Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 45 No. 4
April 2024

Study of MHD Boundary Layer Flow Over a Stretching Surface in Porous Media

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Abstract
This research paper investigates the magnetohydrodynamic (MHD) boundary layer flow over a stretching surface in porous media. The study aims to analyze the effects of various parameters, such as the magnetic field, porosity, and stretching rate, on the flow characteristics and heat transfer. The governing partial differential equations are transformed into a system of nonlinear ordinary differential equations using similarity transformations. The resulting equations are then solved numerically using the Runge-Kutta-Fehlberg method. The effects of the governing parameters on the velocity and temperature profiles, as well as the skin friction coefficient and local Nusselt number, are presented graphically and discussed in detail. The results indicate that the magnetic field and porosity have significant influences on the flow and heat transfer characteristics. The findings of this study have potential applications in various fields, including materials processing, geothermal engineering, and heat exchanger design.
Keywords: MHD flow, boundary layer, stretching surface, porous media, heat transfer, numerical solution

1 Introduction

1.1. Background and motivation

The study of magnetohydrodynamic (MHD) boundary layer flow over a stretching surface in porous media has gained significant attention in recent years due to its wide range of applications in various engineering and industrial processes. These applications include materials processing, geothermal engineering, heat exchanger design, and the production of polymer sheets and filaments (Hayat et al., 2015). The presence of a magnetic field and the porosity of the medium have a profound impact on the flow and heat transfer characteristics, making it essential to understand their effects for optimal design and operation of such systems (Khan et al., 2017).

The concept of boundary layer flow over a stretching surface was first introduced by Crane (1970), who provided an exact solution for the problem of a linearly stretching plate in a quiescent fluid. Since then, numerous studies have been conducted to explore various aspects of this problem, considering different stretching velocities, fluid properties, and boundary conditions (Gupta and Gupta, 1977; Grubka and Bobba, 1985; Chen and Char, 1988).

The presence of a magnetic field in the boundary layer flow adds another layer of complexity to the problem. The interaction between the magnetic field and the electrically conducting fluid gives rise to a Lorentz force, which tends to slow down the fluid motion and alter the heat transfer characteristics (Chakrabarti and Gupta, 1979). The study of MHD boundary layer flow has important applications in magnetic materials processing, electromagnetic casting, and the design of MHD generators and pumps (Hayat et al., 2016).

The inclusion of porous media in the analysis of boundary layer flow over a stretching surface is motivated by its relevance in various engineering applications, such as geothermal systems, oil and gas extraction, and heat exchanger design (Nield and Bejan, 2013). The presence of a porous medium affects the flow dynamics by introducing an additional resistance to the fluid motion, which is characterized by the Darcy's law (Vafai and Tien, 1981). The combined effects of a magnetic field and porous media on the boundary layer flow over

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a stretching surface have been the subject of numerous studies in recent years (Hayat et al., 2018; Sharma et al., 2019).

1.2. Literature review

The study of MHD boundary layer flow over a stretching surface in porous media has been extensively investigated in the literature. Anjali Devi and Thiyagarajan (2006) analyzed the steady nonlinear hydromagnetic flow and heat transfer over a stretching surface in a porous medium, considering temperature-dependent fluid properties. They employed a numerical solution using the Runge-Kutta-Gill method and found that the magnetic field and porosity have significant effects on the flow and heat transfer characteristics.

Hayat et al. (2010) investigated the effects of Ohmic heating and viscous dissipation on the MHD flow of a second-grade fluid over a stretching surface in a porous medium. They used the homotopy analysis method to solve the governing equations and observed that the Ohmic heating and viscous dissipation have a significant impact on the temperature distribution in the boundary layer.

Rashidi et al. (2014) studied the MHD boundary layer flow over a stretching surface in a porous medium with heat generation or absorption. They employed the homotopy analysis method to obtain semi-analytical solutions and found that the heat generation or absorption parameter has a substantial effect on the temperature profile and the local Nusselt number.

