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Solution of Fractional Order Differential-Difference Equation by Using Laplace Transform Method.

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Abstract: In this paper, we have proved solution of a non-homogeneous fractional differential-difference equation of a general fractional order α and difference order 1 with general initial conditions with the help of Laplace Transform where the fractional derivative we are using is in accordance with the definition of fractional derivative as Caputo fractional derivative. A few examples are illustrated that support the results..The theorem is proved on the solution of fractional order Differential difference equation. The existence and uniqueness of solution is given by existence of laplace transform.

Keywords: Caputo fractional derivative, difference Equation, Laplace transform, differential-difference equatioMSC2020-Mathematics Subject Classification:34B15:

1.Inroduction:

Fractional differential equations are very useful for modelling

Kamble, et al.[14] proved existence and uniqueness of solutions for the following equation

$$D^{\alpha} D^{\beta} x(\tau) = f\big(t, x(\tau), \phi x(\tau), \psi x(\tau)\big), \tau\epsilon[0,1], x(0) = x(1) = 0$$

where 0< α ≤ 1, 0 < β ≤ 1, D^{α}, D^{β} are the Caputo fractional derivatives of order α ,β, f : [0, 1] × R^3 ⟶$\mathbb{R}$ is a continuous function, and $\phi x(\tau) = \int_0^{\tau} \lambda(\tau, s) x(s) ds$, $\psi x(\tau) = \int_0^{\tau} \delta(\tau, s) x(s) ds$, λ, δ:[0,1]×[0, 1]→[0,+∞) with $\phi^* = Sup_{t\in[0,1]} \left|\int_0^t \lambda(\tau, s) ds\right| < \infty$ with $\psi^* = Sup_{t\in[0,1]} \left|\int_0^t \delta(\tau, s) ds\right| < \infty$.

In [15], Kamble, et al. have proved the existence and uniqueness of solutions to the initial vlaue problem $D^{\alpha} D^{\beta} D^{\gamma} x(\tau) = f\big(t, x(\tau), \phi x(\tau), \psi x(\tau)\big), \tau\epsilon[0,1]$ $x(0) = x(1) = D^{\beta} D^{\gamma} x(0) = D^{\beta} D^{\gamma} x(1) = D^{\gamma} x(0) = D^{\gamma} x(1) = 0$, where $1 < \alpha \le 2, 1 < \beta \le 2$ and $1 \le \gamma \le 2$, $D^{\alpha}, D^{\beta}, D^{\gamma}$ represent the Caputo fractional derivatives of order α ,β and γ respectively, f : $[0, 1] \times R^3 \longrightarrow R$ is a continuous function, and

$$\phi x(\tau) = \int_0^{\tau} \lambda(\tau, s) x(s) ds,$$

$$\psi x(\tau) = \int_0^{\tau} \delta(\tau, s) x(s) ds$$

$$\phi^* = Sup_{\tau\in[0,1]} \left|\int_0^t \lambda(\tau, s) ds\right| < \infty, \psi^* = Sup_{\tau\in[0,1]} \left|\int_0^t \delta(\tau, s) ds\right| < \infty$$

$\lambda, \delta: [0,1] \times [0, 1] \to [0, +\infty)$.

2. Differential-Difference Equation

An equation which contains the derivatives of an unknown function and some of its derivatives at arguments which differ by a fixed number of values is called a differential-difference equation. The differential order of a differential-difference equation is the order of the highest derivative and the difference order is one less than the number if distinct arguments appearing in the differential-difference equation.

Sugiyama S.[16] has considered the differential-difference equation

$$x'(t) = f[t, x(t), x(t-1)], t \in [0, t_0]$$

with the initial conditions $x(t-1) = \psi(t), 0 \le t < 1$, $x(0) = x_0$ and has proved the existence of a continuous solution $x(t)$ valid for $0 \le t \le \min(t_0, {}^{K}/_{M})$, where it is assumed that

$f(t,x,y)$ is bounded by $M>0$ and continuous in the region $|x-x_0|\le K, |y-y_0|\le K$. However, the continuity of the function f does not guarantee the uniqueness of the solution $x(t)$.

3 Main Results:

As discussed in the earlier sections, many authors have obtained the solutions of fractional differential equations with different types of initial and boundary conditions but none have obtained the solution of a fractional differential equation combined with difference equation. In this section we will solve a general non-homogeneous fractional differential-difference equation of fractional order α and difference order 1 with given initial conditions.

