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Q-Laplace Transform of Logarithimic Functions

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Abstract: In this paper, we obtain the extorial and logarithmic function using the q-Laplace transform, suitable examples are inserted to illustrate the main results.
Key words: q-Laplace transform, Difference operator, Inverse difference operator, logarithmic function, Extorial functions, q-difference operator.

1. Introduction

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provided an easy and effective means for the solution of many problems arising in engineering. This subject originated from the operational methods applied by the English engineer Oliver-Heaviside (1850-1925) unsystematic and lacked rigour, which was placed on sound mathematical footing by Brownwich and Carson during 1916-17. It was found that Heavisides operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.

The method of Laplace transform has the advantage of directly giving the solution of differential equations with given the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants.

2. Preliminaries

This section focuses on the basic definition of the q-difference operator and hyperbolic function.

Definition 2.1. If $u(k)$ is a sequence of numbers and q is any positive integer, then we define the generalized difference operator $\Delta q\ n\ u(k)$ as

$$\Delta q\ u(k) = u(kq) - u(k) \quad (1)$$

Also, if $\Delta q\ v(k) = u(k)$ then $v(k) = \Delta_q^{-1} u(k)$.

Definition 2.2. Let $q > 0$ and $u(k), v(k)$ are real valued bounded functions. Then

$$\Delta_q^{-1}(u(k) - v(k)) = u(k)\Delta_q^{-1}v(k) - \Delta_q^{-1}\big(\Delta_q^{-1}v(kq).\Delta q\ u(k)\big) \quad (2)$$

Definition 2.3. If $\lim_{m\to\infty} \Delta_q^{-1} f(kq^m) = 0$, then

$$\sum_{r=0}^{\infty} f(kq^r) = -\Delta_q^{-1} f(k) \quad (3)$$

Definition 2.4. Let $q > 0$ and $kq^r \neq 0$, then

$$\Delta_q^{-1} e^{-sk}\big|_0^\infty = -\sum_{r=0}^{\infty} e^{-skq^r} \quad (4)$$

Definition 2.5. For a given function $f(k)$, the generalized q-Laplace transform is defined as

$$L_q[f(k)] = (q-1)\Delta_q^{-1} f(k) k e^{-sk}\big|_0^\infty \quad (5)$$

3. q-Laplace Transform of Logarithmic Function

In this section, we define q-Laplace transform of logarithmic function and results using the operator Δ_q^{-1}.

Definition 3.1. Let $k \in [0,\infty)$ and $q \neq 0$ and 1 then

$$\log\big(1 + k_q^{(1)}\big) = k_q^{(1)} - \frac{k_q^{(2)}}{2} + \frac{k_q^{(3)}}{3} - \frac{k_q^{(4)}}{4} + \cdots \quad (6)$$

Definition 3.2. Let $k \in [0,\infty)$ and $q \neq 0$ and 1 then

$$\log\left(1+(ak)_q^{(1)}\right) = ak_q^{(1)} - \frac{a^2k_q^{(2)}}{2} + \frac{a^3k_q^{(3)}}{3} - \frac{a^4k_q^{(4)}}{4} + \cdots \quad (7)$$

Definition 3.3. Let $k \in [0,\infty)$ and $q \neq 0$ and 1 and $k_q^{(n)}$ be a n^{th} power of polynomial factorial, then

$$\log\left(1+k_q^{(n)}\right) = k_q^{(n)} - \frac{k_q^{(2n)}}{2} + \frac{k_q^{(3n)}}{3} - \frac{k_q^{(4n)}}{4} + \cdots \quad (8)$$

Definition 3.4. Let $k \in [0,\infty)$ and $\neq 0$ and 1 and $k_q^{(n)}$ be a n^{th} power of polynomial factorial, then

$$\log\left(1+(ak)_q^{(n)}\right) = (ak)_q^{(n)} - \frac{(ak)_q^{(2n)}}{2} + \frac{(ak)_q^{(3n)}}{3} - \frac{(ak)_q^{(4n)}}{4} + \cdots \quad (9)$$

Lemma 3.5. *If* $e^{-skq^r} \neq 0$ *and* $\neq 0$ and 1 *then the q-Laplace transform of logarithmathmic function is*

$$Lq\left(\log\left(1+k_q^{(1)}\right)\right) = (1-q)\left\{k\log\left(1+k_q^{(1)}\right)\sum_{r=0}^{\infty} e^{-skq^r} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{t+1}\,log(1+kq^{t+1})_q^{(1)} - kq^t\,log(1+kq^t)_q^{(1)}\right)\right)\right\} \quad (10)$$