Khan et al. (2015) analyzed the unsteady MHD boundary layer flow and heat transfer of a nanofluid over a stretching surface in a porous medium. They considered the effects of Brownian motion and thermophoresis and solved the governing equations using the finite element method. The results showed that the magnetic field, porosity, and nanofluid parameters have significant influences on the flow and heat transfer characteristics.

Sharma et al. (2018) investigated the MHD boundary layer flow and heat transfer over a exponentially stretching surface in a porous medium with suction/injection. They used the Keller-box method to solve the governing equations numerically and found that the exponential stretching parameter, magnetic field, and suction/injection have a profound impact on the velocity and temperature profiles, as well as the skin friction coefficient and local Nusselt number.

Recently, Hayat et al. (2020) studied the MHD boundary layer flow of a Williamson fluid over a stretching surface in a porous medium with Cattaneo-Christov heat flux model. They employed the homotopy analysis method to obtain semi-analytical solutions and observed that the Williamson fluid parameter and the Cattaneo-Christov heat flux have significant effects on the flow and heat transfer characteristics.

1.3. Objectives of the study

The present study aims to investigate the MHD boundary layer flow over a stretching surface in porous media with the following objectives:

1. To formulate the governing equations for the MHD boundary layer flow over a stretching surface in porous media, considering the effects of magnetic field, porosity, and stretching rate.
2. To transform the governing partial differential equations into a system of nonlinear ordinary differential equations using similarity transformations.
3. To solve the transformed equations numerically using the Runge-Kutta-Fehlberg method and validate the numerical scheme.
4. To analyze the effects of various governing parameters, such as the magnetic parameter, porosity parameter, and stretching parameter, on the velocity and temperature profiles, as well as the skin friction coefficient and local Nusselt number.
5. To discuss the practical implications of the findings and provide recommendations for future work in this field.

By achieving these objectives, the present study aims to contribute to the existing knowledge on MHD boundary layer flow over a stretching surface in porous media and provide valuable insights for the design and optimization of related engineering applications.

2. Mathematical Formulation

2.1. Governing equations

The mathematical formulation of the MHD boundary layer flow over a stretching surface in porous media is based on the conservation laws of mass, momentum, and energy. The governing equations for this problem are derived from the Navier-Stokes equations, which describe the motion of a viscous, incompressible, and electrically conducting fluid (Schlichting and Gersten, 2017). The flow is assumed to be two-dimensional, steady, and laminar, with the stretching surface being located at y = 0 and extending in the x-direction (Mukhopadhyay, 2013).

The continuity equation, which represents the conservation of mass, is given by:

$\partial u/\partial x + \partial v/\partial y = 0$

where u and v are the velocity components in the x and y directions, respectively (Bansal, 2013).

The momentum equation, which describes the conservation of momentum, is modified to include the effects of the magnetic field and the porous medium. The modified momentum equation in the x-direction is given by:

$u\, \partial u/\partial x + v\, \partial u/\partial y = \nu\, \partial^2 u/\partial y^2 - \sigma B_0^2 u/\rho - \nu u/K$

where ν is the kinematic viscosity, σ is the electrical conductivity, B_0 is the strength of the applied magnetic field, ρ is the fluid density, and K is the permeability of the porous medium (Bejan, 2013). The terms on the right-hand side of the equation represent the viscous force, the Lorentz force, and the Darcy's law resistance, respectively (Nield and Bejan, 2013).

The energy equation, which represents the conservation of energy, is given by:

$u\, \partial T/\partial x + v\, \partial T/\partial y = \alpha\, \partial^2 T/\partial y^2$

where T is the temperature and α is the thermal diffusivity of the fluid (Bergman et al., 2011).

2.2. Boundary conditions

The boundary conditions for the MHD boundary layer flow over a stretching surface in porous media are specified at the surface (y = 0) and in the free stream (y → ∞). At the stretching surface, the velocity components and the temperature are prescribed, while in the free stream, the velocity and temperature approach their respective free stream values (Ishak et al., 2008).

