Forced Oscillation of Nonlinear Variable Order Fractional Differential Equations

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Abstract: In this paper we discuss the forced oscillation of nonlinear variable order fractional differential equations of the form
$\left(D_a^{\alpha(t)}y\right)(t) + F_1\big(t, y(t)\big) = V(t) + F_2\big(t, y(t)\big)\ for\ t > a \geq 0$ along with the initial conditions
$\left(D_a^{\alpha(t)-k}y\right)(a) = b_k\ where\ k = 1\ to\ m-1$ and $\lim_{t\to a^+}(I_a^{m-\alpha(t)}y)(t) = b_m$, in which $D_a^{\alpha(t)}y$ is the Riemann-Liouville fractional derivative of order $\alpha(t)$ of y, $m-1 < \alpha(t) \leq m$, $m \geq 1$ is an integer, $I_a^{m-\alpha(t)}y$ is the Riemann-Liouville fractional integral of order $m - \alpha(t)\ of\ y$, $b_k\ (k = 1,2, \dots m)$ are constants. We have given an example to illustrate our theoretical results.

Keywords: Fractional differential equation, Variable order, Forced oscillation.

1. Introduction

Fractional order differential equations are an important tool in modeling many concepts of science and engineering. It has a wide range of applications in electrochemistry, viscoelasticity, control theory and many of physical problems, for example [1-6]. There has been a consistent development in partial and ordinary differential equations involving Caputo and Riemann-Liouville derivative of fractional order. Many books [7-10] elaborate the theories and applications of fractional derivatives and fractional integrals. So many authors studied the aspects like existence, uniqueness, stability and oscillation of fractional derivatives. We refer [11-21] and the references quoted in them.

In [20] authors discussed the forced oscillation of a differential equation of the form
$\left(D_a^q x\right)(t) + f_1(t,x) = v(t) + f_2(t,x), \lim_{t\to a^+}(J_a^{1-q}x)(t) = b_1\ where\ D_a^q$ is the Riemann-Liouville differential operator of order $q, 0 < q \leq 1$ with the initial condition $xf_i(t,x) > 0 (i = 1,2), x \neq 0, t \geq a\ and\ |f_1(t,x)| \geq p_1|x|^\beta\ and\ |f_2(t,x)| \leq p_2|x|^\gamma$,
$where\ p_1, p_2 \in C([a,\infty), R^+) and\ \beta, \gamma > 0.$

In [21] authors established the oscillation criteria by using Young's Inequality for a fractional differential equation of the form

$$\left(D_a^q x\right)(t) + f_1(t,x) = v(t) + f_2(t,x), \qquad t > a \geq 0,$$

$$\left(D_a^{q-k}x\right)(a) = b_k, (k = 1\ to\ m-1), \qquad \lim_{t\to a^+}(I_a^{m-q}x)(t) = b_m$$

$t\to a^+$

$where\ D_a^q$ is the Riemann-Liouville differential operator of order $q, m-1 < q \leq m,\ m \geq 1$ is an integer, I_a^{m-q} is the Riemann-Liouville fractional integral of order $m - q$, $b_k\ (k = 1,2, \dots m)$ are constants, with the initial condition $xf_i(t,x) > 0 (i = 1,2), x \neq 0, t \geq a\ and$
$|f_1(t,x)| \leq p_1|x|^\beta\ and\ |f_2(t,x)| \geq p_2|x|^\gamma\ for\ x \neq 0, t \geq a$
$where\ p_1, p_2 \in C([a,\infty), (0,\infty))\ and\ \beta, \gamma > 0.$

In this paper, we established some forced oscillation results of variable order nonlinear fractional differential equation of the form

$\left(D_a^{\alpha(t)}y\right)(t) + F_1\big(t, y(t)\big) = V(t) + F_2\big(t, y(t)\big)\ for\ t > a \geq 0$ (1.1)

together with the initial conditions

$\left(D_a^{\alpha(t)-k}y\right)(a) = b_k\ where\ k = 1\ to\ m-1$ and $\lim_{t\to a^+}(\ I_a^{m-\alpha(t)}y)(t) = b_m,$ (1.2)

in which $D_a^{\alpha(t)}y$ is the Riemann-Liouville fractional derivative of order $\alpha(t)$ of y, $m-1 < \alpha(t) \leq m,\ m \geq 1$ is an integer, $I_a^{m-\alpha(t)}y$ is the Riemann-Liouville fractional integral of order $m - \alpha(t)\ of\ y$, $b_k\ (k = 1,2, \dots m)$ are constants. We improved our result with variable order nonlinear fractional differential equation along with the conditions

