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Results on Certain types of Graph Labeling

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Abstract:
In the past few decades abundant changes in the methods of labeling have evolved.one amongst the forms of labeling is square difference labeling, cube difference labeling and Prime labeling of graphs The main focus of this paper is to prove the admittance of square difference labeling, cube difference labeling and Prime labeling for few graphs.
Keywords : square difference labeling, cube difference labeling, Prime labeling, Almost bipartite graph Cube of a path graph and Sierpinski gasket graph.

1. Introduction

Labeling of a graph G is an assignment of labels to vertices or edges or both following certain rules. A dynamic survey on graph labeling by J.A.Gallian (2019) can be found in [5].. A specific type of labeling becomes more stimulating if there arises a number of problems that sparks the interest of the researchers. Prominent among the types of labeling is square difference labeling , cube difference labeling and Prime labeling of graphs.

This paper is organized as follows. In section 2, we give the preliminaries. In section 3, we prove the results of the paper where we prove that the almost bipartite graph $P_m + e$ admits square difference labeling, the cube of a path graph $P_n^3, n \geq 4$ admits cube difference labeling and the graph obtained by duplicating arbitrary vertex of Sierpinski gasket graph S_n , $n = 2$ is a Prime graph. In section 4, we provide the concluding remarks of the paper.

For number theory concept refer [4] and for basic definitions refer [1], [2] , [3], [6], [7], [8], {9] , [10].

2. Preliminaries

Definition 2.1. [1],[6]

Let G = (V (G),E(G)) be a graph. G is said to be a Square difference labeling if there exists a bijection f : V (G) → {0, 1, 2, ..., p − 1} such that the induced function f *: E(G) → N is given by f *(uv) = |[f(u)]^2 − [f(v)]^2| for every uv ∈ E(G) are all distinct. Any graph which admits square difference labeling is said to be square difference labeling graph.

Definition 2.2. [2]

Let G = (V (G),E(G)) be a graph. G is said to be a cube difference labeling if there exists a bijection f : V (G) → {0, 1, 2, ..., p − 1} such that the induced function f * : E(G) → N is given by f *(uv) = |[f(u)]^3 − [f(v)]^3| for every uv ∈ E(G) are all distinct. Any graph whichadmits cube difference labeling is said to be cube difference labeling graph.

Definition 2.3. [9], [10]

Let G be a graph. A bijection f : V → {1, 2, ..., |V |} is called a prime labeling if for each edge, e = uv in E, we have GCD{f(u), f(v)} = 1. A graph that admits a prime labeling is said to be a prime graph.

Definition 2.4. [3]

An almost-bipartite graph is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite.

Definition 2.5. [7]

The cube of a path is the graph obtained by joining every pair of vertices of distance three in the path.

Definition 2.6. [8]

The Sierpinski gasket also called the Sierpinski triangle or the Sierpinski sieve, is a fractal and attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively in to smaller equilateral triangles.

The Sierpinski gasket graphs, are defined geometrically as the graphs whose vertices are the intersection points of the line segments of the finite Sierpinski gasket and line segments of the gasket as edges.

3. Main Results

Theorem 3.1. The Almost bipartite graph $P_m + e$ admits square difference labeling.

Proof. Let $P_m + e$ be an almost bipartite graph with m path vertices

We denote the almost bipartite graph $P_m + e$ by G having vertices

$w_0, w_1, w_2, \ldots\ldots\ldots, w_{m-1}$, and

edges $e_1, e_2, \ldots\ldots\ldots, e_{m-1}, e$.

We find that $|V(G)| = m$.

Define $f: V(G) \to \{0,1,2, \ldots\ldots, m-1\}$

by $f(w_i) = i, 0 \leq i \leq m-1$

f induces square difference labeling on G.