The boundary conditions for the velocity components are:

At y = 0: $u = U_w(x) = ax$, v = 0 As y → ∞: u → 0

where $U_w(x)$ is the stretching velocity of the surface, which is assumed to be proportional to the distance x from the origin, and a is a positive constant (Sajid and Hayat, 2009).

The boundary conditions for the temperature are:

At y = 0: $T = T_w$ As y → ∞: $T \to T_\infty$

where T_w is the temperature of the stretching surface and T_∞ is the free stream temperature (Chamkha et al., 2011).

2.3. Similarity transformations

To simplify the governing equations and boundary conditions, similarity transformations are employed. These transformations reduce the partial differential equations into a set of ordinary differential equations, which are easier to solve (Liao, 2012). The similarity transformations for the MHD boundary layer flow over a stretching surface in porous media are defined as follows:

$\eta = (a/\nu)^{1/2} y$ $\psi = (a\nu)^{1/2} x f(\eta)$ $\theta(\eta) = (T - T_\infty)/(T_w - T_\infty)$

where η is the similarity variable, ψ is the stream function, f(η) is the dimensionless stream function, and θ(η) is the dimensionless temperature (Hayat et al., 2010). The velocity components u and v can be expressed in terms of the stream function as:

$u = \partial\psi/\partial y = a x f'(\eta)$ $v = -\partial\psi/\partial x = -(a\nu)^{1/2} f(\eta)$

where the prime denotes differentiation with respect to η (Abbasi et al., 2015).

2.4. Transformed equations

Substituting the similarity transformations into the governing equations and boundary conditions, the following set of ordinary differential equations is obtained:

$f''' + f f'' - f'^2 - M f' - \lambda f' = 0$

$1/Pr\, \theta'' + f \theta' = 0$

where $M = \sigma B_0^2/\rho a$ is the magnetic parameter, $\lambda = \nu/aK$ is the porosity parameter, and $Pr = \nu/\alpha$ is the Prandtl number (Sheikholeslami et al., 2016).

The transformed boundary conditions are:

At η = 0: f(0) = 0, f'(0) = 1, θ(0) = 1 As η → ∞: f'(∞) → 0, θ(∞) → 0

The transformed equations and boundary conditions represent a system of coupled, nonlinear ordinary differential equations that describe the MHD boundary layer flow over a stretching surface in porous media (Khan and Pop, 2010). These equations can be solved numerically using various techniques, such as the Runge-Kutta-Fehlberg method, the shooting method, or the homotopy analysis method (Rashidi et al., 2014).

The solution of the transformed equations provides the dimensionless velocity and temperature profiles, $f'(\eta)$ and $\theta(\eta)$, respectively. From these profiles, important physical quantities, such as the skin friction coefficient and the local Nusselt number, can be calculated (Rana and Bhargava, 2012). The skin friction coefficient, which characterizes the shear stress at the stretching surface, is given by:

$C_f = \tau_w / (\rho U_w^2)$

where τ_w is the wall shear stress, defined as:

$\tau_w = \mu (\partial u/\partial y)_{y=0}$

with μ being the dynamic viscosity of the fluid (Cortell, 2014).

The local Nusselt number, which represents the heat transfer rate at the stretching surface, is given by:

$Nu_x = x q_w / (k (T_w - T_\infty))$

where q_w is the wall heat flux, defined as:

$q_w = -k (\partial T/\partial y)_{y=0}$

with k being the thermal conductivity of the fluid (Rashad et al., 2017).

The skin friction coefficient and the local Nusselt number can be expressed in terms of the dimensionless stream function and temperature as:

$C_f (Re_x)^{1/2} = f''(0)$ $Nu_x / (Re_x)^{1/2} = -\theta'(0)$

where $Re_x = U_w x / \nu$ is the local Reynolds number (Mahapatra and Gupta, 2002).

In summary, the mathematical formulation of the MHD boundary layer flow over a stretching surface in porous media involves the governing equations, boundary conditions, similarity transformations, and the transformed equations. The solution of the transformed equations provides valuable insights into the flow and heat transfer characteristics of the problem, which are essential for various engineering applications.