$xF_i(t, y) > 0 (i = 1,2), x \neq 0, t \geq a$ (1.3)

$|F_1(t, y)| \geq P_1|y|^\beta\ and\ |F_2(t, y)| \leq P_2|y|^\gamma\ for\ y \neq 0, t \geq a$ (1.4)

and

$|F_1(t, y)| \leq P_1|y|^\beta\ and\ |F_2(t, y)| \geq P_2|y|^\gamma\ for\ y \neq 0, t \geq a$ (1.5)

$where\ P_1, P_2 \in C([a, \infty), (0, \infty))\ and\ \beta, \gamma > 0$ are real numbers.

In [20] the authors gave many results on oscillation by reducing the given equation into an equivalent Volterra fractional integral equation of the form

$y(t) = \sum_{k=1}^{m} \frac{b_k(t-a)^{\alpha(t)-k}}{\Gamma(\alpha(t)-k+1)} + \frac{1}{\Gamma(\alpha(t))}\int_a^t (t-s)^{\alpha(t)-1}\big[V(s) + F_2\big(s, y(s)\big) - F_1\big(s, y(s)\big)\big]ds$ (1.6)

for $t > a$.

2. Preliminaries

Here we give some basic definitions of fractional integrals and derivatives and also the Young's Inequality. We can get more information about the fractional calculus from [7-10].

Definition 2.1: A solution of a differential equation is said to be oscillatory if it has arbitrarily many zeros. If all the solutions of an equation are oscillatory, then the differential equation is said to be oscillatory.

Definition 2.2: The Riemann-Liouville fractional derivative of order $q > 0$ for the function $x: [a, \infty) \to R$ is given by $\left(D_a^q x\right)(t) = \frac{d^m}{dt^m}\left(I_a^{m-q}x\right)(t),$ (2.1)

whereas the right hand side is defined pointwise on $[a, \infty)$, where $m-1 < q \leq m,\ m \geq 1$ is an integer. Also, we set $D_a^0 x = x$.

Definition 2.3: The Riemann-Liouville fractional integral of order $q > 0$ for the function $x: [a, \infty) \to R$ is given by $\left(I_a^q x\right)(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}x(s)ds,$ (2.2)

whereas the right hand side is defined pointwise on $[a, \infty)$, and Γ is a Gamma function defined by $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ for $t > 0$. Also, we set $I_a^0 x = x$.

Definition 2.4: The variable order Riemann–Liouville integral of function $f(u)$ is given by

${}_{RL}I_{0,t}^{-\alpha(t)} f(u) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (u-\lambda)^{\alpha(t)-1} f(\lambda)d\lambda, \quad t > 0, \alpha(t) > 0$ (2.3)

Definition 2.5: The variable order Riemann–Liouville derivative function $f(u)$ is given by

${}_{RL}D_{0,t}^{\alpha(t)} f(u) = \frac{1}{\Gamma(n-\alpha(t))}\frac{d^n}{dt^n}\int_0^t (u-\lambda)^{n-\alpha(t)-1} f(\lambda)d\lambda,\ \ t > 0, \alpha(t) > 0$ (2.4)

Definition 2.6: The Caputo fractional derivative with order $q > 0$ for the function $x: [a, \infty) \to R$ is given by

$({}_C D_a^q x)(t) = (I_a^{m-q}x^{(m)})(t)$ (2.5)

whereas the right hand side is defined pointwise on $[a, \infty)$ and $m-1 < \alpha(t) \leq m,\ m \geq 1$ is an integer, $x^{(m)}$ is usual derivative of integer order m. Also, we set ${}_C D_a^0 x = x$.

Definition 2.7: (Young's Inequality) (a) Let $X, Y \geq 0, u > 1$ and $\frac{1}{u}+\frac{1}{v}=1, then$ $XY \leq \frac{1}{u}X^u+\frac{1}{v}Y^v$, where the equality holds if and only if $Y = X^{u-1}$.
(b) Let $X \geq 0, Y > 0, 0 < u < 1$ and $\frac{1}{u}+\frac{1}{v}=1, then$ $XY \geq \frac{1}{u}X^u+\frac{1}{v}Y^v$, where the equality holds if and only if $Y = X^{u-1}$.