3.1 Theorem : *Let φ be a function which is piecewise continuous and of exponential order. Then the fractional differential-difference equation*

$$\varphi^{\alpha}(\tau)-\varphi(\tau-w)=\psi(\tau) \quad ...(1)$$

of fractional differential order $\alpha, n-1<\alpha\le n, n\in\mathbb{N}$ and difference order 1, w is any positive real number with initial conditions of the form $\varphi^{(k)}(0)=\varphi_k,\ k=0\,1,2,\dots,n-1,\ \varphi(\tau)=0\ for\ \tau<0$ has a unique solution, given by

$$\varphi(\tau)=\int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\psi(\tau-\eta)\ d\eta+\int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{\alpha k-1}}{\Gamma(\alpha k)}\sum_{k=0}^{n-1}\frac{\varphi_k(\tau-\eta)^k}{k!}\ d\eta$$

where $\varphi^{\alpha}(\tau)$ represents the Caputo fractional derivative of order α and $\varphi^{(k)}(0)$ represents the ordinary derivative of integer order k at $\tau=0$ of the function φ.

Proof: We define the Laplace transforms of the functions φ and ψ and use the following notations

$$\mathcal{L}[\varphi(\tau)]=\int_0^{\infty}e^{-s\tau}\varphi(\tau)d\tau=\Phi(s)$$

$$\mathcal{L}[\psi(\tau)]=\int_0^{\infty}e^{-s\tau}\psi(\tau)d\tau=\Psi(s)$$

Taking the Laplace transform of the equation (1) and using the above notations, we have

$$s^{\alpha}\Phi(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}\varphi^{(k)}(0)-e^{-ws}\Phi(s)=\Psi(s)$$

Using the initial conditions $\varphi^{(k)}(0)=\varphi_k,\ k=1,2,\dots,n-1, \varphi(\tau)=0$ for $\tau<0$, and solving for $\Phi(s)$, we get

$$\Phi(s)=\frac{\Psi(s)}{s^{\alpha}-e^{-ws}}+\sum_{k=0}^{n-1}\frac{\varphi_k\,s^{\alpha-k-1}}{s^{\alpha}-e^{-ws}} \quad ...(2)$$

Consider

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}-e^{-ws}}\right]=\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\frac{1}{1-\frac{e^{-ws}}{s^{\alpha}}}\right]$$

$$=\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\left(1-\frac{e^{-ws}}{s^{\alpha}}\right)^{-1}\right]$$

$$=\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}+\frac{e^{-ws}}{s^{2\alpha}}+\frac{e^{-2ws}}{s^{3\alpha}}+\frac{e^{-3ws}}{s^{4\alpha}}+\cdots\right]$$

$$=\mathcal{L}^{-1}\left[\sum_{k=0}^{\infty}\frac{e^{-kws}}{s^{(k+1)\alpha}}\right]$$

$$=\begin{cases}\sum_{k=0}^{\infty}\frac{(\tau-wk)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, & \text{for } \tau\ge \text{wk}\\ 0, & \text{otherwise}\end{cases}$$

$$=\sum_{k=0}^{[\tau]}\frac{(\tau-wk)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}=f(\tau)$$

where $[\tau]$ is the greatest integer not greater than τ.

By convolution theorem for Laplace transform

$$\mathcal{L}^{-1}\left[\frac{\Psi(s)}{s^{\alpha}-e^{-ws}}\right] = \int_0^{\tau} f(\eta)\psi(\tau-\eta)\,d\eta$$

$$= \int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\psi(\tau-\eta)\;d\eta \qquad ...(3)$$

Also, we have

$$\mathcal{L}^{-1}\left[\sum_{k=0}^{n-1}\frac{\varphi_k\, s^{\alpha-k-1}}{s^{\alpha}-e^{-ws}}\right] = \mathcal{L}^{-1}\left[\sum_{k=0}^{n-1}\frac{\varphi_k}{s^{k+1}}\,\frac{1}{1-\frac{e^{-ws}}{s^{\alpha}}}\right]$$

$$= \mathcal{L}^{-1}\left[\sum_{k=0}^{n-1}\frac{\varphi_k}{s^{k+1}}\left(1-\frac{e^{-ws}}{s^{\alpha}}\right)^{-1}\right] \quad ... \quad (4)$$

Also, we note that

$$\mathcal{L}^{-1}\left[\sum_{k=0}^{n-1}\frac{\varphi_k}{s^{k+1}}\right] = \sum_{k=0}^{n-1}\frac{\varphi_k\tau^k}{k!} = \nu(\tau)$$

and

$$\mathcal{L}^{-1}\left[\left(1-\frac{e^{-ws}}{s^{\alpha}}\right)^{-1}\right] = \mathcal{L}^{-1}\left[\sum_{k=0}^{\infty}\frac{e^{-swk}}{s^{\alpha k}}\right]$$

$$= \begin{cases}\sum_{k=0}^{\infty}\frac{(\tau-wk)^{\alpha k-1}}{\Gamma(\alpha k)}, & \text{for } \tau \geq wk\\ 0, & \text{otherwise}\end{cases}$$

$$= \sum_{k=0}^{[\tau]}\frac{(\tau-wk)^{\alpha k-1}}{\Gamma(\alpha k)} = \mu(\tau)$$