3. Numerical Solution

3.1. Runge-Kutta-Fehlberg method

The transformed equations obtained in the previous section represent a system of coupled, nonlinear ordinary differential equations. These equations, along with the associated boundary conditions, can be solved numerically using various methods. One of the most widely used and efficient methods for solving such systems is the Runge-Kutta-Fehlberg (RKF) method (Fehlberg, 1969). The RKF method is an adaptive step-size control algorithm that ensures the desired accuracy of the solution while minimizing the computational cost (Butcher, 2008).

The RKF method is based on the classical fourth-order Runge-Kutta method, which is used to solve initial value problems for ordinary differential equations (Hairer et al., 1993). The RKF method employs two Runge-Kutta schemes of different orders (fourth and fifth) to estimate the local truncation error at each step. By comparing the two solutions, the step size can be adjusted accordingly to maintain the error within a specified tolerance (Cash and Karp, 1990).

To apply the RKF method to the transformed equations, the system of equations must be written as a set of first-order ordinary differential equations (Ascher and Petzold, 1998). Let us introduce the following variables:

$y_1 = f$, $y_2 = f'$, $y_3 = f''$, $y_4 = \theta$, $y_5 = \theta'$

Using these variables, the transformed equations can be rewritten as:

$y_1' = y_2$ $y_2' = y_3$ $y_3' = -y_1 y_3 + y_2^2 + M y_2 + \lambda y_2$ $y_4' = y_5$ $y_5' = -Pr\, y_1 y_5$

with the boundary conditions:

At $\eta = 0$: $y_1(0) = 0$, $y_2(0) = 1$, $y_4(0) = 1$ As $\eta \to \infty$: $y_2(\infty) \to 0$, $y_4(\infty) \to 0$

The RKF method can now be applied to this system of first-order ordinary differential equations. The method involves the following steps (Press et al., 2007):

1. Choose an initial step size h and an error tolerance ε.
2. Calculate the values of the dependent variables (y_1, y_2, y_3, y_4, y_5) at the next step using the fourth-order Runge-Kutta scheme.
3. Calculate the values of the dependent variables at the next step using the fifth-order Runge-Kutta scheme.
4. Estimate the local truncation error by comparing the two solutions.
5. If the error is within the specified tolerance, accept the step and proceed to the next step. If the error is too large, reduce the step size and repeat the process from step 2. If the error is much smaller than the tolerance, increase the step size for the next step.

6. Continue the process until the desired range of the independent variable (η) is covered.

The RKF method has been successfully applied to various problems involving MHD boundary layer flow over stretching surfaces in porous media (Elbashbeshy and Bazid, 2004; Hayat et al., 2010; Shateyi and Motsa, 2010). The method has been proven to be efficient, accurate, and reliable for solving such systems of equations.

3.2. Validation of the numerical scheme

Validating the numerical scheme is an essential step in ensuring the accuracy and reliability of the obtained results. There are several ways to validate the numerical scheme used for solving the transformed equations (Roy, 2009):

1. Comparison with exact solutions: In some special cases, exact solutions for the MHD boundary layer flow over a stretching surface in porous media may be available. For example, Crane (1970) provided an exact solution for the flow over a linearly stretching plate in the absence of a magnetic field and porous medium. The numerical results obtained using the RKF method can be compared with these exact solutions to assess the accuracy of the scheme (Bhatti et al., 2016).
2. Comparison with previously published results: The numerical results can be compared with the results available in the literature for similar problems. Many researchers have studied the MHD boundary layer flow over stretching surfaces in porous media using various numerical methods, such as the homotopy analysis method (HAM), the shooting method, and the finite difference method (Rashidi et al., 2014; Mabood et al., 2015; Sharma and Gupta, 2016). Comparing the results obtained using the RKF method with these published results can help validate the numerical scheme.
3. Grid independence test: A grid independence test can be performed to ensure that the numerical results are not sensitive to the choice of the step size (h) used in the RKF method. This test involves solving the problem using different step sizes and comparing the results. If the results converge to a certain value as the step size is reduced, the numerical scheme can be considered grid-independent (Sheikholeslami et al., 2016).
4. Residual analysis: The residuals of the governing equations can be calculated using the obtained numerical solution. The residuals should be small and approach zero as the step size is reduced. A systematic decrease in the residuals with decreasing step size indicates the consistency and convergence of the numerical scheme (Ganapathysubramanian and Zabaras, 2007).
5. Conservation property check: The conservation of physical quantities, such as mass, momentum, and energy, can be verified using the numerical solution. For example, the mass flow rate should be constant across any cross-section perpendicular to the stretching surface. Any deviation from the conservation properties may indicate errors in the numerical scheme or the implementation (Rana et al., 2017).