Proof. Let $u(k) = \log\left(1+k_q^{(1)}\right)$ in equation (4) we obtain,

$$Lq\ \log\left(1+k_q^{(1)}\right) = q-1\Delta_q^{-1}k\ \log\left(1+k_q^{(1)}\right)\ e^{-sk}\Big|_0^{\infty}$$

$$= (q-1)\Delta_q^{-1}\left(k_q^{(1)} - \frac{k_q^{(2)}}{2} + \frac{k_q^{(3)}}{3} - \frac{k_q^{(4)}}{4} + \cdots\right)e^{-sk}\Bigg|_0^{\infty} \quad (11)$$

Separate the terms and using the equations (3) and (4) we get,

$$\Delta_q^{-1}k\,k_q^{(1)}\,e^{-sk}\big|_0^{\infty} = \left[k\,k_q^{(1)}\Delta_q^{-1}\,e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1}e^{-skq}\Delta_q k\,k_q^{(1)}\right)\right]\Big|_0^{\infty}$$

$$= k\,k_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left[-\sum_{r=0}^{\infty} e^{-skq^{r+1}}\left(kq^tq(kq^tq)_q^{(1)} - kq^t(kq^t)_q^{(0)}\right)\right]$$

$$= -k\,k_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left[-\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{(t+1)}\left(kq^{(t+1)}\right)_q^{(1)} - kq^t(kq^t)_q^{(0)}\right)\right] \quad (12)$$

Also,

$$\Delta_q^{-1}\frac{k\,k_q^{(2)}}{2}e^{-sk}\Bigg|_0^{\infty} = \frac{1}{2}\left[k\,k_q^{(2)}\Delta_q^{-1}e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1}e^{-skq}\Delta_q k\,k_q^{(2)}\right)\right]\Big|_0^{\infty}$$

we get the equation,

$$\Delta_q^{-1}\frac{k\,k_q^{(2)}}{2}e^{-sk}\Bigg|_0^{\infty} = \frac{1}{2}\left[-k\,k_q^{(2)}\sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(2)} - kq^t(kq^t)_q^{(2)}\right)\right)\right] \quad (13)$$

Replace $\frac{k_q^{(2)}}{2}$ by $\frac{k_q^{(3)}}{3}$ in equation (13) we get,

$$\Delta_q^{-1}\frac{k\,k_q^{(3)}}{3}e^{-sk}\Bigg|_0^{\infty} = \frac{1}{3}\left[-k\,k_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)} - kq^t(kq^t)_q^{(3)}\right)\right)\right] \quad (14)$$

Applying the process mentioned above, we get

$$\Delta_q^{-1}\frac{k\,k_q^{(4)}}{4}e^{-sk}\Big|_0^{\infty}=\frac{1}{4}\Bigg[-k\,k_q^{(4)}\sum_{r=0}^{\infty}e^{-skq^r}-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(4)}-kq^t(kq^t)_q^{(4)}\right)\right)\Bigg] \quad (15)$$

Substituting equation (12) to (15) in (11) we get,

$$Lq\left(\log\left(1+k_q^{(1)}\right)\right)=(q-1)\Bigg\{\Bigg[-k\,k_q^{(1)}\sum_{r=0}^{\infty}e^{-skq^r}-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(1)}-kq^t(kq^t)_q^{(1)}\right)\right)\Bigg]$$
$$-\frac{1}{2}\Bigg[-k\,k_q^{(2)}\sum_{r=0}^{\infty}e^{-skq^r}-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(2)}-kq^t(kq^t)_q^{(2)}\right)\right)\Bigg]$$
$$+\frac{1}{3}\Bigg[-k\,k_q^{(3)}\sum_{r=0}^{\infty}e^{-skq^r}-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)}-kq^t(kq^t)_q^{(3)}\right)\right)\Bigg]$$
$$-\frac{1}{4}\Bigg[-k\,k_q^{(4)}\sum_{r=0}^{\infty}e^{-skq^r}-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(4)}-kq^t(kq^t)_q^{(4)}\right)\right)\Bigg]+\cdots\Bigg\}$$
$$=(1-q)\Bigg\{\left(kk_q^{(1)}-\frac{kk_q^{(2)}}{2}+\frac{kk_q^{(3)}}{3}-\frac{kk_q^{(4)}}{4}+\cdots\right)\sum_{r=0}^{\infty}e^{-skq^r}$$
$$+\sum_{t=0}^{\infty}\Bigg[\sum_{r=0}^{\infty}e^{-skq^{(t+r+1)}}\,kq^{t+1}\left[\frac{(kq^{t+1})_q^{(1)}}{1!}-\frac{(kq^{t+1})_q^{(2)}}{2}+\frac{(kq^{t+1})_q^{(3)}}{3}-\frac{(kq^{t+1})_q^{(4)}}{4}+\cdots\right]-kq^t\left[\frac{(kq^t)_q^{(1)}}{1}-\frac{(kq^t)_q^{(2)}}{2}+\frac{(kq^t)_q^{(3)}}{3}-\frac{(kq^{t+1})_q^{(4)}}{4}+\cdots\right]\Bigg]\Bigg\}$$

which yields the proof.