From equation (4), we have

$$\mathcal{L}^{-1}\left[\sum_{k=0}^{n-1}\frac{\varphi_k\, s^{\alpha-k-1}}{s^{\alpha}-e^{-s}}\right] = \int_0^{\tau}\mu(\eta)\,\nu(\tau-\eta)\,d\eta$$

$$= \int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{\alpha k-1}}{\Gamma(\alpha k)}\sum_{k=0}^{n-1}\frac{\varphi_k(\tau-\eta)^k}{k!}\;d\eta \qquad ...(5)$$

Taking the inverse Laplace transform of the equation (2), using equations (3) and (5) and using the convolution theorem, the solution of the equation (1) is given by

$$\varphi(\tau) = \int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\psi(\tau-\eta)\;d\eta + \int_0^{\tau}\sum_{k=0}^{[\eta]}\frac{(\eta-wk)^{\alpha k-1}}{\Gamma(\alpha k)}\sum_{k=0}^{n-1}\frac{\varphi_k(\tau-\eta)^k}{k!}\;d\eta$$

This completes the proof of the theorem.

Applications of the method *: Let φ be a function which is piecewise continuous and of exponential order. Then the fractional differential-difference equation*

$$\varphi^{\alpha}(\tau)-\varphi(\tau-w)=\psi(\tau)\,, \qquad \psi(\tau)=t^n\,,\quad n>0 \qquad ...(1)$$

of fractional differential order α, $n-1<\alpha\leq n$ $,n\in\mathbb{N}, w$ is any pasitive real number and difference order 1 with initial conditions of the form

$\varphi^{(k)}(0)=0,\; k=,1,2,\dots,n-1,\; \varphi(\tau)=0\; for\; \tau\leq 0$

has a unique solution, given by

$$\varphi(\tau) = \Gamma(n+1)\sum_{k=0}^{[\tau]}\frac{(\tau-wk)^{(k+1)\alpha+n}}{\Gamma(\alpha(k+1)+n+1)}\;\; where\; [\tau]$$

$$denotes\; gretest\; integer\; function.$$

Proof:

$$L[\varphi^{\alpha}(\tau)-\varphi(\tau-w)] = L[\tau^n]$$

$s^{\alpha}\Phi(s)-e^{-ws}\Phi(s)=\frac{\Gamma(n+1)}{s^{n+1}}$

$$\Phi(s)=\frac{\Gamma(n+1)}{s^{n+1}(s^{\alpha}-e^{-ws})}$$

$\Phi(s)=\frac{\Gamma(n+1)}{s^{\alpha+n+1}}(1-\frac{e^{-ws}}{s^{\alpha}})^{-1}$

$\Phi(s)=\frac{\Gamma(n+1)}{s^{\alpha+n+1}}(1+\frac{e^{-ws}}{s^{\alpha}}+\frac{e^{-2ws}}{s^{2\alpha}}+\frac{e^{-3ws}}{s^{3\alpha}}+\cdots$ ....

$\Phi(s)=\Gamma(n+1)\,(\frac{1}{s^{\alpha+n+1}}+\frac{e^{-ws}}{s^{2\alpha+n+1}}+\frac{e^{-2ws}}{s^{3\alpha+n+1}}+\frac{e^{-3ws}}{s^{4\alpha+n+1}}+\cdots$ ....

$\Phi(s)=\Gamma(n+1)\,\sum_{k=0}^{\infty}\frac{e^{-swk}}{s^{\alpha+n+1+\alpha k}}$

Taking inverse laplace transform

$$\varphi(\tau)=\Gamma(n+1)L^{-1}\left[\sum_{k=0}^{\infty}\frac{e^{-swk}}{s^{(k+1)\alpha+n+1}}\right]$$

$$\varphi(\tau)=\begin{Bmatrix}\Gamma(n+1)\sum_{k=0}^{\infty}\frac{(\tau-wk)^{(k+1)\alpha+n}}{\Gamma((k+1)\alpha+n+1)}, & t\geq wk\\ 0 & ,otherwise\end{Bmatrix}$$

Thus if $[\tau]$ denotes the gretest integer less than or equal to τ, we find

$\varphi(\tau)=\Gamma(n+1)\sum_{k=0}^{[\tau]}\frac{(\tau-wk)^{(k+1)\alpha+n}}{\Gamma(\alpha(k+1)+n+1)}$.

4.Some Applications:

In this section, we present some examples as the applications of the above result.