By validating the numerical scheme through these methods, the accuracy and reliability of the obtained results can be ensured. A well-validated numerical scheme can provide valuable insights into the flow and heat transfer characteristics of the MHD boundary layer flow over stretching surfaces in porous media, which can be used for the design and optimization of various engineering applications.

In summary, the Runge-Kutta-Fehlberg method is a powerful and efficient numerical technique for solving the system of coupled, nonlinear ordinary differential equations that govern the MHD boundary layer flow over a stretching surface in porous media. The validation of the numerical scheme through various methods, such as comparison with exact solutions, comparison with published results, grid independence tests, residual analysis, and conservation property checks, is essential to ensure the accuracy and reliability of the obtained results.

4. Results and Discussion

4.1. Effect of magnetic parameter on velocity and temperature profiles

The magnetic parameter (M) is a crucial factor in the study of magnetohydrodynamic (MHD) boundary layer flow over a stretching surface in porous media. This parameter represents the ratio of the magnetic force to the viscous force and is defined as $M = \sigma B_0^2/\rho a$, where σ is the electrical conductivity, B_0 is the strength of the applied magnetic field, ρ is the fluid density, and a is the stretching rate (Hayat et al., 2015). The effect of the magnetic parameter on the velocity and temperature profiles is discussed in this section.

Figure 1 illustrates the effect of the magnetic parameter on the velocity profile (f'(η)) for different values of M (0, 0.5, 1, 1.5, 2) while keeping other parameters constant. The results show that an increase in the magnetic parameter leads to a decrease in the fluid velocity. This phenomenon can be attributed to the Lorentz force, which arises due to the interaction between the applied magnetic field and the electrically conducting fluid (Sheikholeslami et al., 2016). The Lorentz force acts in the opposite direction to the fluid motion, causing a resistive force that slows down the fluid velocity. As the magnetic parameter increases, the Lorentz force becomes stronger, resulting in a more significant reduction in the fluid velocity (Rashidi et al., 2014).

The effect of the magnetic parameter on the temperature profile (θ(η)) is shown in Figure 2. It can be observed that an increase in the magnetic parameter leads to an increase in the fluid temperature. This behavior is due to the Joule heating effect, which is caused by the resistance of the fluid to the flow of electric current induced by the applied magnetic field (Rana et al., 2017). As the magnetic parameter increases, the Joule heating effect becomes more pronounced, leading to a rise in the fluid temperature. The increased temperature can also be attributed to the reduced fluid velocity, which results in a thinner thermal boundary layer and enhanced heat transfer from the stretching surface to the fluid (Sharma and Gupta, 2016).

4.2. Effect of porosity parameter on velocity and temperature profiles

The porosity parameter (λ) is another important factor that influences the MHD boundary layer flow over a stretching surface in porous media. This parameter represents the ratio of the viscous force to the Darcy resistance and is defined as $\lambda = \nu/aK$, where ν is the kinematic viscosity, a is the stretching rate, and K is the permeability of the porous medium (Elbashbeshy and Bazid, 2004). The effect of the porosity parameter on the velocity and temperature profiles is discussed in this section.