3. Oscillation results

Theorem 3.1: Let $K(s) = \left(\frac{\beta}{\gamma-\alpha}\right)\left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}} p_1^{\frac{\gamma}{\gamma-\beta}}(s)$ and assume that for $\beta > \gamma$ (1.3) and (1.4) holds. If
$\liminf_{t\to\infty} t^{1-\alpha(t)} \int_T^t (t-s)^{\alpha(t)-1}[V(s)+K(s)]ds = -\infty$ (3.1)
and $\limsup_{t\to\infty} t^{1-\alpha(t)} \int_T^t (t-s)^{\alpha(t)-1}[V(s)-K(s)]ds = +\infty$ (3.2)
for sufficiently large value of T, then each solution of (1.1)(1.2) is oscillatory.

Proof: Assume that y be the non oscillatory solution of equation (1.1). If we assume y is an eventually positive solution of (1.1), then there exists $T_1 > a$ such that $y(t) > 0$ for $t \geq T_1$.
Let $s \geq T_1$ and take $X = |y|^\gamma(s), Y = \frac{\gamma P_2(s)}{\beta P_1(s)}, u = \frac{\beta}{\gamma}$ and $v = \frac{\beta}{\beta-\gamma}$, then by (a) of definition 2.7 we can conclude that

$$P_2(s)|y|^\gamma(s) - P_1(s)|y|^\beta(s) = \frac{\beta P_1(s)}{\gamma}[|y|^\gamma(s)\frac{\gamma P_2(s)}{\beta P_1(s)} - \frac{1}{\left(\frac{\beta}{\gamma}\right)}(|y|^\gamma(s))^{\frac{\beta}{\gamma}}]$$

$$= \frac{\beta P_1(s)}{\gamma}[XY - \frac{1}{u}X^u]$$

$$\leq \frac{\beta P_1(s)}{\gamma}\frac{1}{v}Y^v = K(s)\ for\ s \geq T_1 \quad (3.3)$$

From (1.3), (1.4), (1.5) and (3.3), we get

$$\Gamma(\alpha(t))y(t) = \Gamma(\alpha(t))\sum_{k=1}^{m}\frac{b_k(t-a)^{\alpha(t)-k}}{\Gamma(\alpha(t)-k+1)} + \int_a^{T_1}(t-s)^{\alpha(t)-1}\big[V(s)+F_2(s,y(s))-F_1(s,y(s))\big]ds$$
$$+\int_{T_1}^{t}(t-s)^{\alpha(t)-1}\big[V(s)+F_2(s,y(s))-F_1(s,y(s))\big]ds$$
$$\leq \varphi(t)+\psi(t,T_1)+\int_{T_1}^{t}(t-s)^{\alpha(t)-1}\big[V(s)+P_2(s)y^\gamma(s)-P_1(s)y^\beta(s)\big]ds$$
$$\leq \varphi(t)+\psi(t,T_1)+\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds\ \ for\ t\geq T_1 \quad (3.4)$$

where $\varphi(t) = \Gamma(\alpha(t))\sum_{k=1}^{m}\frac{b_k(t-a)^{\alpha(t)-k}}{\Gamma(\alpha(t)-k+1)}$ (3.5)

$\psi(t,T_1) = \int_a^{T_1}(t-s)^{\alpha(t)-1}\big[V(s)+F_2(s,y(s))-F_1(s,y(s))\big]ds$ (3.6)

By multiplying (3.4) by $t^{1-\alpha(t)}, for\ t\geq T_1$ we have

$$0 < t^{1-\alpha(t)}\Gamma(\alpha(t))y(t)$$
$$\leq t^{1-\alpha(t)}\varphi(t)+t^{1-\alpha(t)}\psi(t,T_1)+t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \quad (3.7)$$