Case-1. $\boldsymbol{m}$ **is even,** $e = w_0 w_{m-2}$

Let V_1 and V_2 be the bipartition of the vertex set V of G

Where $V_1 = \{ w_1, w_3, \dots\dots, w_{m-1} \}$ and
$V_2 = \{ w_0, w_2, \dots\dots, w_{m-2} \}$
The edge set $E = E_1 \cup E_2$
$E_1 = \{e_p / e_p = w_p w_{p+1}, 0 \le p \le m-2 \}$, $E_2 = \{e/e = w_0 w_{m-2}\}$
Case-2. $\boldsymbol{m}$ **is odd,** $e = w_0 w_{m-1}$
Let V_1 and V_2 be the bipartition of the vertex set V of G

Where $V_1 = \{ w_1, w_3, \dots\dots, w_{m-2} \}$,
$V_2 = \{ w_0, w_2, \dots\dots, w_{m-1} \}$
The edge set $E = E_1 \cup E_2$
$E_1 = \{e_p / e_p = w_p w_{p+1}, 0 \le p \le m-2 \}$,$E_2 = \{e/e = w_0 w_{m-1}\}$
For if, f^* be the induced function defined by $f^*: E(G) \to N$ such that
$f^*(w_p w_q) = \left| [f(w_p)]^2 - [f(w_q)]^2 \right|$
To prove that f^* is injective in E
For both the cases
Claim 1 : f^* is injective in E_1.
Let $e_1, e_2, \dots\dots, e_{m-1}$ be the $m-1$ edges of E_1.
It is visible that $f(w_0) < f(w_1) < f(w_2) < \cdots < f(w_{m-2}) < f(w_{m-1})$
$\Rightarrow [f(w_0)]^2 < [f(w_1)]^2 < [f(w_2)]^2 <\ldots< [f(w_{m-2})]^2 < [f(w_{m-1})]^2$
So
$|[f(w_0)]^2 - [f(w_1)]^2| < |[f(w_1)]^2 - [f(w_2)]^2| <\ldots< |[f(w_{m-2})]^2 - [f(w_{m-1})]^2|$
Hence
$f^*(w_0 w_1) < f^*(w_1 w_2) <\ldots< f^*(w_{m-2} w_{m-1})$
$f^*(e_1) < f^*(e_2) < \cdots < f^*(e_{m-1})$
Thus f^* is injective in E_1.
Claim 2 : f^* is injective in E_2
Since E_2 has only one dege e f^* is injective in E_2
Claim 3 : f^* is injective in E_1 and E_2 when $\boldsymbol{m}$ **is even**
Let $e_1, e_2, \dots\dots, e_{m-1}$, e be the m edges of E.
It is visible that $f(w_0) < f(w_1) < f(w_2) < \cdots < f(w_{m-2}) < f(w_{m-1})$
Then $[f(w_0)]^2 < [f(w_1)]^2 < [f(w_2)]^2 <\ldots\ldots< [f(w_{m-2})]^2 < [f(w_{m-1})]^2$
So
$|[f(w_0)]^2 - [f(w_1)]^2| < |[f(w_1)]^2 - [f(w_2)]^2| < \cdots < |[f(w_{m-2})]^2 - [f(w_{m-1})]^2| < |[f(w_0)]^2 - [f(w_{m-2})]^2|$
Hence

$f^*(w_0 w_1) < f^*(w_1 w_2) < \cdots < f^*(w_{m-2} w_{m-1}) < f^*(w_0 w_{m-2})$
$f^*(e_1) < f^*(e_2) <.< f^*(e_{m-1}) < f^*(e)$
Thus f^* is injective in E_1 and E_2 .
Claim 4 : : f^* is injective in E_1 and E_2 when $\boldsymbol{m}$ **is odd**
Let $e_1, e_2, \dots\dots, e_{m-1}$, e be the m edges of E.
It is visible that
$f(w_0) < f(w_1) < f(w_2) < \cdots < f(w_{m-2}) < f(w_{m-1})$
Then $[f(w_0)]^2 < [f(w_1)]^2 < [f(w_2)]^2 <.< [f(w_{m-2})]^2 < [f(w_{m-1})]^2$
So
$|[f(w_0)]^2 - [f(w_1)]^2| < |[f(w_1)]^2 - [f(w_2)]^2| < \cdots < |[f(w_{m-2})]^2 - [f(w_{m-1})]^2| < |[f(w_0)]^2 - [f(w_{m-1})]^2|$
Hence

$f^*(w_0 w_1) < f^*(w_1 w_2) < \cdots < f^*(w_{m-2} w_{m-1}) < f^*(w_0 w_{m-1})$

$f^*(e_1) < f^*(e_2) << f^*(e_{m-1}) < f^*(e)$
Thus f^* is injective in E_1 and E_2 .
Hence all the edge labels in E are distinct. Thus f^* is injective in E.
So the almost bipartite graph $P_m + e$ admits square difference labeling.