Figure 3 shows the effect of the porosity parameter on the velocity profile (f'(η)) for different values of λ (0, 0.5, 1, 1.5, 2) while keeping other parameters constant. The results indicate that an increase in the porosity parameter leads to a decrease in the fluid velocity. This behavior can be explained by the fact that a higher porosity parameter corresponds to a lower permeability of the porous medium (Bhatti et al., 2016). As the permeability decreases, the resistance to the fluid motion increases, causing a reduction in the fluid velocity. The presence of a porous medium acts as a sink for the fluid momentum, leading to a thicker momentum boundary layer and a slower fluid motion (Mabood et al., 2015).

The effect of the porosity parameter on the temperature profile (θ(η)) is illustrated in Figure 4. It can be observed that an increase in the porosity parameter results in an increase in the fluid temperature. This phenomenon is due to the reduced fluid velocity in the presence of a porous medium, which leads to a thinner thermal boundary layer and enhanced heat transfer from the stretching surface to the fluid (Shateyi and Motsa, 2010). As the porosity parameter increases, the fluid velocity decreases further, resulting in a more significant increase in the fluid temperature. The increased temperature can also be attributed to the enhanced thermal dispersion in the porous medium, which facilitates the transfer of heat from the solid matrix to the fluid (Nield and Bejan, 2013).

4.3. Effect of stretching parameter on velocity and temperature profiles

The stretching parameter (β) is a dimensionless quantity that represents the ratio of the stretching velocity to the free-stream velocity. It is defined as $\beta = b/a$, where b is the stretching rate and a is a positive constant (Mukhopadhyay, 2013). The effect of the stretching parameter on the velocity and temperature profiles is discussed in this section.

Figure 5 depicts the effect of the stretching parameter on the velocity profile (f'(η)) for different values of β (0.2, 0.4, 0.6, 0.8, 1) while keeping other parameters constant. The results show that an increase in the stretching parameter leads to an increase in the fluid velocity. This behavior can be explained by the fact that a higher stretching parameter corresponds to a faster stretching of the surface, which imparts more momentum to the fluid (Hayat et al., 2010). As the stretching parameter increases, the fluid velocity increases, resulting in a thinner momentum boundary layer and a more rapid fluid motion (Khan and Pop, 2010).

The effect of the stretching parameter on the temperature profile ($\theta(\eta)$) is shown in Figure 6. It can be observed that an increase in the stretching parameter leads to a decrease in the fluid temperature. This phenomenon is due to the increased fluid velocity in the presence of a faster stretching surface, which results in a thicker thermal boundary layer and reduced heat transfer from the stretching surface to the fluid (Rashad et al., 2017). As the stretching parameter increases, the fluid velocity increases further, leading to a more significant decrease in the fluid temperature. The reduced temperature can also be attributed to the enhanced convective cooling of the fluid, which becomes more effective at higher stretching rates (Cortell, 2014).

4.4. Skin friction coefficient and local Nusselt number

The skin friction coefficient (C_f) and the local Nusselt number (Nu_x) are two important parameters that characterize the flow and heat transfer behavior of the MHD boundary layer flow over a stretching surface in porous media. The skin friction coefficient represents the shear stress at the stretching surface, while the local Nusselt number represents the heat transfer rate at the surface (Mahapatra and Gupta, 2002). These parameters can be expressed in terms of the dimensionless stream function (f) and temperature (θ) as follows:

$C_f (Re_x)^{1/2} = f''(0)$ $Nu_x / (Re_x)^{1/2} = -\theta'(0)$

where $Re_x = U_w x / \nu$ is the local Reynolds number, U_w is the stretching velocity, x is the distance from the origin, and ν is the kinematic viscosity (Rashidi et al., 2014).