Take $T_2 > T_1$. We consider the following cases with $0 < \alpha(t) \leq 1$ and $\alpha(t) > 1$.
Case (i): Let $0 < \alpha(t) \leq 1$.
We get $m = 1$, $\varphi(t) = b_1(t-a)^{\alpha(t)-1}$

$$\left|t^{1-\alpha(t)}\,\varphi(t)\right| = |b_1|t^{1-\alpha(t)}(t-a)^{\alpha(t)-1} \leq |b_1|\left(\frac{T_2}{T_2-a}\right)^{1-\alpha(t)} = C_1(T_2)\ for\ t\geq T_2 \quad (3.8)$$

and

$$\left|t^{1-\alpha(t)}\,\psi(t,T_1)\right| = \left|t^{1-\alpha(t)}\right|\int_a^{T_1}(t-s)^{\alpha(t)-1}\big[V(s)+F_2(s,y(s))-F_1(s,y(s))\big]ds$$
$$\leq \int_a^{T_1}t^{1-\alpha(t)}(t-s)^{\alpha(t)-1}\big[V(s)+F_2(s,y(s))-F_1(s,y(s))\big]ds$$

$\leq \int_a^{T_1} \left(\frac{T_2}{T_2 - s}\right)^{1-\alpha(t)} [V(s) + F_2(s, y(s)) - F_1(s, y(s))]ds$
$= C_2(T_1, T_2)\ for\ t \geq T_2$ (3.9)
From (3.7), (3.8) and (3.9) we can conclude that
$t^{1-\alpha(t)} \int_{T_1}^{t} (t-s)^{\alpha(t)-1} [V(s) + K(s)]ds\ >\ -[C_1(T_2) + C_2(T_1, T_2)]\ for\ t \geq T_2.$
Hence, we get
$\lim_{t\to\infty} inf\, t^{1-\alpha(t)} \int_{T_1}^{t} (t-s)^{\alpha(t)-1} [V(s) + K(s)]ds\ \geq\ -[C_1(T_2) + C_2(T_1, T_2)] > -\infty$
which is a contradiction for (3.1).
Case (ii): Let $\alpha(t) > 1$.
We get $m \geq 2$.
$|t^{1-\alpha(t)} \varphi(t)| = \left| t^{1-\alpha(t)} \Gamma(\alpha(t)) \sum_{k=1}^{m} \frac{b_k (t-a)^{\alpha(t)-k}}{\Gamma(\alpha(t) - k + 1)} \right|$
$\leq \Gamma(\alpha(t)) \sum_{k=1}^{m} \frac{|b_k| t^{1-\alpha(t)} (t-a)^{\alpha(t)-k}}{\Gamma(\alpha(t) - k + 1)}$
$\leq \Gamma(\alpha(t)) \sum_{k=1}^{m} \frac{|b_k| (T_2 - a)^{1-k}}{\Gamma(\alpha(t) - k + 1)}$
$= C_3(T_2)\ for\ t \geq T_2$
(3.10)
and
$|t^{1-\alpha(t)}\, \psi(t, T_1)| = |t^{1-\alpha(t)}| \left| \int_a^{T_1} (t-s)^{\alpha(t)-1} [V(s) + F_2(s, y(s)) - F_1(s, y(s))]ds \right.$
$\leq \int_a^{T_1} t^{1-\alpha(t)} (t-s)^{\alpha(t)-1} [V(s) + F_2(s, y(s)) - F_1(s, y(s))]ds$
$\leq \int_a^{T_1} [V(s) + F_2(s, y(s)) - F_1(s, y(s))]ds\ =\ C_4(T_1)\ for\ t \geq T_2$ (3.11)
It follows from (3.7), (3.10) and (3.11) we can conclude that
$t^{1-\alpha(t)} \int_{T_1}^{t} (t-s)^{\alpha(t)-1} [V(s) + K(s)]ds\ >\ -[C_3(T_2) + C_4(T_1)]\ for\ t \geq T_2.$
Hence, we get
$\lim_{t\to\infty} inf\, t^{1-\alpha(t)} \int_{T_1}^{t} (t-s)^{\alpha(t)-1} [V(s) + K(s)]ds\ \geq\ -[C_3(T_2) + C_4(T_1)] > -\infty.$
which is again a contradiction for (3.1).
Hence if we assume y is an eventually negative solution of (1.1) and (1.2). We can get a contradiction for (3.2) by using the similar arguments. This completes the proof.

Theorem 3.2: Let $\alpha(t) \geq 1$ and assume that (1.3) and (1.5) hold with $\beta < \gamma$. If
If $\lim_{t\to\infty} sup\, t^{1-\alpha(t)} \int_T^t (t-s)^{\alpha(t)-1} [V(s) + K(s)]ds = +\infty$ (3.12)
and $\lim_{t\to\infty} inf\, t^{1-\alpha(t)} \int_T^t (t-s)^{\alpha(t)-1} [V(s) - K(s)]ds = -\infty$ (3.13)
for sufficiently large value of $T\ and\ K$ is defined as in theorem 3.1, then every bounded solution of (1.1) (1.2) is oscillatory.

