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The Steiner Boundary Distance in Graphs

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Abstract:
For any two vertices u and v in a connected graph G, the distance d(u,v) between u and v is the length of a shortest path joining them. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. A vertex v is an eccentric vertex of u if e(u) =d(u,v). It is clear that d(u,w) ≤ d(u,v) for all $w \in N(v)$) if v is an eccentric vertex of u. However, a vertex may have this property without being an eccentric vertex. A vertex v is a boundary vertex of a vertex u if d(u,w) ≤ d(u,v) for all $w \in N(v)$. For a set S ⊆V(G), the Steiner distance d(S) is defined to be the minimum size of a connected subgraph containing S. We define B(u) as the set of all boundary vertices of u, and we call it as the boundary vertex set of u. The closed boundary vertex set of u is B[u]=B(u) U {u}. we define the Steiner–boundary distance dsb(u,v) between u and v is the minimum size of a connected subgraph containing B[u] U B[v]. The highlights of this paper are

1. Initiates a new distance parameter called Steiner-boundary distance.
2. It follows the Nassi-Schneiderman style proof to prove the main theorem.

Keywords: Boundary Vertex, Boundary distance, Steiner boundary distance.

1.Introduction

The graphs considered here are nontrivial and simple connected graphs with the vertex set V and the edge set E. For other graph theoretical notation and terminology, We follow Buckley and Harary[1]. In a graph G the distance d(u,v) between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. The radius r(G) of G is defined by r(G)= min{e(u): $u \epsilon V(G)$} and the diameter d(G) of G is defined by d(G)= max{e(u): $u \epsilon V(G)$}. The center C(G) of graph G is the set of vertices of minimum eccentricity, and such vertices are called central vertices. The periphery P(G) is the set of vertices of maximum eccentricity, and those vertices are called peripheral. A vertex v is called an eccentric vertex of a vertex u if d(u,v)=e(u). A vertex v of G is called an eccentric vertex of G if it is an eccentric vertex of some vertex of G. Two vertices u and v are called antipodal vertices of G if d(u,v)=d(G). Chartrand et al. [2] introduced the concept of boundary vertices in graphs. If v is an eccentric vertex of a vertex u and w is a neighbor of v, then d(u,w) ≤ d(u,v) for all $w \in N(v)$. A vertex v may have this property, however without being an eccentric vertex of u. A vertex v is a boundary vertex of u if d(u,w) ≤ d(u,v) for all . We define B(u) as the set of all boundary vertices of u and we call it as the boundary vertex set of u. The closed boundary vertex set of u is B[u] = B(u) U{u}. A vertex $u \epsilon V(G)$ is complete if < N(v) > is complete. Two vertices u and v are called mutually boundary vertices if they are boundary vertices to each other.

Parthasarathy and Nandakumar[6] introduced the notion of unique eccentric point graph. If each vertex of a graph G has exactly one eccentric vertex, then G is called a unique eccentric point graph. A simple class of unique eccentric point graphs is the paths P2n on an even number of vertices. The notion of even graphs was studied by [5].

A nontrivial connected graph G is called even if for each vertex v of G there is a unique vertex v, the buddy of v, such that d(u,v) = diam G. Even graphs are also referred as for diametrical graphs and self-centered unique eccentric point graphs.

Marimuthu and Sivanandha Saraswathy [3] introduced the concept of Boundary graphs. A vertex v is a boundary vertex of a vertex u if d(u,w) ϵ d(u,v) for all $w\,\epsilon$N (v) . The boundary graph B(G) based on a connected graph G is a simple graph which has the vertex set as in G. Two vertices u and v are adjacent in B(G) if either u is a boundary of v , or v is a boundary of u. If G is disconnected, then each v vertex in a component is adjacent to all the vertices in the other components and is adjacent to all of its boundary vertices within the component. A graph G is called a boundary graph if there exists a graph H such that B(H)=G.

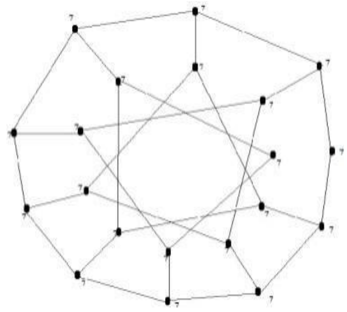
In 1989, Chartrand et al.[4] introduced a generalization of distance called Steiner distance. For a set $S \subseteq V,$ the Steiner distance d(S) among the vertices of S is the minimum size among all connected sub graphs whose vertex sets contain S. Based on this we introduce a new generalization of distance called Steiner boundary distance and we gave the characterization of unique boundary vertex graph using Steiner boundary distance. Let F_{11}, F_{12}, F_{22}, F_{23}, F_{24} and F_3 denote the set of all graphs G such that r(G)=1 and d(G)=1; r(G)=1 and d(G)=2; r(G)=2 and d(G)=2 ; r(G)=2 and d(G)=3; r(G)=2 and d(G)=4 and r(G) ≥ 3 respectively .