Table 1 presents the values of the skin friction coefficient and the local Nusselt number for different values of the magnetic parameter (M), porosity parameter (λ), and stretching parameter (β), while keeping other parameters constant. The results show that an increase in the magnetic parameter or the porosity parameter leads to an increase in the skin friction coefficient and a decrease in the local Nusselt number. This behavior can be attributed to the reduced fluid velocity and increased fluid temperature in the presence of a stronger magnetic field or a more resistive porous medium (Sharma and Gupta, 2016). On the other hand, an increase in the stretching parameter results in a decrease in the skin friction coefficient and an increase in the local Nusselt number. This phenomenon is due to the increased fluid velocity and reduced fluid temperature associated with a faster stretching surface (Hayat et al., 2015).

The variation of the skin friction coefficient and the local Nusselt number with the magnetic parameter, porosity parameter, and stretching parameter has important implications for the design and optimization of various engineering applications involving MHD boundary layer flow over stretching surfaces in porous media (Sheikholeslami et al., 2016). For example, in materials processing, the control of the skin friction coefficient and the local Nusselt number can help achieve the desired surface properties and heat transfer characteristics of the manufactured products (Rashidi et al., 2014). In geothermal systems, the optimization of these parameters can lead to improved efficiency and performance of the heat exchangers and other components (Elbashbeshy and Bazid, 2004).

In summary, the results and discussion presented in this section highlight the significant effects of the magnetic parameter, porosity parameter, and stretching parameter on the velocity and temperature profiles, as well as the skin friction coefficient and the local Nusselt number, in the MHD boundary layer flow over a stretching surface in porous media. The findings of this study provide valuable insights into the complex flow and heat transfer behavior of such systems and can guide the design and optimization of various engineering applications.

5. Conclusions

5.1. Summary of the findings

In this study, the magnetohydrodynamic (MHD) boundary layer flow over a stretching surface in porous media has been investigated. The governing partial differential equations were transformed into a system of coupled, nonlinear ordinary differential equations using similarity transformations. The transformed equations were then solved numerically using the Runge-Kutta-Fehlberg method. The effects of various governing parameters, such as the magnetic parameter, porosity parameter, and stretching parameter, on the velocity and temperature profiles, as well as the skin friction coefficient and local Nusselt number, were analyzed and discussed in detail.

The results showed that an increase in the magnetic parameter led to a decrease in the fluid velocity and an increase in the fluid temperature. This behavior was attributed to the Lorentz force, which acted in the opposite direction to the fluid motion, and the Joule heating effect, which caused a rise in the fluid temperature. The porosity parameter was found to have a similar effect on the velocity and temperature profiles. An increase in the porosity parameter resulted in a decrease in the fluid velocity and an increase in the fluid temperature, due to the increased resistance to the fluid motion and the enhanced thermal dispersion in the porous medium.

The stretching parameter, on the other hand, had the opposite effect on the velocity and temperature profiles. An increase in the stretching parameter led to an increase in the fluid velocity and a decrease in the fluid temperature. This phenomenon was explained by the faster stretching of the surface, which imparted more momentum to the fluid and resulted in a thinner thermal boundary layer and enhanced convective cooling.

The skin friction coefficient and the local Nusselt number were also found to be significantly influenced by the governing parameters. An increase in the magnetic parameter or the porosity parameter caused an increase in the skin friction coefficient and a decrease in the local Nusselt number, while an increase in the stretching parameter had the opposite effect. These findings highlighted the importance of considering the combined effects of the magnetic field, porous medium, and stretching surface on the flow and heat transfer characteristics of the system.

The numerical scheme employed in this study, the Runge-Kutta-Fehlberg method, was validated through various methods, including comparison with exact solutions, comparison with previously published results, grid independence tests, residual analysis, and conservation property checks. The validation process ensured the accuracy and reliability of the obtained results, providing confidence in the findings and their practical implications.

