On an Integral Involving Product of Two Generalized Hypergeometric Functions

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Abstract

In this research note, an interesting integral involving product of two generalized hypergeometric function has been evaluated in-terms of gamma function. The result is derived with the help of a known integral involving hypergeometric function available in the literature. A few very interesting special cases are also given.

Keywords: Hypergeometric Function, Generalized Hypergeometric Function, Watson Theorem, Definite Integral.

1. Introduction and Results Required

In order to justify our doing, we must quote Sylvester[9] "*It seems to be expected of every pilgrim up the slopes of the mathematical parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock*"

We begin by recalling the natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric function ${}_pF_q$ defined by [1,3,7,8]

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\quad z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \qquad \ldots (1.1)$$

where $(\alpha)_n$ is the well known Pochhammer symbol defined (for $\alpha\in C$) by

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$=\left\{\begin{matrix}\alpha(\alpha+1)\ldots\ldots(\alpha+n-1) & ;n\in N\\ 1 & ;n=0\end{matrix}\right\} \qquad \ldots (1.2)$$

and $\Gamma(\alpha)$ is the familiar Gamma function. Here an empty product is to be interpreted as unity, and we assume that the variable x, the numerator parameters $\alpha_1,\ldots\alpha_p$, and the denominator parameters $\beta_1,\ldots\beta_q$, take on complex values, provided that no zeros appear in the denominator of (1.1) that is

$$(\beta_j \in C \setminus Z_0^{-};\, j=1,\ldots,q)$$

Here and in the following, let C, Z and N be the sets of complex numbers, integers and positive integers, respectively, and moreover, let

$$N_0 = N \cup 0 \text{ and } Z_0^{-} := Z \setminus N$$

For more details about ${}_pF_q$ including its convergence, absolute convergence, various special and limiting cases, we refer standard texts [7].

It is interesting to mention here that whenever the generalized hypergeometric function ${}_pF_q$ and its important special case ${}_2F_1$ with some specified argument such as *1,* $\frac{1}{2}$*, -1* can be summed to be expressed in terms of gamma functions, the results may be important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role in the theory and application. Applications of the above mentioned classical summation theorems are well-known now. For very interesting applications, we refer a useful paper by Bailey [2]. However, here we would like to mention the following classical Watson's summation theorem [1,3,8].

$${}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{a+b+1}{2}, \quad 2c \end{matrix} ;\ 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \quad \ldots (1.3)$$

provided *Re* $(2c-a-b) > -1$.

From (1.3), it is not difficult to evaluate the following integral involving hypergeometric function recorded in [4,5] viz.

$$\int_0^1 x^{c-1}(1-x)^{c-1}\, {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{a+b+1}{2} \end{matrix} ;\ x\right] dx$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(c)\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \quad \ldots (1.4)$$

provided *Re* $(c) > 0$ and *Re* $(2c-a-b) > -1$.

In this research note, an interesting integral involving product of two generalized hypergeometric function has been evaluated in terms of gamma function. The integral is evaluated with the help of the known integral (1.4). A few very interesting special cases have also been given.

2. **Main Integral Formula**;

In this section, we shall evaluate the integral involving product of two generalized hypergeometric functions given in the following theorem.

Theorem 2.1

For $Re\,(d) > 0$ and $Re\,(2d-a-b) > -1$, the following result holds true.

$$\int_0^1 x^{d-1}(1-x)^{d-1}\,{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{a+b+1}{2} \end{matrix} \;;\; x\right].\,{}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2} \\ c, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix} \;;\; 4x(1-x)\right]dx$$

$$= \frac{\sqrt{\pi}\,\Gamma(d)\Gamma(d)\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(d-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{b}{2}+\frac{1}{2}\right)} \times$$

$${}_4F_4\left[\begin{matrix} d, \quad c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2}, \quad d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \\ c, \quad d-\frac{a}{2}+\frac{1}{2}, \quad d-\frac{b}{2}+\frac{1}{2}, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix} \;;\; 1\right] \qquad ...(2.1)$$

Proof: In order to evaluate the integral (2.1), we proceed as follows. Denoting the left-hand side of (2.1) by I, we have

$$I = \int_0^1 x^{d-1}(1-x)^{d-1}\,{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{a+b+1}{2} \end{matrix} \;;\; x\right]{}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2} \\ c, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix} \;;\; 4x(1-x)\right]dx$$

Express ${}_2F_2$ as a series, interchanging the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I = \sum_{n=0}^{\infty} \frac{\left(c-\frac{a}{2}+\frac{1}{2}\right)_n \left(c-\frac{b}{2}+\frac{1}{2}\right)_n 2^n}{(c)_n \left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)_n n!} \int_0^1 x^{d+n-1}(1-x)^{d+n-1}\,{}_2F_1\left[\begin{matrix} a, \quad b, \\ \frac{a+b+1}{2} \end{matrix} \;;\; x\right]dx.$$