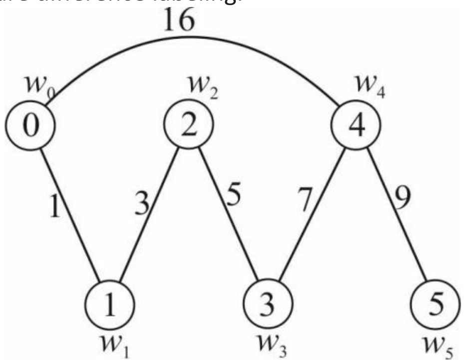

Figure. 1 square difference labeling of the almost bipartite graph $\mathrm{P}_6 + \mathrm{e}$

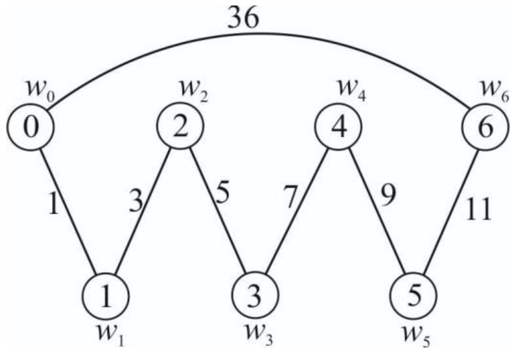

Figure. 2 square difference labeling of the almost bipartite graph $\mathrm{P}_7 + \mathrm{e}$

Theorem 3.2. The cube of a path graph $P_n^3, n \ge 4$ admits cube difference labeling
Proof. Let P_n^3 be an cube of a path graph for $n \ge 4$.
We denote the cube of a path graph for $n \ge 4$ by G having vertices
$v_1, v_2, \dots\dots, v_n$, and edges $e_1, e_2, \dots\dots, e_{2n-4}$.
We find that $|V(G)| = n$,
$|E(G)| = 2n - 4$.

Define $\mathcal{U}: V(G) \to \{0,1,2, \ldots\ldots, n-1\}$ by $\mathcal{U}(v_i) = i-1, 1 \leq i \leq n$.
$\mathcal{U}$ induces a cube difference labeling on *G*.
For if, $\mathcal{U}^*$ be the induced function defined by $\mathcal{U}^*: E(G) \to N$ such that

$$\mathcal{U}^*(v_r v_s) = |\,[\mathcal{U}(v_r)]^3 - [\mathcal{U}(v_s)]^3\,|$$

Let $E = E_1 \cup E_2$ Where
$E_1 = \{e_u / e_u = v_p v_{p+1}, 1 \leq p \leq n-1\}$
$E_2 = \{e_u / e_u = v_p v_{p+3}, 1 \leq p \leq n-3\}$
To prove that $\mathcal{U}^*$ is injective in E.
Claim 1. $\mathcal{U}^*$ is injective in E_1.
Let $e_1, e_2, \ldots\ldots\ldots, e_{n-1}$ be the n-1 edges of E_1.
It is visible that
$\mathcal{U}(v_1) < \mathcal{U}(v_2) < \mathcal{U}(v_3) < \cdots < \mathcal{U}(v_{n-1}) < \mathcal{U}(v_n)$
Then
$[\mathcal{U}(v_1)]^3 < [\mathcal{U}(v_2)]^3 < [\mathcal{U}(v_3)]^3 < \ldots\ldots < [\mathcal{U}(v_{n-1})]^3 < [\mathcal{U}(v_n)]^3$
$\Rightarrow |[\mathcal{U}(v_1)]^3 - [\mathcal{U}(v_2)]^3| < |[\mathcal{U}(v_2)]^3 - [\mathcal{U}(v_3)]^3| < \cdots < |[\mathcal{U}(v_{n-1})]^3 - [\mathcal{U}(v_n)]^3|$
Hence
$\mathcal{U}^*(v_1v_2) < \mathcal{U}^*(v_2v_3) < \cdots < \mathcal{U}^*(v_{n-1}v_n)$
$\mathcal{U}^*(e_1) < \mathcal{U}^*(e_2) < \cdots < \mathcal{U}^*(e_{n-1})$
Thus $\mathcal{U}^*$ is injective in E_1.
Claim 2. $\mathcal{U}^*$ is in injective in E_2.
Let us consider any two edges
$e_1 = v_1v_4, e_2 = v_2v_5$ where $e_1, e_2 \in E_2$
It is visible that
$\mathcal{U}(v_1) < \mathcal{U}(v_2) < \mathcal{U}(v_4) < \mathcal{U}(v_5)$

