vertically upwards. We consider that the general solution of the equations (9) - (12) can be written in the form

$$\boldsymbol{u}(x,y,z,t)=\boldsymbol{0}+\bar{w}(x)\,\hat{z}+\varepsilon\boldsymbol{u}^{\dagger}(x,y,z,t)\,,$$

(14)

$$C(x,y,z,t)=C_0+\bar{C}(x)+\varepsilon C^{\dagger}(x,y,z,t)\,,$$

(15)

$$p(x,y,z,t)=p_h+\bar{p}(x)+\varepsilon\,p^{\dagger}(x,y,z,t)\,,$$

(16)

$$T(x,y,z,t)=T_h+\bar{T}(x)+\varepsilon T^{\dagger}(x,y,z,t)\,,$$

(17)

such that the variables T_h, p_h have hydrostatic contribution and given by

$$T_h = T_0 + \frac{(z - z_0)}{\sigma R} ,$$
(18)

$$p_h = p_0 - \frac{\left(z - z_0\right)}{\beta \tilde{C}} + \frac{\left(z - z_0\right)^2}{2\sigma R} .$$
(19)

The variables $\bar{w}(x),\ \bar{C}(x),\ \bar{p}(x),\ \bar{T}(x)$ are basic state variables dependent only on a horizontal length, x , and the variables $\boldsymbol{u}^\dagger,\ C^\dagger,\ p^\dagger,\ T^\dagger$ indicate a perturbation of the amplitude $\varepsilon << 1$.(see figure 1).

The boundary conditions of this physical system are that all variables of the system are finite away from the plume interfaces, the heat and momentum fluxes are continuous across the interfaces, and the interface between the plumes and surrounding fluid is a material surface.

Substituting the equations (14) - (17) into the system (9) - (12) , and neglecting the terms independent of ε gives the following basic state equations

$$-\frac{d\bar{p}}{dx}\hat{x} + \left(\frac{d^2\bar{w}}{dx^2} + \bar{C} + \bar{T}\right)\hat{z} = 0 ,$$
(20)

$$\frac{d^2\bar{T}}{dx^2} = \bar{w}(x) ,$$
(21)

$$\frac{d^2\bar{C}}{dx^2} = 0 ,$$
(22)

and equation (10) is automatically satisfied.

Equation (20) gives $\frac{d\bar{p}}{dx} = 0$. Using the boundary conditions, we found that $\bar{p}\left(x\right) = 0$ and $\bar{C}\left(x\right)$ has the general form

$$\bar{C}\left(x\right) = Ax + B .$$
(23)

In the present study, we take a concentration profile of the so-called top-hat profile

$$\bar{C} = \begin{cases} 1 , & |x| \le x_0 \\ 0 , & |x| > x_0 \end{cases} .$$
(24)

This defines a column of thickness $2x_0$ and concentration 1 rising in an infinite fluid of concentration 0.

Combining the $z-$component of equation (20) and equation (21) gives the following second-order ordinary differential equation

$$\frac{d^2F}{dx^2} - \mathrm{i}F = \mathrm{i}\bar{C} ,$$
(25)

where

$$F\left(x\right) = \bar{T}\left(x\right) - \mathrm{i}\bar{w}\left(x\right) .$$
(26)

This equation is solved subject to the conditions that F and $\frac{dF}{dx}$ are continuous across the interfaces, and F is finite away from the interfaces.

3. The stability analysis

The solution of the basic state equations (25) and (26), was obtained by Eltayeb and Loper (1991) as

$$\bar{w}\left(x\right) = \frac{1}{2}\left\{\exp\left(-K_+\right)\sin\left(K_+\right) - \exp\left(-\left|K_-\right|\right)\sin\left(K_-\right)\right\} ,$$
(27)

$$\bar{T}\left(x\right) = \frac{1}{2}\left\{\exp\left(-K_+\right)\cos\left(K_+\right) - 1 - \mathrm{sgn}\left(K_-\right)\left[\exp\left(-\left|K_-\right|\right)\cos\left(K_-\right) - 1\right]\right\} ,$$
(28)

where

$$K_\pm = \left(\frac{|x| \pm x_0}{\sqrt{2}}\right) .$$
(29)

The flow (27) and the concentration (24) define a plume enclosed by the two planes $x = \pm x_0$ rising in an infinite fluid. We intend here to investigate the stability of such a plume.