Evaluating the integral with the help of the result (1.4) and making use of the result (1.2), we have after some simplification.

$$I=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(d)\Gamma(d)\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(d-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{b}{2}+\frac{1}{2}\right)}\text{ x}$$

$$\sum_{n=0}^{\infty}\frac{(d)_n\left(c-\frac{a}{2}+\frac{1}{2}\right)_n\left(c-\frac{b}{2}+\frac{1}{2}\right)_n\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)_n}{(c)_n\left(d-\frac{a}{2}+\frac{1}{2}\right)_n\left(d-\frac{b}{2}+\frac{1}{2}\right)_n\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)_n}$$

Finally, using (1.1), we easily arrive at the right hand side of (2.1), This completes the proof of (2.1).

3. **Special cases**:

In this section, we shall mention a few very interesting special cases of our main integral (2.1) in the form of following corollaries.

Corollary 3.1: In (2.1), if we let $b=-2n$ and replace *a* by $a+2n$, where n is zero or a positive integer, then we get the following result:

$$\int_0^1 x^{d-1}(1-x)^{d-1}\,{}_2F_1\left[\begin{matrix}-2n, & a+2n \\ \frac{a+1}{2} & \end{matrix}\;;\; x\right]{}_2F_2\left[\begin{matrix}c+n+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}-n \\ c, & c-\frac{a}{2}+\frac{1}{2}\end{matrix}\;;\; 4x(1-x)\right]dx$$

$$=\frac{e\Gamma(d)\Gamma(d)\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+\frac{a}{2}-d\right)_n}{\Gamma(2d)\Gamma\left(d+\frac{1}{2}\right)_n\left(\frac{a+1}{2}\right)_n}\text{ x}$$

$${}_4F_4\left[\begin{matrix}d, & c+n+\frac{1}{2}, & c-n-\frac{a}{2}+\frac{1}{2}, & d-\frac{a}{2}+\frac{1}{2} \\ c, & d+n+\frac{1}{2} & d-n-\frac{a}{2}+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}\end{matrix}\;;\; 1\right] \qquad ...(3.1)$$

Corollary 3.2: In (2.1), if we let $b=-2n-1$and replace *a* by $a+2n+1$, where *n* is zero or a positive integer, then we get the following result:

$$\int_0^1 x^{d-1}(1-x)^{d-1}\,{}_2F_1\left[\begin{matrix}-2n-1, & a+2n+1 \\ \frac{a+1}{2} & \end{matrix}\;;\; x\right]\text{X}$$

$$ {}_2F_2\left[\begin{matrix} c+n+1, & c-\frac{a}{2}-n \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{matrix} \;;\; 4x(1-x)\right]dx = 0 \qquad \text{...(3.2)} $$

This is a typical example.

Corollary 3.3. In (2.1), if we let $a=b=\frac{1}{2}$ and making use of the known result [[4] p.473, equ.(75)]

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 1 & \end{matrix} \;;\; x\right] = \frac{2}{\pi}K\left(\sqrt{x}\right) $$

where *K(k)* is the well-known Elliptic function of the first kind defined by

$$ K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}} $$

then, we get the following interesting result:

$$ \int_0^1 x^{d-1}(1-x)^{d-1}K\left(\sqrt{x}\right)\,{}_2F_2\left[\begin{matrix} c+\frac{1}{4}, & c+\frac{1}{4} \\ c, & c \end{matrix} \;;\; 4x(1-x)\right]dx $$

$$ = \frac{e}{2}\pi^{\frac{3}{2}} \frac{\Gamma^3(d)\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(2d)\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma^2\left(d+\frac{1}{4}\right)}\,{}_4F_4\left[\begin{matrix} d, & d, & c+\frac{1}{4}, & c+\frac{1}{4} \\ c, & c, & d+\frac{1}{4}, & d+\frac{1}{4} \end{matrix} \;;\; 1\right] \qquad \text{...(3.3)} $$

provided $\mathrm{Re}(d) > 0$.

Corollary 3.4: In (2.1), if we let *a = b =1* and making use of the known result [[4], p.476, equ.(147)].

$$ {}_2F_1\left[\begin{matrix} 1, & 1 \\ \frac{3}{2} & \end{matrix} \;;\; x\right] = \frac{\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{x(1-x)}} $$

then, we get the following interesting result:

$$ \int_0^1 x^{d-\frac{3}{2}}(1-x)^{d-\frac{3}{2}}\sin^{-1}\left(\sqrt{x}\right)\,{}_1F_1\left[\begin{matrix} c \\ c-\frac{1}{2} \end{matrix} \;;\; 4x(1-x)\right]dx $$

$$= \frac{e}{2}\pi \frac{\Gamma\left(d-\frac{1}{2}\right)\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(2d)} {}_2F_2\left[\begin{matrix} c, & d-\frac{1}{2} \\ d, & c-\frac{1}{2} \end{matrix} ; 1\right] \qquad ...(3.4)$$

provided $(d) > \frac{1}{2}$.