$\Rightarrow [\mathcal{U}(v_1)]^3 < [\mathcal{U}(v_2)]^3 < [\mathcal{U}(v_4)]^3 < [\mathcal{U}(v_5)]^3$
Hence
$|[\mathcal{U}(v_1)]^3 - [\mathcal{U}(v_4)]^3| < |[\mathcal{U}(v_2)]^3 - [\mathcal{U}(v_5)]^3|$
$\mathcal{U}^*(v_1v_4) < \mathcal{U}^*(v_2v_5)$
$\mathcal{U}^*(e_1) < \mathcal{U}^*(e_2)$
$\mathcal{U}^*(e_1) \neq \mathcal{U}^*(e_2)$
Thus $\mathcal{U}^*$ is injective in E_2.
Hence all the edge labelings in E_2 are distinct.
Claim 3. $\mathcal{U}^*$ is in injective in E_1 and E_2.
Let us consider any two edges
$e_1 = v_1v_2, e_2 = v_3v_6$ where $e_1 \in E_1$ and $e_2 \in E_2$
It is visible that
$\mathcal{U}(v_1) < \mathcal{U}(v_2) < \mathcal{U}(v_3) < \mathcal{U}(v_6)$
$\Rightarrow [\mathcal{U}(v_1)]^3 < [\mathcal{U}(v_2)]^3 < [\mathcal{U}(v_3)]^3 < [\mathcal{U}(v_6)]^3$
Hence
$|[\mathcal{U}(v_1)]^3 - [\mathcal{U}(v_2)]^3| < |[\mathcal{U}(v_3)]^3 - [\mathcal{U}(v_6)]^3|$
$\mathcal{U}^*(v_1v_2) < \mathcal{U}^*(v_3v_6)$
$\mathcal{U}^*(e_1) < \mathcal{U}^*(e_2)$
$\mathcal{U}^*(e_1) \neq \mathcal{U}^*(e_2)$
Thus $\mathcal{U}^*$ is injective in E_1 and E_2
Hence all the edge labelings in E_1 and E_2 are distinct.
Hence all the edge labels in E are distinct. Thus $\mathcal{U}^*$ is injective in E.

So the cube of a path graph $P_n^3, n \geq 4$ admits cube difference labeling .

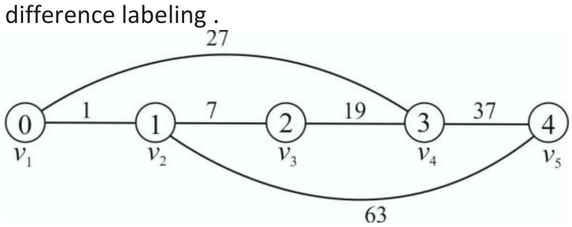

Figure. 3. square difference labeling of cube of a path graph P_5^3

Theorem-3.3. The graph obtained by duplicating arbitrary vertex of Sierpinski gasket graph $S_n, n = 2$ is a Prime graph.
Proof. Let G denote the Sierpinski gasket graph $S_n, n = 2$
Case-1. Duplication of the vertex v_1
Let G_1 be the graph obtained by duplicating the vertex v_1
Define $f: V(G_1) \to \{1,2, \ldots, 7\}$ by
$f(v_1') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \leq i \leq 6$.
Evidently all the vertex labels are distinct
For edges in G_1
GCD ($f(v_i), f(v_{i+1})) = 1, 1 \leq i \leq 5$
GCD ($f(v_1), f(v_3)) = 1$,
GCD ($f(v_2), f(v_5)) = 1$
GCD ($f(v_2), f(v_6)) = 1$,
GCD ($f(v_3), f(v_5)) = 1$
GCD ($f(v_1'), f(v_2)) = 1$,
GCD ($f(v_1'), f(v_3)) = 1$
Thus f is a Prime labeling on G_1.Hence G_1 is a prime graph.