2.The steiner boundary distance in graphs

Definition 2.1: Let G = (V,E) be a simple connected graph. The Steiner boundary distance d_{sb}(u,v) between two vertices u and v is the minimum size of a connected subgraph containing B[u] U B[v].

Definition 2.2: A vertex v is a Steiner boundary eccentric vertex of u if d_{sb}(u,v) = e_{sb}(u).

Definition 2.3: For a simple connected graph G, the Steiner boundary radius r_{sb}(G) = min{ e_{sb}(v) :vϵ V(G)}and the Steiner boundary diameter d_{sb}(G) = max{ e_{sb}(v) :vϵ V(G)}. Next we give an example to illustrate the idea presented above.



Petersen graph P(10,3)

Fig.1

In the above graph r_{sb}(G) = d_{sb}(G) = 7. It is well known that the usual distance defined on a connected graph G is a metric on its vertex set. But it does not hold true in the case of Steiner boundary distance. Now, we give the properties of Steiner boundary distance.

Proposition 2.4 For any graph G, the Steiner boundary distance between any two vertices always remains positive, that is dsb(u,v) > 0 for any u, v ϵV(G).

Proof: Since every vertex has at least one boundary vertex, d_{sb}(u,v) > 0.

Proposition 2.5 For any vertex in a graph v ϵV(G), e_{sb}(v) ≥1.

Proof. The proof follows from the definition.

Proposition 2.6 d_{sb}(u,v) = d_{sb}(v,u) for all u, v ϵV(G).

Proposition 2.7 *d* (*u*, *v*) ≤ d_{sb} (*u*, *v*) for all u, v ϵV(G).

Proposition 2.8 d_{sb}(u,v) =d(u,v) if and only if u and v are mutually boundary vertices.

Proposition 2.9 d_{sb} (*u*, *w*) ≤d_{sb} (*u*, *v*) +d_{sb} (*v*, *w*) for all u, v, w ϵV(G).

Proof: Let S be a connected graph and $S \subseteq V(G), where\, S \neq \phi$ implies d(S)≥1.

Let S_1, S_2, S_3 be subsets of V(G) such that $\phi \neq S \subseteq S_1 \cup S_2$ and $S_1 \cap S_2 \neq \phi$.

Then d_{sb}(S) ≤ d_{sb}(S_1) + d_{sb}(S_2).

Proposition 2.10 In a graph G, d_{sb} (*G*) ≤ d_{sb} (*u*, *w*) +1 where *w* is a non-peripheral vertex.

Observation 2.11 For a connected graph G, and for an integer c such that r_{sb}(G) < c < d_{sb}(G) there does not exists a vertex v ofG such that e(v)= c.

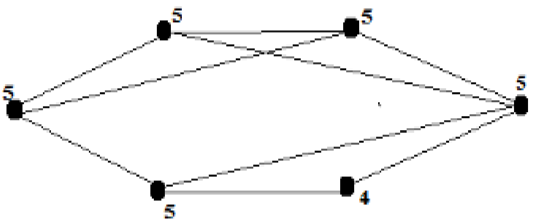


Fig.2

Theorem 2.12: For a connected graph, the Steiner boundary radius satisfies the inequality $r_{sb}(G) \leq d_{sb}(G) \leq r_{sb}(G) + 1$.

Proof: The first inequality follows from the definitions. For the second inequality, consider the vertices u and v of G such that $d_{sb}(u,v) = d_{sb}(G)$ and let $w \in V(G)$ where w is a non-peripheral vertex such that $e_{sb}(w) = r_{sb}(G)$.

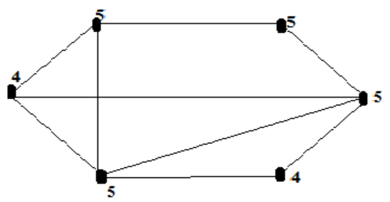
By the property $d_{sb}(G) \leq d_{sb}(u,w) + 1$.