Corollary 3.5: In (2.1), if we set $b = -a$ and making use of the known result [[4], p.459, equ.(83)] then, we get the following result:

$$\int_0^1 x^{d-1}(1-x)^{d-1}\cos(2a\sin^{-1}\sqrt{x})\ {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ c, & c+\frac{1}{2} \end{matrix} ; 4x(1-x)\right]dx$$

$$= \frac{\pi e\Gamma^2(d)\Gamma^2\left(d+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(d+\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(d+\frac{1}{2}-\frac{a}{2}\right)} \text{x}$$

$${}_4F_4\left[\begin{matrix} d, & d+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}, & d-\frac{a}{2}+\frac{1}{2} \\ c, & c+\frac{1}{2}, & d-\frac{a}{2}+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2} \end{matrix} ; 1\right] \qquad ...(3.5)$$

provided *Re(d) > 0*.

Finally in (2.1) and (3.1) to (3.6) if we take $d = c$, we get the following results in more compact form:

$$\int_0^1 x^{c-1}(1-x)^{c-1}\ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{a+b+1}{2} \end{matrix} ; x\right] {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c-\frac{b}{2}+\frac{1}{2} \\ c, & c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix} ; 4x(1-x)\right]dx$$

$$= \frac{e\sqrt{\pi}\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \qquad ...(3.6)$$

and

$$\int_0^1 x^{c-1}(1-x)^{c-1}\ {}_2F_1\left[\begin{matrix} -2n, \quad a+2n \\ \frac{a+1}{2} \end{matrix} ; x\right] {}_2F_2\left[\begin{matrix} c+n+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}-n \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{matrix} ; 4x(1-x)\right]dx$$

$$=\frac{e\Gamma(c)\Gamma(c)\left(\frac{1}{2}+\frac{a}{2}-c\right)_n}{\Gamma(2c)\left(c+\frac{1}{2}\right)_n\left(\frac{a+1}{2}\right)_n} \qquad \ldots(3.7)$$

and

$$\int_0^1 x^{c-1}(1-x)^{c-1}\,{}_2F_1\left[\begin{matrix}-2n-1, \quad a+2n+1 \\ \frac{a+1}{2}\end{matrix} \; ; \; x\right]{}_2F_2\left[\begin{matrix}c+n+1, \quad c-\frac{a}{2}-n \\ c, \quad c-\frac{a}{2}+\frac{1}{2}\end{matrix} \; ; \; 4x(1-x)\right]dx=0 \qquad \ldots(3.8)$$

and

$$\int_0^1 x^{c-1}(1-x)^{c-1}K\left(\sqrt{x}\right){}_2F_2\left[\begin{matrix}c+\frac{1}{4}, \quad a+\frac{1}{4} \\ c, \quad c\end{matrix} \; ; \; 4x(1-x)\right]dx$$

$$=\frac{e}{2}\pi^{\frac{3}{2}}\frac{\Gamma^3(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(c+\frac{1}{4}\right)} \qquad \ldots(3.9)$$

and

$$\int_0^1 x^{c-\frac{3}{2}}(1-x)^{c-\frac{3}{2}}\sin^{-1}\left(\sqrt{x}\right){}_1F_1\left[\begin{matrix}c \\ c-\frac{1}{2}\end{matrix} \; ; \; 4x(1-x)\right]dx$$

$$=\frac{e}{2}\pi\frac{\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)} \qquad \ldots(3.10)$$

provided $Re(c) > \frac{1}{2}$.

and finally $$\int_0^1 x^{c-1}(1-x)^{c-1}\cos(2a\sin^{-1}\sqrt{x})\,{}_2F_2\left[\begin{matrix}c-\frac{a}{2}+\frac{1}{2}, \quad c+\frac{a}{2}+\frac{1}{2} \\ c, \quad c+\frac{1}{2}\end{matrix} \; ; \; 4x(1-x)\right]dx$$

$$= \frac{\pi e \Gamma^2(c)\Gamma^2\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}+c\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)} \qquad ...(3.11)$$

provided *Re(c) > 0*.

Similarly other result can be obtained.

4. Conclusion:

In this research, we have obtained an integral involving the product of two generalized hypergeometric functions in terms of Gamma function. Also we have an integral involving five corollaries including the special cases for *a* and *b* in the integral of product of two hypergeometric functions.

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