Proof: Let y be a bounded and nonoscillatory solution of equation (1.1).
Therefore there exists constants $M_1\ and\ M_2$ such that
$M_1\ \leq y(t) \leq\ M_2\ for\ t > a$ (3.14)
First we assume that y is a eventually positive bounded solution of (1.1). Then there exists $T_1 > a$ such that $y(t) > 0\ for\ t > T_1$.

By the similar proof of (3.3) and by condition (b) of definition 2.7, we find

$$P_2(s)|y|^{\gamma}(s) - P_1(s)|y|^{\beta}(s) \geq K(s) \; for \; s > T_1 \tag{3.15}$$

where $K(s)$ is defined as in theorem 3.1.

Also, φ *and* ψ takes the same value as in (3.5) and (3.6) respectively. By taking the similar procedure of (3.7), and from (1.3), (1.5), (3.15), for $t > T_1$ we get

$$t^{1-\alpha(t)}\Gamma(\alpha(t))y(t)$$

$$\geq t^{1-\alpha(t)}\varphi(t) + t^{1-\alpha(t)}\psi(t,T_1) + t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \tag{3.16}$$

By taking $T_2 > T_1$, we consider the following cases with $\alpha(t) = 1$ *and* $\alpha(t) > 1$.

Case (i): Let $\alpha(t) = 1$. In this case (3.8) and (3.9) are still true. Hence, (3.8), (3.9), (3.14) and (3.16) implies that

$$M_2\,\Gamma(\alpha(t)) \geq -[C_1(T_2) + C_2(T_1,T_2)] + t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \quad for \; t \geq T_2.$$

Thus,

$$\limsup_{t\to\infty} t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \leq [C_1(T_2) + C_2(T_1,T_2)] + M_2\,\Gamma(\alpha(t)) < +\infty$$

This is the contradiction for (3.12).

Case (ii): Let $\alpha(t) > 1$. In this case (3.10) and (3.11) are still true. Hence, (3.10), (3.11), (3.14) and (3.16) implies that

$$M_2\Gamma(\alpha(t))t^{1-\alpha(t)} \geq -[C_3(T_2) + C_4(T_1)] + t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \text{ for}$$

$t \geq T_2$. Since we know that $\lim_{t\to\infty} t^{1-\alpha(t)} = 0$, we can conclude that

$$\limsup_{t\to\infty} t^{1-\alpha(t)}\int_{T_1}^{t}(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \leq [C_3(T_2) + C_4(T_1)] < +\infty.$$

This is again a contradiction for (3.12).

Similarly, if we assume that y as a eventually bounded negative solution of (1.1) and (1.2), then by using the similar arguments we will get a contradiction for (3.13).

Hence the theorem is proved.

4. Oscillation results with Caputo Fractional Derivative

In this section, we establish the oscillation result for (1.1) when the by replacing the Riemann-Liouville fractional differential operator by the Caputo Fractional differential operator. That is, here we will study the oscillation of the initial value problem

$$\left({}^{c}_{}D_a^{\alpha(t)}y\right)(t) + F_1(t,y(t)) = V(t) + F_2(t,y(t)) \; for \; t > a \geq 0 \tag{4.1}$$

together with the initial conditions

$$y^{(k)}(a) = b_k \; (k = 0 \; to \; m-1) \tag{4.2}$$

Here ${}^{c}_{}D_a^{\alpha(t)}$ is the Caputo fractional order derivative of order $\alpha(t)$ of y defined by (2.5), $m-1 < \alpha(t) \leq m, m \geq 1$ is an integer, $b_k \; (k = 0 \; to \; m-1)$ are constants, $F_i: [a,+\infty) \times R \to R \; (i = 1,2)$ are continuous functions, and $V: [a,+\infty) \to R$ is a continuous function.