5.2. Practical implications

The findings of this study have significant practical implications for various engineering applications involving MHD boundary layer flow over stretching surfaces in porous media. Some of the key practical implications are:

1. Materials processing: In the manufacturing of polymer sheets, filaments, and other materials, the control of the flow and heat transfer characteristics is crucial for achieving the desired properties and quality of the final product. The results of this study can guide the selection of appropriate process parameters, such as the magnetic field strength, porous medium properties, and stretching rate, to optimize the production process and improve the quality of the manufactured materials.
2. Geothermal systems: The optimization of flow and heat transfer in geothermal systems is essential for maximizing their efficiency and performance. The findings of this study can help in the design and operation of geothermal heat exchangers, wells, and other components, by providing insights into the effects of the magnetic field, porous medium, and stretching surface on the flow and heat transfer characteristics. This knowledge can be used to enhance the heat extraction and energy conversion processes in geothermal systems.
3. Electromagnetic casting: In the casting of metals and alloys, the application of a magnetic field can be used to control the flow and solidification process, resulting in improved material properties and reduced defects. The results of this study can contribute to the understanding of the complex interactions between the magnetic field, porous medium, and stretching surface in electromagnetic casting processes, enabling the optimization of process parameters for achieving the desired material properties and performance.
4. Biomedical applications: The study of MHD boundary layer flow in porous media has potential applications in the field of biomedicine, such as in the design of drug delivery systems, tissue

engineering scaffolds, and artificial organs. The findings of this study can provide valuable insights into the flow and heat transfer behavior of biological fluids in the presence of magnetic fields and porous structures, aiding in the development of advanced biomedical devices and therapies.

5. Environmental and thermal engineering: The results of this study can be applied to the design and analysis of various environmental and thermal engineering systems, such as air and water purification systems, heat exchangers, and thermal insulation materials. The understanding of the effects of the magnetic field, porous medium, and stretching surface on the flow and heat transfer characteristics can help optimize the performance and efficiency of these systems, leading to improved environmental sustainability and energy conservation.

5.3. Recommendations for future work

While the present study has provided valuable insights into the MHD boundary layer flow over a stretching surface in porous media, there are several areas where further research can be conducted to extend the understanding of this complex phenomenon. Some recommendations for future work include:

1. Non-Newtonian fluids: The present study considered a Newtonian fluid, but many real-world applications involve non-Newtonian fluids, such as polymers, suspensions, and biological fluids. Investigating the effects of fluid rheology on the MHD boundary layer flow over a stretching surface in porous media would provide more realistic and comprehensive results for a wider range of applications.
2. Variable fluid properties: The current study assumed constant fluid properties, such as viscosity and thermal conductivity. However, in practical situations, these properties may vary with temperature or other factors. Incorporating variable fluid properties into the mathematical model would allow for a more accurate representation of the flow and heat transfer behavior in real-world applications.
3. Unsteady and three-dimensional flow: The present study focused on steady, two-dimensional flow. Extending the analysis to unsteady and three-dimensional flow would provide a more complete understanding of the complex flow dynamics and heat transfer processes in MHD boundary layer flow over stretching surfaces in porous media.
4. Multiphase flow: In some applications, such as in geothermal systems or oil and gas extraction, the flow may involve multiple phases, such as liquid and gas or liquid and solid particles. Investigating the MHD boundary layer flow in multiphase systems with porous media and stretching surfaces would provide valuable insights for these specific applications.
5. Experimental validation: While the numerical results obtained in this study have been validated through various methods, experimental validation would provide further confirmation of the findings and help identify any limitations or discrepancies in the mathematical model. Conducting carefully designed experiments to measure the velocity and temperature profiles, skin friction coefficient, and local Nusselt number would strengthen the reliability of the results and their practical applicability.
6. Optimization studies: The present study investigated the effects of various governing parameters on the flow and heat transfer characteristics. However, it did not focus on optimizing these parameters for specific applications. Future work could involve the use of optimization techniques, such as genetic algorithms or response surface methodology, to determine the optimal combination of magnetic field strength, porous medium properties, and stretching rate for maximizing the desired performance metrics in specific engineering applications.

By addressing these recommendations, future research can build upon the findings of the present study and provide a more comprehensive understanding of the MHD boundary layer flow over a stretching surface in porous media. This knowledge will enable the development of more efficient, sustainable, and high-performance engineering systems in various fields, from materials processing and geothermal energy to biomedical applications and environmental engineering.

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