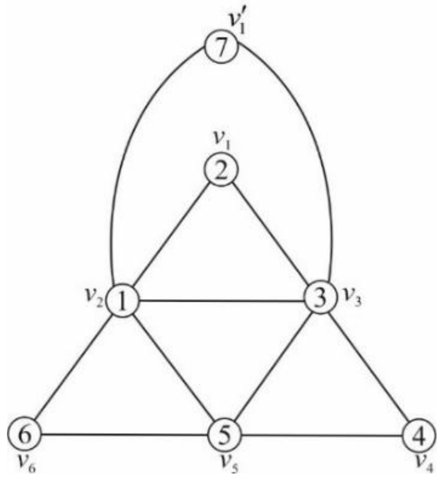

Figure - 4. Prime labeling of duplication of vertex v_1 in Sierpinski gasket graph S_2

Case-2. Duplication of the vertex v_2
Let G_2 be the graph obtained by duplicating the vertex v_2
Define $f: V(G_1) \to \{1,2, \ldots, 7\}$ by
$f(v_2') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \leq i \leq 6$.
Evidently all the vertex labels are distinct

For edges in G_2,
GCD ($f(v_i)$, $f(v_{i+1})) = 1, 1 \leq i \leq 5$
GCD ($f(v_1)$, $f(v_3)) = 1$,
GCD ($f(v_2)$, $f(v_5)) = 1$
GCD ($f(v_2)$, $f(v_6)) = 1$,
GCD ($f(v_3)$, $f(v_5)) = 1$
GCD ($f(v_2')$, $f(v_1)) = 1$,
GCD ($f(v_2')$, $f(v_6)) = 1$
GCD ($f(v_2')$, $f(v_3)) = 1$,
GCD ($f(v_2')$, $f(v_5)) = 1$
Thus f is a Prime labeling on G_2.Hence G_2 is a prime graph.

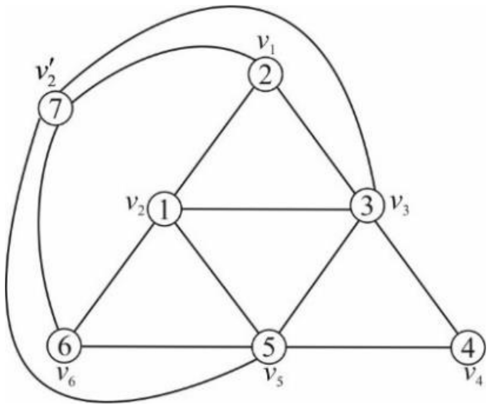

Figure - 5. Prime labeling of duplication of vertex v_2 in Sierpinski gasket graph S_2

Case-3. Duplication of the vertex v_3
Let G_3 be the graph obtained by duplicating the vertex v_3
Define $f: V(G_3) \rightarrow \{1,2, \dots ,7\}$ by
$f(v_3') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \leq i \leq 6$.
Evidently all the vertex labels are distinct
For edges in G_3
GCD ($f(v_i)$, $f(v_{i+1})) = 1, 1 \leq i \leq 5$
GCD ($f(v_1)$, $f(v_3)) = 1$,
GCD ($f(v_2)$, $f(v_5)) = 1$
GCD ($f(v_2)$, $f(v_6)) = 1$,
GCD ($f(v_3)$, $f(v_5)) = 1$
GCD ($f(v_3')$, $f(v_1)) = 1$,
GCD ($f(v_3')$, $f(v_2)) = 1$
GCD ($f(v_3')$, $f(v_4)) = 1$,
GCD ($f(v_3')$, $f(v_5)) = 1$
Thus f is a Prime labeling on G_3. Hence G_3 is a prime graph.