Therefore $d_{sb}(G) \leq r_{sb}(G) + 1$.

Result 2.13: For any graph G, $Ecc(v_i) \subseteq S_i$ where $v_i \in V(G)$ and $Ecc(v_i)$ denote the set of all eccentric vertices of v_i.

Proof: Since each eccentric vertex is a boundary vertex $Ecc(v_i)$ are all boundary vertices of v_i. Hence the result follows.

Observation 2.14: A self – centered graph need not be a Sb self- centered. The following graph illustrates the above observation.



The graph G

Observation 2.15: Let G be a connected graph. If each Steiner boundary (Sb) set has the eccentric vertices alone the boundary vertices then G is Sb-self-centered.

From the above observation it is clear that every regular graph is sb self-centered. Since each vertex in a regular graph has eccentric vertices only their boundary vertices.

Lemma 2.16: Every regular graph is self-centered.

Proof: Let G be a regular graph. This implies G has a spanning cycle. Since each cycle is self-centered, G is self-centered.

Theorem2.17: Let G be a regular graph, then G is both self-centered and Sb self-centered.

Proof: The proof follows from observation 2.15 and Lemma 2.16.

Theorem 2.18: Every graph is the Sb-periphery of some graph.

Proof: Let G be a graph. We show that G is the sb-periphery of some graph. First, add two new vertices u and v to G and join them to every vertices of G but not to each other. Next, we add two other vertices u_1 and v_1 where we join u_1 to u and v_1 to v. The resulting graph is F. Since $e_{sb}(u_1) = e_{sb}(v_1) = 4$, $e_{sb}(u) = e_{sb}(v) = 4$ and $e(x) = 4+$ minimum size of a connected subgraph containing B[x] in G, it follows that V(G) is the set of sb-peripheral vertices of F and so sb -peri(F) = G.

Definition 2.19: Let T be any double star. There are two vertices u and v such that each pendant vertex is adjacent with either u or v. For a given double star T, the T_n, $n \geq 0$, by subdividing uv 'n' times. It is obvious that T_0 is the given double star.

Proposition 2.20: In T_n $d_{sb}(u,v) = q(T_n)$.

Proof: Since each pendant vertex in T_n is a boundary vertex to all other vertices, the Steiner boundary distance between u and v is the minimum size of the connected graph contain all the pendant vertices together with u and v. This includes all the edges in T_n. The result is true for n = 0.

Lemma 2.21: If $G \in F_{11} \cup F_{12}$, then G is a Sb-self -centered graph of Sb-radius n-1.

Proof: If $G \in F_{11}$, then $B[u] \cup B[v] = V(G)$ for any two vertices u and v of G. This shows that $d_{sb}(u,v)$ =n, where n is the order of G. Suppose that $G \in F_{12}$. The boundary vertices of a vertex u of degree n-1 are V – {u}. This implies that $e_{sb}(u) = n-1$. Hence the lemma.

Lemma 2.22: Let $N(u)$ -{v} ⊆ $N(v)$ -{u} nor $N(v)$ -{u} ⊆$N(u)$ -{v} then G is Sb – self – centered graph.

Proof: With the assumption of the condition mentioned for any two adjacent vertices, each $S_i = V - N(v_i)$ and therefore $|S_i|$ ⍰ n ⍰1 where n denotes the order of the graph. Also for all S_i there exists at least one S_j, where $j \neq i$ such that $|S_i \cup S_j| = n$ implies

$d(S_i) = n-1$ and this holds true for all v_i.

Hence G is Sb – self – centered graph.

Lemma 2.23: If G is a complete bipartite graph $K_{m,n}$

then G is a sb self-centered graph of sb radius m+n-1.

Proof: The boundary vertices of a vertex v is the set B[v] = V-N(v). Let $u \in N(v)$. Then B[u] = V-N(u).

Now d_{sb}(u,v) = d_{sb}(B[u] U B[v]) = d_{sb}(V-N(u)) U (V-N(v)) = d_{sb}(V) = m+n-1.

Next we consider the (n; p) graph corresponding to the Steiner boundary distance.

Definition 2.24 Let n and p be integers with 2 ≤ *n* ≤ *p.* A graph G of order p is called an (n; p) graph if it is of minimum size with the property that d(S) = n-1 for all sets S of vertices of G with |S| = n.

Theorem 2.25: Every complete graph is a (n; p) graph.

Proof: Since every vertex in a complete graph is of degree n-1, |S| = n. This holds for all S. Hence the result.

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