Example 4.1) $\varphi^{1/2}(\tau)-\varphi(\tau-1)=\tau^2$, $\varphi(\tau)=0$ $,for\ \ \tau<0$ $\alpha=\frac{1}{2}$, $0<\frac{1}{2}<1$,

$\varphi(\tau)=2\sum_{k=0}^{[\tau]}\frac{(\tau-k)^{(k+1)\frac{1}{2}+2}}{\Gamma((k+1)\frac{1}{2}+3}$

$$\varphi(2)=2\sum_{k=0}^{2}\frac{(2-k)^{(k+1)\frac{1}{2}+2}}{\Gamma((k+1)\frac{1}{2}+3}$$

$$\varphi(2)=2\left\{\frac{2^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}+\frac{(1)^3}{\Gamma(4)}\right\}$$

$$\varphi(3)=2\sum_{k=0}^{3}\frac{(3-k)^{(k+1)\frac{1}{2}+2}}{\Gamma\left((k+1)\frac{1}{2}+3\right)}$$

$=\varphi(3)=2\left\{\frac{3^{\frac{5}{2}}}{\Gamma(\frac{7}{2})}+\frac{(2)^3}{\Gamma(4)}+\frac{(1)^{\frac{7}{2}}}{\Gamma(\frac{9}{2})}\right\}$ is the solution.

Example4 .2) $\varphi^{\frac{3}{2}}(\tau)-\varphi(\tau-1)=\tau^2$, $\varphi(\tau)=0, for\ \tau<0$ $\alpha=\frac{3}{2}$, $1<\frac{3}{2}<2$

$$\varphi(\tau)=2\sum_{k=0}^{[\tau]}\frac{(\tau-k)^{(k+1)\frac{3}{2}+2}}{\Gamma\left(\frac{3}{2}(k+1)+3\right)}$$

If we take $\tau=\frac{5}{2}$ $[\frac{5}{2}]=2$

$\varphi(\frac{5}{2})=2\sum_{k=0}^{[\tau]}\frac{(\frac{5}{2}-k)^{(k+1)\frac{3}{2}+2}}{\Gamma\left(\frac{3}{2}(k+1)+3\right)}$

$$=2\sum_{k=0}^{2}\frac{(\frac{5}{2}-k)^{(k+1)\frac{3}{2}+2}}{\Gamma\left(\frac{3}{2}(k+1)+3\right)}$$

$$\varphi(\tfrac{5}{2})=2\left\{\frac{(\frac{5}{2})^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}+\frac{(\frac{3}{2})^{5}}{\Gamma(6)}+\frac{(\frac{1}{2})^{\frac{9}{2}}}{\Gamma\left(\frac{14}{2}\right)}\right\}$$

Example 4.3) Special case when we take $\alpha=1$ the equation is converted to differential-difference equation.

$\varphi'(\tau)-\varphi(\tau-1)=\tau^n$, $,\varphi(\tau)=0$, $for\ \tau\leq 0$ and the solution is given by

$$\varphi(\tau)=\Gamma(n+1)\sum_{k=0}^{[\tau]}\frac{(\tau-k)^{(k+1)\alpha+n}}{\Gamma(\alpha(k+1)+n+1)}$$

putting $\alpha=1$

$$\varphi(\tau)=\Gamma(n+1)\sum_{k=0}^{[\tau]}\frac{(\tau-k)^{(k+1)+n}}{\Gamma((k+1)+n+1)}$$

so our proved result is valid for differential-difference equations for this type of problem.[17]

Example 4.3.1): $\varphi'(\tau)-\varphi(\tau-1)=\tau^2$, $,\varphi(\tau)=0, for\ \tau\leq 0$, $n=2$,

$$\varphi(\tau)=2\sum_{k=0}^{[\tau]}\frac{(\tau-k)^{k+3}}{\Gamma(k+4)}$$

If t=4 , [4]=4 wehave

$\varphi(4)=2\sum_{k=0}^{4}\frac{(4-k)^{k+3}}{\Gamma(k+4)}$

$2\left\{\frac{4^3}{\Gamma(4)}+\frac{3^4}{\Gamma(5)}+\frac{2^5}{\Gamma(6)}+\frac{1^6}{\Gamma(7)}\right\}$=28.62($aproximatly$)

If $\tau=\pi$, $[\pi]=3$

$\varphi(\pi)=2\sum_{k=0}^{4}\frac{(\pi-k)^{k+3}}{\Gamma(k+4)}$

$=2\left\{\frac{\pi^3}{\Gamma(4)}+\frac{(\pi-1)^4}{\Gamma(5)}+\frac{(\pi-2)^5}{\Gamma(6)}+\frac{(\pi-3)^6}{\Gamma(7)}\right\}=12.12($ $aproximatly$)

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