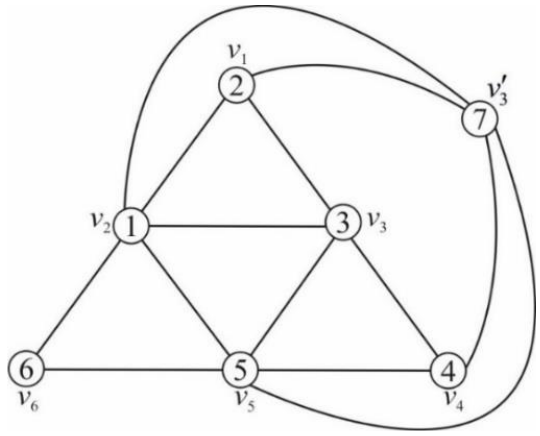

Figure - 6. Prime labeling of duplication of vertex v_3 in Sierpinski gasket graph S_2

Case-4. Duplication of the vertex v_4
Let G_4 be the graph obtained by duplicating the vertex v_4
Define $f: V(G_4) \rightarrow \{1,2, \dots ,7\}$ by
$f(v_4') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \leq i \leq 6$.
Evidently all the vertex labels are distinct
For edges in G_4
GCD ($f(v_i)$, $f(v_{i+1})) = 1, 1 \leq i \leq 5$
GCD ($f(v_1)$, $f(v_3)) = 1$,
GCD ($f(v_2)$, $f(v_5)) = 1$
GCD ($f(v_2)$, $f(v_6)) = 1$,
GCD ($f(v_3)$, $f(v_5)) = 1$
GCD ($f(v_4')$, $f(v_3)) = 1$,
GCD ($f(v_4')$, $f(v_5)) = 1$
Thus f is a Prime labeling on G_4.
Hence G_4 is a prime graph.

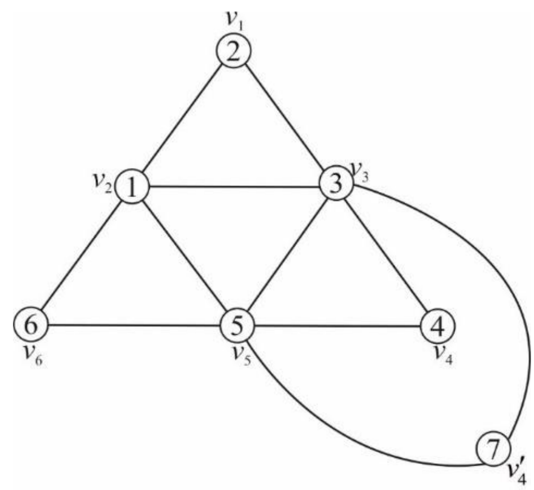

Figure – 7. Prime labeling of duplication of vertex v_4 in Sierpinski gasket graph S_2

Case-5. Duplication of the vertex v_5
Let G_5 be the graph obtained by duplicating the vertex v_5
Define $f: V(G_5) \rightarrow \{1,2, \dots ,7\}$ by
$f(v_5') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \leq i \leq 6$.

Evidently all the vertex labels are distinct For edges in G_5
GCD ($f(v_i), f(v_{i+1})) = 1, 1 \le i \le 5$
GCD ($f(v_1), f(v_3)) = 1$,
GCD ($f(v_2), f(v_5)) = 1$
GCD ($f(v_2), f(v_6)) = 1$,
GCD ($f(v_3), f(v_5)) = 1$
GCD ($f(v_5'), f(v_2)) = 1$,
GCD ($f(v_5'), f(v_3)) = 1$
GCD ($f(v_5'), f(v_4)) = 1$,
GCD ($f(v_5'), f(v_6)) = 1$
Thus f is a Prime labeling on G_5. Hence G_5 is a prime graph

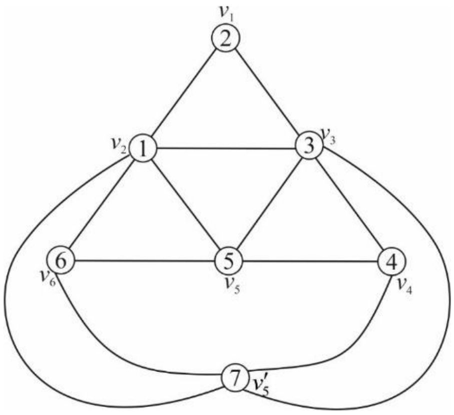

Figure – 8. Prime labeling of duplication of vertex v_5 in Sierpinski gasket graph S_2