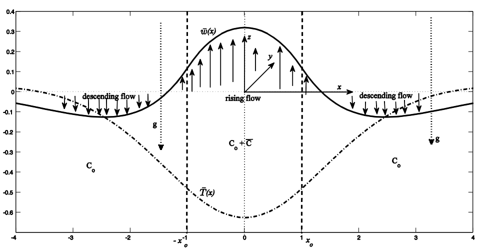

Figure 1. The geometry of the system drawn here for x_0 = 1.0 and showing a typical plume flow, $\bar{w}(x)$, and associated temperature , $\bar{T}(x)$ profile.

Now to obtain the perturbation equations, we subtract the system (20)-(22) from perturbation equations (9)-(12) after substituting the expressions (14)-(17). Then neglecting the terms of order ε^2 to get the linearized perturbation equations

$$R\left[\frac{\partial \boldsymbol{u}^\dagger}{\partial t}+\bar{w}\,\hat{\boldsymbol{z}}.\nabla\boldsymbol{u}^\dagger+\left(\boldsymbol{u}^\dagger.\nabla\bar{\varpi}\right)\hat{\boldsymbol{z}}\right]=-\nabla p^\dagger + \nabla^2\boldsymbol{u}^\dagger+\left(T^\dagger+C^\dagger\right)\hat{\boldsymbol{z}} , \quad (30)$$

$$\nabla.\boldsymbol{u}^\dagger=0 , \quad (31)$$

$$\sigma R\left[\frac{\partial T^\dagger}{\partial t}+\bar{w}\frac{\partial T^\dagger}{\partial z}+\boldsymbol{u}^\dagger.\nabla\bar{T}\right] + \boldsymbol{u}^\dagger.\hat{\boldsymbol{z}} = \nabla^2 T^\dagger , \quad (32)$$

$$R\,S_c\left[\frac{\partial C^\dagger}{\partial t}+\bar{w}\frac{\partial C^\dagger}{\partial z}+\boldsymbol{u}^\dagger.\nabla\bar{C}\right] = \nabla^2 C^\dagger . \quad (33)$$

The system (30)-(33) is solved subject to the boundary conditions that all perturbation variables decay to zero away from the interfaces, all variables except the concentration are continuous across the interfaces, the momentum flux and the heat flux are continuous across the interfaces , and the interfaces between the two fluids are material surfaces.

We consider that the interface at $x=x_0$ is given a small harmonic disturbance of the form

$$x=x_0+\varepsilon\,\exp\left(\Omega t+\mathrm{i}(my-nz)\right)+c.c. , \quad (34)$$

where n and m are the vertical and horizontal wavenumbers , $c.c.$ refers to the complex conjugate, and Ω is given by

$$\Omega=\Omega_r+\mathrm{i}\Omega_\mathrm{i} , \quad (35)$$

such that Ω_r and Ω_i are the real and imaginary parts. The stability of the plume is determined by the sign of the real part of the growth rate, Ω_r .

The disturbance (34) propagates into the dynamical system, and affect the interface at $x=-x_0$, and the variables of the system to make the perturbation. The perturbation equations and the boundary conditions of the system allow two categories of solution one of which is even in x and other is odd. The even solution is associated with the interfaces at $x=\pm x_0$ propagating out-of-phase and hence this solution is referred as the varicose solution, while the odd solution is called the sinuous (or meandering) mode because the two interfaces propagate in-phase. Due to this parity property, the system is solved in the half domain, $x\geq 0$.

The perturbation variables produced by the disturbance (34) have the harmonic dependence

$$\left\{\boldsymbol{u}^\dagger, C^\dagger, T^\dagger, p^\dagger\right\}=\left\{-\mathrm{i}n\,u^\beta, nm\,v^\beta, w^\beta, C^\beta, T^\beta, -\mathrm{i}n\,p^\beta\right\}\exp\left(\Omega t+\mathrm{i}(my-nz)\right)+c.c., \quad (36)$$

where we have introduced the parameter

$$\beta=\begin{cases} v , & \text{for the varicose mode} \\ s , & \text{for the sinuous mode}\end{cases} , \quad (37)$$

in order to help us present the two categories of solution concisely, and the factors nm , $-\mathrm{i}n$, and $-\mathrm{i}n$ are introduced in v^β, u^β and p^β for convenience.