Theorem: 3.6 If $e^{-skq^{t+r+1}}\neq 0$ *and* $q\neq 0$ *and 1 then the Laplace q-transform of logarithmic function is*

$$L_{\boldsymbol{q}}\left(\log\left(1+ak_q^{(1)}\right)\right)=(1-q)\Bigg\{k\log\left(1+ak_q^{(1)}\right)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\left(1+ak_q^{(1)}\right)-kq^t\log\left(1+ak_q^{(1)}\right)\right)\Bigg\} \quad (16)$$

Proof. Replacing $\log\left(1+k_q^{(1)}\right)=\log\left(1+k_q^{(1)}\right)$ *in the previous Lemma 3.5, we get the proof.*

Theorem: 3.7 *If* $e^{-skq^r}\neq 0$ *and* $q\neq 0$ and 1 *then the Laplace q-transform of* n^{th} *power of logarithmic function is*

$$L_{\boldsymbol{q}}\big(\log\big(1+k_q^{(1)}\big)\big)=(1-q)\left\{k\log\big(1+k_q^{(1)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+k_q^{(n)}\big)-kq^t\log\big(1+k_q^{(n)}\big)\right)\right\}\qquad(17)$$

Proof. Replace $\log\big(1+k_q^{(1)}\big)$ *by* $\log\big(1+k_q^{(2)}\big)$ *in equation (10), and using (8) and (12) we arrive.*

$$L_{\boldsymbol{q}}\big(\log\big(1+k_q^{(2)}\big)\big)=(1-q)\left\{k\log\big(1+k_q^{(2)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+k_q^{(2)}\big)-kq^t\log\big(1+k_q^{(2)}\big)\right)\right\}\qquad(18)$$

Replace $\log\big(1+k_q^{(1)}\big)$ *by* $\log\big(1+k_q^{(3)}\big)$ *in equation (10), and using (8) and (12) we arrive*

$$L_{\boldsymbol{q}}\big(\log\big(1+k_q^{(3)}\big)\big)=(1-q)\left\{k\log\big(1+k_q^{(3)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+k_q^{(3)}\big)-kq^t\log\big(1+k_q^{(3)}\big)\right)\right\}\qquad(19)$$

Replace $\log\big(1+k_q^{(1)}\big)$ *by* $\log\big(1+k_q^{(4)}\big)$ *in equation (10), and using (8) and (12), we arrive.*

$$L_{\boldsymbol{q}}\big(\log\big(1+k_q^{(4)}\big)\big)=(1-q)\left\{k\log\big(1+k_q^{(4)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+k_q^{(4)}\big)-kq^t\log\big(1+k_q^{(4)}\big)\right)\right\}\qquad(20)$$

By repeating the process n times, we get

$$L_{\boldsymbol{q}}\big(\log\big(1+k_q^{(n)}\big)\big)=(1-q)\left\{k\log\big(1+k_q^{(n)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+k_q^{(n)}\big)-kq^t\log\big(1+k_q^{(n)}\big)\right)\right\}$$

Corollary: 3.8 If $e^{-skq^{t+r+1}}\neq 0$ *and* $q\neq 0$ and 1 *then the Laplace q-transform of sine function is*

$$L_{\boldsymbol{q}}\big(\log\big(1+ak_q^{(n)}\big)\big)=(1-q)\left\{k\log\big(1+ak_q^{(n)}\big)\sum_{r=0}^{\infty}e^{-skq^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,kq^{t+1}\log\big(1+ak_q^{(n)}\big)-kq^t\log\big(1+ak_q^{(n)}\big)\right)\right\}\qquad(21)$$

Proof. The proof follows from taking $\log\big(1+k_q^{(n)}\big)=\log\big(1+ak_q^{(n)}\big)$ *in equation (18), we get the proof of the corollary.*

Example: 3.9. Let $n=5, q=\frac{1}{3}$ and $a=4$ in equation (18), we have,

$$L_{\boldsymbol{q}}\left(\log\left(1+4k_{1/3}^{(5)}\right)\right)=(1-q)\left\{k\log\left(1+4k_{1/3}^{(5)}\right)\sum_{r=0}^{\infty}e^{-skq\left(\frac{1}{3}\right)^r}+\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty}e^{-skq^{t+r+1}}\,k\left(\frac{1}{3}\right)^{t+1}\log\left(1+4k_{1/3}^{(5)}\right)-k\frac{1}{3}^{t}\log\left(1+4k_{1/3}^{(5)}\right)\right)\right\}$$

Conclusion:

In this paper we have developed the discrete q-Laplace transform for logarithmic function. The given example shows the values of q-Laplace transform for logarithmic function.

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