Case-6**.** Duplication of the vertex v_6
Let G_6 be the graph obtained by duplicating the vertex v_3
Define $f: V(G_6) \to \{1,2, \dots ,7\}$ by
$f(v_6') = 7, f(v_1) = 2, f(v_2) = 1$ and $f(v_i) = i, 3 \le i \le 6$.
Evidently all the vertex labels are distinct
For edges in G_6
GCD ($f(v_i), f(v_{i+1})) = 1, 1 \le i \le 5$
GCD ($f(v_1), f(v_3)) = 1$,
GCD ($f(v_2), f(v_5)) = 1$
GCD ($f(v_2), f(v_6)) = 1$,
GCD ($f(v_3), f(v_5)) = 1$
GCD ($f(v_6'), f(v_2)) = 1$,
GCD ($f(v_6'), f(v_5)) = 1$
Thus f is a Prime labeling on G_6. Hence G_6 is a prime graph

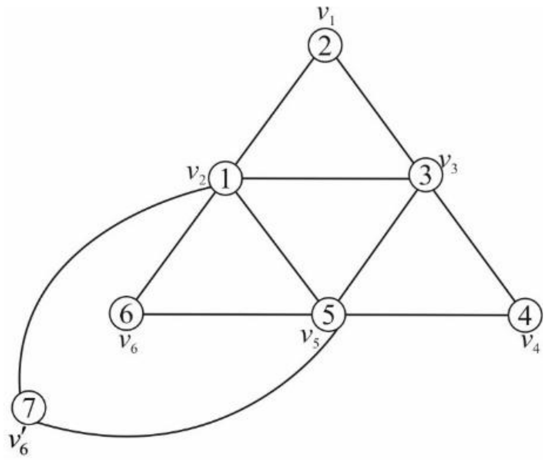

Figure - 9. Prime labeling of duplication of vertex v_6 in Sierpinski gasket graph S_2

Thus, in all the cases the graph obtained by duplication of any arbitrary vertex of *Sierpinski gasket graph* S_n , $n = 2$ is a Prime graph.

4. Conclusion

It is highly inspiring to study graphs which admit square difference labeling , cube difference labelling and Prime labeling. To study comparable results for different types of graphs is an open area of research.

References

[1] Ajitha,V., Princy, K.L,. Lokesha, V., and Ranjini, P.S.; On Square Difference Graphs, International J.Math. Combin. Vol.1(2012), 31-40.
[2] Amuda,G., Meena,S.; Cube Difference Labeling Of Some Cycle related Graphs, IJISET International Journal of Innovative Science, Engineering & Technology, Vol.2 Issue 1, ISSN 2348 – 7968 ,January (2015) 461-471.
[3] Blinco, A., Zanati, El, S.I., and Vanden Eynden, C.; On the cyclic decomposition of complete graphs into almost- bipartite graphs, Discrete Mathematics, 284(2004), 71-81.
[4] Burton,D.M.; Elementary number theory, Second Edition, Wm. C. Brown Company publishers, (1980)

[5]Gallian,J.;Adynamicsurveyofgraphlabeling.Electron.J.Combin.https://www.combinatorics.org/ojs/index.php/eljc/article/viewFile/DS6/pdf (2019)
[6] Shiama, J.; Square Difference Labeling for Some Graphs, International Journal of Computer Applications (0975 – 8887) Volume 44– No.4, April 2012.
[7] Sreenivasan, R., and Paulraj, M.S.; Vertex Antimagic Edge Labeling of Cube of A Path Graph, International Journal of Innovative Technology and Exploring Engineering (IJITEE)

ISSN: 2278-3075 (Online), Volume-9 Issue-2, December 2019.

[8] Teguia A.M and Godbole A.P, Sierpin'ski gasket graphs and some of their properties, arXiv:math/0509259vl, 12 September 2005, Australasian Journal of Combinatorics, 35, 181-192, 2006

[9] A.Tout A.N.Dabboucy and K.Howalla *Prime labeling of graphs*. Nat.Acad.Sci letters 11 (1982)365-368.

[10] Vaidya, S. K and Kanmani, K.K (2010), Prime Labeling for some Cycle Related Graphs, Jornals of Mathematics Research Vol.2. No.2., 98-104.