Substituting the variables (36) in (30)-(33) gives the following ordinary differential equations in x

$$\frac{du^\beta}{dx}-m^2v^\beta+w^\beta=0 , \quad (38)$$

$$\left(\frac{d^2}{dx^2}-a^2\right)u^\beta-\frac{dp^\beta}{dx}=R\,\bar{\Omega}^\beta\,u^\beta , \quad (39)$$

$$\left(\frac{d^2}{dx^2}-a^2\right)v^\beta-p^\beta=R\,\bar{\Omega}^\beta\,v^\beta , \quad (40)$$

$$\left(\frac{d^2}{dx^2}-a^2\right)w^\beta+T^\beta+C^\beta+n^2p^\beta=R\left(\bar{\Omega}^\beta\,w^\beta-\mathrm{i}n\,\frac{d\bar{w}}{dx}u^\beta\right) , \quad (41)$$

$$\left(\frac{d^2}{dx^2}-a^2\right)T^\beta-w^\beta=\sigma R\left(\bar{\Omega}^\beta\,T^\beta-\mathrm{i}n\,\frac{d\bar{T}}{dx}u^\beta\right) , \quad (42)$$

$$\left(\frac{d^2}{dx^2}-a^2\right)C^\beta=R\,S_c\left(\bar{\Omega}^\beta\,C^\beta-\mathrm{i}n\,\frac{d\bar{C}}{dx}u^\beta\right) , \quad (43)$$

where

$$a^2=m^2+n^2 , \qquad \bar{\Omega}^\beta=\Omega^\beta-\mathrm{i}n\,\bar{w}(x) , \quad (44)$$

subject to the independent boundary conditions

$$u^\beta, v^\beta, w^\beta, T^\beta, C^\beta, \text{ and } p^\beta\to 0 \quad \text{as} \quad x\to\infty , \quad (45)$$

$$u^\beta, w^\beta, T^\beta, p^\beta, C^\beta, \text{ and } \frac{dT^\beta}{dx} \text{ are continuous across } x=x_0 , \quad (46)$$

$$\frac{dw^{\beta}}{dx}\left(x_0^+\right)-\frac{dw^{\beta}}{dx}\left(x_0^-\right)=-1\,,\quad \frac{dC^{\beta}}{dx}\left(x_0^+\right)-\frac{dC^{\beta}}{dx}\left(x_0^-\right)=\,\mathrm{i}n\,R\,S_c\;u^{\beta}(x_0)\,,$$
(47)

$$-\mathrm{i}n\,u^{\beta}(x_0)=\Omega^{\beta}-\mathrm{i}n\,\overline{w}\,(x_0)\;,$$
(48)

$$\left.\begin{array}{ll}\text{(i)} & u^{v}(0)=\frac{dv^{v}}{dx}(0)=\frac{dw^{v}}{dx}(0)=\frac{dT^{v}}{dx}(0)=\frac{dC^{v}}{dx}(0)=\frac{dp^{v}}{dx}(0)=0\\ \text{(ii)} & \frac{du^{s}}{dx}(0)=v^{s}(0)=w^{s}(0)=T^{s}(0)=C^{s}(0)=p^{s}(0)=0\end{array}\right\}.$$

(49)

Following Eltayeb and Loper (1994) , we assume the perturbation variables of the system and the growth rate , Ω , are written as expansions of the small parameter $R\,(\ll 1)$:

$$f\,(x,y,z,t)=\sum_{r=0}^{\infty}f_r\,(x,y,z,t)R^r\;,\qquad \Omega^{\beta}=\sum_{r=1}^{\infty}\Omega^{\beta}_r\,R^{\,r-1}\;,$$

(50)

where $f\,(x,y,z,t)$ indicates the perturbation variable u^{β}, v^{β}, w^{β}, p^{β}, T^{β}, and C^{β}. Substituting the variables (50) into the equations (38)-(43) and the relevant boundary conditions (45)-(49) , and equating the coefficients of R^r (r = 0 , 1 , 2 , ….) to zero to get systems of ordinary differential equations which can be solved to find the the growth rate. The two systems for $r=0$ and $r=1$ are sufficient to find the stability of the interface.

When $r=0$, the system consists of the equations

$$\frac{du_0^{\beta}}{dx}-m^2v_0^{\beta}+w_0^{\beta}=0\;,\tag{51}$$

$$\left(\frac{d^2}{dx^2}-a^2\right)u_0^{\beta}-\frac{dp_0^{\beta}}{dx}=0\;,\tag{52}$$

$$\left(\frac{d^2}{dx^2}-a^2\right)v_0^{\beta}-p_0^{\beta}=0\;,\tag{53}$$

$$\left(\frac{d^2}{dx^2}-a^2\right)w_0^{\beta}+T_0^{\beta}+C_0^{\beta}+n^2p_0^{\beta}=0\,,\tag{54}$$

$$\left(\frac{d^2}{dx^2}-a^2\right)T_0^{\beta}-w_0^{\beta}=0\;,\tag{55}$$

$$\left(\frac{d^2}{dx^2}-a^2\right)C_0^{\beta}=0\,,\tag{56}$$

and the independent boundary conditions

$$u_0^{\beta},v_0^{\beta},w_0^{\beta},C_0^{\beta},T_0^{\beta}\,,\text{ and }\;p_0^{\beta}\to 0\quad\text{as}\quad x\to\infty\;,\tag{57}$$

$$u_0^{\beta},\,w_0^{\beta},C_0^{\beta},T_0^{\beta},\;p_0^{\beta}\,,\text{ and }\frac{dT_0^{\beta}}{dx}\text{ are continuous across } x=x_0\,,\tag{58}$$

$$\frac{dw_0^{\beta}}{dx}\left(x_0^-\right)-\frac{dw_0^{\beta}}{dx}\left(x_0^+\right)=1\,,\tag{59}$$

$$\frac{dC_0^{\beta}}{dx}\left(x_0^-\right)-\frac{dC_0^{\beta}}{dx}\left(x_0^+\right)=0\;,\tag{60}$$

$$-\mathrm{i}n\,u_0^{\beta}(x_0)=\Omega_1^{\beta}-\mathrm{i}n\,\overline{w}\,(x_0)\,,\tag{61}$$

$$\left.\begin{array}{ll}\text{(i)} & u_0^{v}(0)=\frac{dv_0^{v}}{dx}(0)=\frac{dw_0^{v}}{dx}(0)=\frac{dT_0^{v}}{dx}(0)=\frac{dC_0^{v}}{dx}(0)=\frac{dp_0^{v}}{dx}(0)=0\\ \text{(ii)} & \frac{du_0^{s}}{dx}(0)=v_0^{s}(0)=w_0^{s}(0)=T_0^{s}(0)=C_0^{s}(0)=p_0^{s}(0)=0\end{array}\right\}.\tag{62}$$

The solution of this system is given by

$$\left\{w_0^{\beta},T_0^{s},p_0^{\beta},v_0^{s}\right\}=\sum_{j=1}^{3}\frac{K_j}{2}\left\{\mu_j^3,\mu_j^2,\mu_j,1\right\}\left(Be^{-\lambda_j(x+x_0)}+\begin{cases}e^{\lambda_j(x-x_0)}\\ e^{-\lambda_j(x-x_0)}\end{cases},\begin{vmatrix}0\le x<x_0\\ x_0\le x<\infty\end{vmatrix}\right),\tag{63}$$

$$u_0^{\beta}=\sum_{j=1}^{3}\frac{-K_j\lambda_j}{2}\left(B\,e^{-\lambda_j(x+x_0)}+\begin{Bmatrix}-e^{\lambda_j(x-x_0)}\\ e^{-\lambda_j(x-x_0)}\end{Bmatrix},\;\begin{vmatrix}0\le x<x_0\\ x_0\le x<\infty\end{vmatrix}\right),\tag{64}$$

$$C_0^{\beta}=0\;,\tag{65}$$

where

$$K_j=\frac{\mu_j^2}{\lambda_j\left(2\mu_j+3n^2\right)}\;,\qquad \lambda_j=\sqrt{\mu_j+a^2}\;,\tag{66}$$

and μ_j ($j=1,2,3$) are the roots of the cubic equation

$$\mu^3+\mu+n^2=0\;,\tag{67}$$

$$B=\begin{cases}\;1\;, & \text{for the varicose (even) mode}\\ -1\;, & \text{for the sinuous (odd) mode}\end{cases}.$$
(68)

The expression of the growth rate Ω_1^{β} is given by

$$\Omega_1^{\beta}\left(m,n,x_0,B\right)=\frac{\mathrm{i}n}{2}\left(S+B\,N_2\right),\tag{69}$$

where

$$S=e^{-\sqrt{2}x_0}\sin\left(\sqrt{2}\;x_0\right)\;,\tag{70}$$

$$N_k = \sum_{j=1}^{3} \frac{\mu_j^k}{(3n^2+2\mu_j)} E_j \ , \qquad E_j = \exp\left(-2\lambda_j x_0\right) .$$
(71)

It should be noted here that the nature of the roots of the equation (67) make the expressions for N_k (k = 1 , 2 , 3 , ……) real. It then follows that the expression for Ω_1^β is purely imaginary which indicates that the disturbance is a neutral wave which has a vertical phase speed V defined by

$$V = \frac{\Omega_1^\beta}{\mathrm{i}n} = \frac{1}{2}\left(S + B\, N_2\right) . \tag{72}$$

It is then necessary to investigate the next order problem to determine the stability.

When $r = 1$, the system is

$$\frac{du_1^\beta}{dx} - m^2 v_1^\beta + w_1^\beta = 0 \ , \tag{73}$$

$$\Delta u_1^\beta = M^\beta(x) \ , \tag{74}$$

$$\Delta v_1^\beta - p_1^\beta = M_v^\beta \ , \tag{75}$$

$$\Delta w_1^\beta + T_1^\beta + n^2 p_1^\beta = M_w^\beta \ , \tag{76}$$

$$\Delta T_1^\beta - w_1^\beta = M_T^\beta \ , \tag{77}$$

$$\Delta C_1^\beta = 0 \ , \tag{78}$$

where

$$\Delta = \left(\frac{d^2}{dx^2} - a^2 \right) \ , \tag{79}$$

$$M^\beta(x) = \frac{dp_1^\beta}{dx} + \left(\Omega_1^\beta - \mathrm{i}n\, \bar{w}(x)\right) u_0^\beta \ , \tag{80}$$

$$M_v^\beta = \left(\Omega_1^\beta - \mathrm{i}n\, \bar{w}(x)\right) v_0^\beta \ , \tag{81}$$

$$M_w^\beta = \left(\Omega_1^\beta - \mathrm{i}n\, \bar{w}(x)\right) w_0^\beta - \mathrm{i}n \frac{d}{dx}\left(\bar{w}(x)\right) u_0^\beta - C_1^\beta \ , \tag{82}$$

$$M_T^\beta = \sigma\left(\Omega_1^\beta - \mathrm{i}n \bar{w}(x)\right) T_0^\beta - \mathrm{i}n\sigma \frac{d}{dx}\left(\bar{T}(x)\right) u_0^\beta \ , \tag{83}$$

and the boundary conditions are

$$u_1^\beta, v_1^\beta, w_1^\beta, T_1^\beta, C_1^\beta \text{ and } p_1^\beta \to 0 \quad \text{as} \quad x \to \infty \ , \tag{84}$$

$$u_1^\beta, C_1^\beta, w_1^\beta, T_1^\beta,\ p_1^\beta,\ \frac{dw_1^\beta}{dx} \text{ and } \frac{dT_1^\beta}{dx} \text{ are continuous across } x = x_0 \ , \tag{85}$$

$$\frac{dC_1^\beta}{dx}\left(x_0^-\right) - \frac{dC_1^\beta}{dx}\left(x_0^+\right) = -\mathrm{i}n S_c\, u_0^\beta\left(x_0\right) , \tag{86}$$

$$\Omega_2^\beta = -\mathrm{i}n u_1^\beta(x_0) \ , \tag{87}$$

$$\left.\begin{array}{ll} \text{(i)} & u_1^v(0) = \frac{dv_1^v}{dx}(0) = \frac{dw_1^v}{dx}(0) = \frac{dT_1^v}{dx}(0) = \frac{dC_1^v}{dx}(0) = \frac{dp_1^v}{dx}(0) = 0 \\ \text{(ii)} & \frac{du_1^s}{dx}(0) = v_1^s(0) = w_1^s(0) = T_1^s(0) = C_1^s(0) = p_1^s(0) = 0 \end{array}\right\} . \tag{88}$$

Operating on equation (73) with Δ , and using equations (74)-(76) and (80)-(82) we obtain

$$\Delta p_1^\beta - T_1^\beta = M_p^\beta \ , \tag{89}$$

where

$$M_p^\beta = 2\mathrm{i}n \frac{d}{dx}\left(\bar{w}(x)\right) u_0^\beta + C_1^\beta \ . \tag{90}$$

This system of linear ordinary differential equations is nonhomogeneous, and it can be obtained as the sum of particular and complementary solutions for each variable. Then the application of the boundary conditions leads to the growth rate Ω_2^β . However, in order for a solution to exist, the system must obey a solvability condition. It turns out that the solvability condition provides the required expression for Ω_2^β . We therefore will not find explicit expressions for the variables of the first order problem. The expression can be written in the form

$$\Omega_2^\beta\left(x_0,\sigma,Sc,B;m,n\right) = \Omega_{2o}^\beta\left(x_0,\sigma,Sc,B;m,n\right) + Sc\ \Omega_{2d}\left(x_0;m,n\right) , \tag{91}$$

where Ω_{2o}^β is the growth rate that was found in the absence of the material diffusion and it is real, and Ω_{2d} is the growth rate introduced by the addition of the material diffusion. The expression for Ω_{2d} can be expressed concisely as

$$\Omega_{2d} = \frac{n^2}{4}\ N_1 N_2 \ , \tag{92}$$

this shows that it is real and then it affects the growth rate of the disturbance.

We mention here that the case of a single interface has a different stability property compared to that of the Cartesian plume when material diffusion is present. This is due to the fact that expressing the basic concentration for a single interface by

$$\bar{C} = -\frac{1}{2}\operatorname{sgn}(x) \,. \qquad (93)$$

We note that the zeroth order solution has an odd $u_0(x)$ which vanishes at the interface $x = 0$. Hence $C_1^{\beta} = 0$ and consequently $\Omega_{2d} = 0$. Thus material diffusion does not modify the growth rate of single interface at order R^1, as in the case of the Cartesian plume.

4. Discussion

The expression of the growth rate (91) is composed of two terms. The first term Ω_{20}^{β} represents the expression obtained in the absence of material diffusion while the term Ω_{2d} is entirely due to the presence of material diffusion. The properties of Ω_{20}^{β} have been investigated by Eltayeb and Loper (1994). Here we want to isolate the influence of material diffusion by examining in detail the dependence of Ω_{2d} on the parameters of the problem, and then discuss the effect of Ω_{2d} on the total growth rate Ω_2^{β}. In the calculations of the growth rates Ω_2^{β}, Ω_{20}^{β} and Ω_{2d} a scaling factor of 10^2 is used in order to facilitate comparison with the results in the absence of material diffusion.

We immediately note that the expressions (92) and (71) for Ω_{2d} are independent of the parity of the solution and of σ. However, Ω_{2d} depends strongly on the thickness of the plume. The isolines of Ω_{2d} in the (m, n) plane are plotted in figure 2 for different representative values of x_0. It is found that Ω_{2d} is negative except in a small region with small n and moderate m provided x_0 is moderate or large. We therefore find it informative to examine in detail the expression (92) as $n \to 0$. The calculations show that

$$\tilde{\Omega}_{2d} = N_{10} N_{20}, \qquad n \to 0, \qquad (94)$$

where

$$\tilde{\Omega}_{2d} = \frac{4}{n^2}\,\Omega_{2d}, \qquad (95)$$

$$N_{10} = -e^{-2m x_0} + e^{-2x_0\sqrt{r}\cos\left(\frac{\theta}{2}\right)} \cos\left(2x_0\sqrt{r}\sin\left(\frac{\theta}{2}\right)\right) + o\left(n^2\right), \qquad (96)$$

$$N_{20} = e^{-2x_0\sqrt{r}\cos\left(\frac{\theta}{2}\right)} \sin\left(2x_0\sqrt{r}\sin\left(\frac{\theta}{2}\right)\right) + o\left(n^2\right), \qquad (97)$$

$$r = \sqrt{m^4 + 1}, \qquad \theta = \tan^{-1}\left(\frac{1}{m^2}\right). \qquad (98)$$

Numerical computations of N_{10} and N_{20} showed that N_{10} is negative for all values of m and x_0 while N_{20} takes positive values in a number of regions determined by the argument of the sine function in (97). As a result, $\tilde{\Omega}_{2d}$, which is given by the product of $N_{10} N_{20}$ in (94), takes positive values outside the regions in which $N_{20} > 0$. This is illustrated in figure 3.

Since Ω_{2d} is mostly negative, it is of interest to see how the presence of diffusion affects the total growth rate Ω_2^{β} for different values of S_c. This can be first illustrated by plotting the isolines of Ω_2^{β} in the (m, n) plane for sample values of S_c, x_0 and σ for both varicose and sinuous modes (see figures 4-6). For fixed x_0 and small σ (□ 1.0), the varicose mode is 3-dimensional when material diffusion is absent (i.e. $S_c = 0.0$). As S_c is increased it remains 3-dimensional until S_c reaches a value, S_{σ}, dependent on σ when the mode becomes 2-dimensional. The value of S_{σ} increases with σ. For example, when $\sigma = 10.0$, S_{σ} is greater than 100.0.

The sinuous mode has a 2-dimensional maximum when $S_c = 0.0$. As S_c increases, the maximum becomes 3-dimensional in a range (S_{c1}, S_{c2}) of S_c beyond which the maximum reverts to its 2-

dimensional nature. As σ increases, the interval (S_{c1}, S_{c2}) expands until the 2-dimensional maximum disappears completely at a value $\sigma(x_0)$, dependent on x_0, and the maximum becomes 3-dimensional for all values of S_c. It is then observed that the vertical wavenumber, n_c, of the maximum growth rate is increased by the presence of material diffusion. This results in a smaller vertical wavelength. This behavior is unusual since the influence of diffusion is normally to stabilize short wavelengths. However, the fact that such shortening of the wavelength is associated with 2–dimensional motions for both parities indicates that the two effects may be linked.

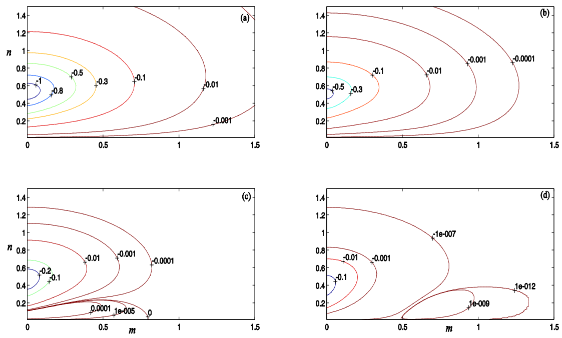

Figure 2. The contours of the growth rate contribution, Ω_{2d}, due to material diffusion in the (m, n) plane for different values of x_0. (a), (b), (c), (d) refer to $x_0 =$ 1.0, 2.0, 3.0, 5.0, respectively. Note that Ω_{2d} is negative except for a small area occurring when x_0 exceeds about 2.0 and n is small.

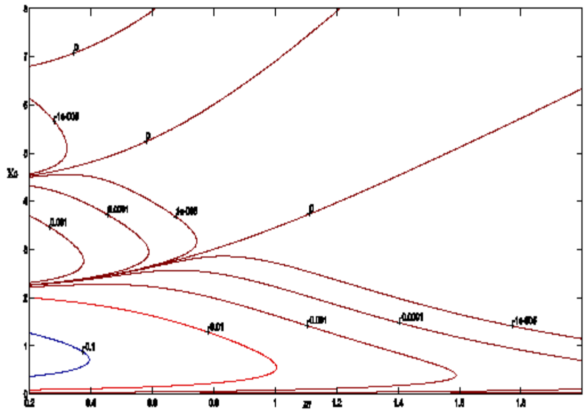

Figure 3. The growth rate of material diffusion, $\tilde{\Omega}_{2d}$, isolines in the (m, x_0) plane when $n \to 0$ (see equation (94)). Note that $\tilde{\Omega}_{2d}$ is positive for x_0 in the interval $\approx (2.3,\ 4.5)$.

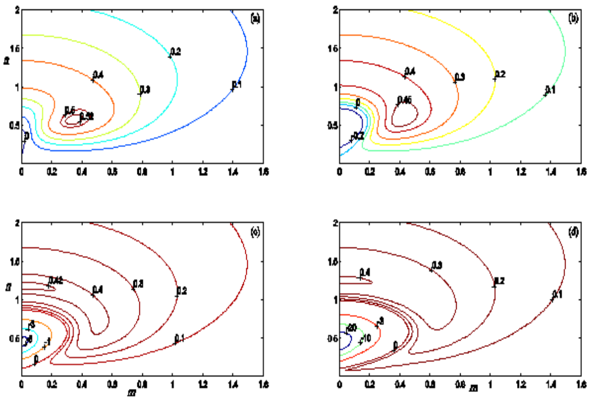

Figure 4. Counters of the growth rate of the varicose mode, Ω_2^v, for the Cartesian plume in the (*m,n*) plane for fixed values of $x_0 =$ 3.0 and $\sigma =$ 1.0 and for four different values of the Schmidt number; (a) $S_c =$ 0.0, (b) 5.0, (c) 20.0, and (d) 100.0. Note that the area of instability decreases when the Schmidt number increases and the value of the maximum decreases.

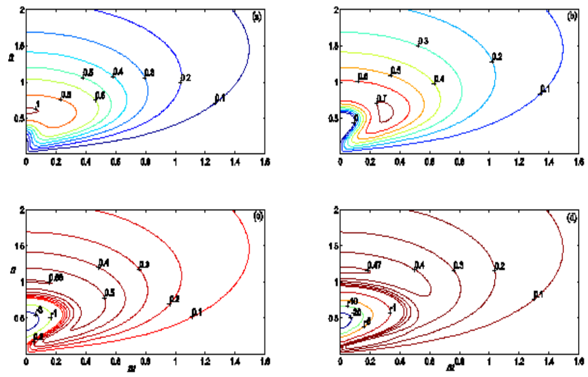

Figure 5. Counters of the growth rate of the sinuous mode , Ω_2^s , for the Cartesion plume in the (*m,n*) plane for the same parameters and notation as in figure 4. Note that the area of instability decreases when the Schmidt number increases and the value of the maximum decreases.

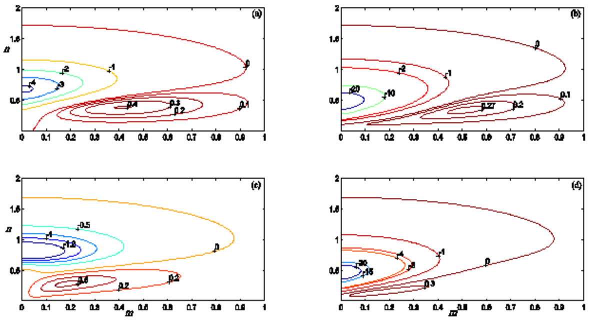

Figure 6. illustration of the dependence of the growth rate Ω_2^{β} on the parity of the solution. $x_0 = 3.0$, $\sigma = 10.0$ and $S_c = 0.0$ in (a) , (c) , and 100.0 in (b) and (d). (a) , (b) for varicose and (c) , (d) for sinuous. Compare with figures 4(a) , (d) and 5(a) , (d) for $\sigma = 1.0$.

5. Conclusion

The linear stability of a compositional plume of finite width $2x_0$ rising in an infinite fluid of the same thermal diffusivity, viscosity, and material diffusion has been studied. The main purpose is to investigate the effect of material diffusion on the stability problem. The stability analysis showed that the growth rate of the disturbance of the interfaces of the plume is a sum of two terms one of which , Ω_{2o} , represents the growth rate in the absence of material diffusion while the other is entirely due to the presence of material diffusion, $S_c\,\Omega_{2d}$ (see equation (91)). Detailed investigations of the term Ω_{2d} due to material diffusion showed that it is negative except in a small region for moderate values of x_0 and horizontal wavenumber , m , and small values of the vertical wavenumber, n . Another characteristic property of Ω_{2d} is that it is independent of the parity of the solution so that the two possible modes of sinuous and varicose are equally affected by material diffusion.

When the total growth rate is computed in the parameter space $(x_0, \sigma, S_c; m, n)$ for both sinuous and varicose modes, it is found that material diffusion has a strong effect on the properties of growth rate, Ω_c , of the preferred mode of instability. The growth rate Ω_c is reduced in value and the wavenumbers, m_c , n_c , are also altered when S_c is increased from 0 . In some cases, the 3-dimensional nature of the preferred mode is destroyed, and 2-dimensional modes become preferred for large values of S_c (see, e.g., figure 4).

When the thickness of the plume increases to large values (i.e. $x_0 \to \infty$) , the plume resembles the single interface model studied by Eltayeb and Loper (1991). In the present study, we find that material diffusion has no effect on the growth rate of the single interface problem, to the same leading order. This indicates that the thickness of the plume plays an important role in the stability of the plume.

Another unexpected result produced by material diffusion is an increase in the vertical wavenumber leading to a smaller wavelength at large Schmidt number. It is noted that such effect always occurred when a 3–dimensional mode is transformed into a 2–dimensional one.

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