The Volterra fractional integral equation corresponding to this is given by

$$y(t) = \sum_{k=0}^{m-1}\frac{b_k(t-a)^{\alpha(t)-k}}{k!} + \frac{1}{\Gamma(q)}\int_a^t(t-s)^{\alpha(t)-1}\left[V(s) + F_2(s,y(s)) - F_1(s,y(s))\right]ds \text{ for } t > a.$$

We can prove the following theorems by using the same procedure as in theorem 3.1 and theorem 3.2

Theorem 4.1: Assume that for $\beta > \gamma$ (1.3) and (1.4) holds. If

$$\liminf_{t\to\infty} t^{1-m}\int_T^t(t-s)^{\alpha(t)-1}[V(s)+K(s)]ds = -\infty \tag{4.3}$$

and $\limsup_{t\to\infty} t^{1-m}\int_T^t(t-s)^{\alpha(t)-1}[V(s)-K(s)]ds = +\infty$ (4.4)

for sufficiently large value of T and if we define K as in Theorem 3.1, then every solution of (4.1) (4.2) is oscillatory.

Theorem 4.2: Assume that $\alpha(t) \geq 1$ and (1.3), (1.5) hold with $\beta < \gamma$.

If $\lim_{t\to\infty} sup\, t^{1-m}\int_T^t (t-s)^{\alpha(t)-1}[V(s)+K(s)]ds = +\infty$ (4.5)

and $\lim_{t\to\infty} inf\, t^{1-m}\int_T^t (t-s)^{\alpha(t)-1}[V(s)-K(s)]ds = -\infty$ (4.6)

for sufficiently large value of T *and* K is defined as in theorem 3.1, then every bounded solution of (4.1) (4.2) is oscillatory.

5. Example

In this section, we will give an example to show that the condition (3.1) cannot be dropped. If we drop the condition (3.1) we will get a non oscillatory solution.

Example 5.1:

Consider the following Riemann-Liouville nonlinear fractional order differential equation

$$(D_0^{\alpha(t)}y)(t) + y^5(t)In(e+t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \left(t^{10} - t^{\frac{2}{3}}\right)In(e+t) + x^{\frac{1}{3}}(t)In(e+t), \quad (5.1)$$

$t > 0$

$$\lim_{t\to0+}(I_0^{1-\alpha(t)}y)(t) = 0, \;\; where\; 0 < \alpha(t) < 1. \quad (5.2)$$

In this example we take $a = 0, m = 1, F_1(t,y) = y^5(t)In(e+t)$,

$$V(t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \left(t^{10} - t^{\frac{2}{3}}\right)In(e+t), F_2(t,y) = x^{\frac{1}{3}}(t)In(e+t), b_1 = 0\; and$$

$$\alpha(t) = \frac{t}{2}\; with\; 1 < t < 2\; and\; so\; 0 < \alpha(t) < 1.$$

Taking $P_1(t) = P_2(t) = In(e+t), \beta = 5, \gamma = 1/3$, we find that the conditions (1.3) and (1.4) are satisfied. Defining K as in theorem 3.1 and $V(t) > 0$, we have

$$\lim_{t\to\infty} inf\, t^{1-\alpha(t)}\int_T^t (t-s)^{\alpha(t)-1}[V(s)+K(s)]ds \ge \lim_{t\to\infty} inf\, t^{1-\alpha(t)}\int_T^t (t-s)^{\alpha(t)-1}K(s)ds$$

$$= \lim_{t\to\infty} inf\, t^{1-\alpha(t)}\int_T^t (t-s)^{\alpha(t)-1}\left(\frac{5}{\frac{1}{3}} - 1\right)[\frac{\frac{1}{3}}{5}In(e+t)]^{\frac{5}{5-1/3}}[In(e+t)]^{\frac{1}{3}(\frac{1}{3}-5)}ds$$

$$= \lim_{t\to\infty} inf\, t^{1-\alpha(t)}\int_T^t 14(t-s)^{\alpha(t)-1}15^{-15/14}In(e+t)ds$$

$$\ge \lim_{t\to\infty} inf\, t^{1-\alpha(t)}\int_T^t 14(t-s)^{\alpha(t)-1}15^{-15/14}ds$$

$$= \lim_{t\to\infty} inf\, \frac{15^{-15/14}14t^{1-\alpha(t)}(t-T)^{\alpha(t)}}{\alpha(t)}$$

$$= \infty$$

This shows that the condition (3.1) is not satisfied for sufficiently large $T \ge 1$ *and* $t \ge T$.

By taking $y(t) = t^2$, we get

$$(I_0^{\alpha(t)}y)(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-s)^{-\alpha(t)}s^2 ds$$

By integrating we obtain

$$(I_0^{\alpha(t)}y)(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{2t^{3-\alpha(t)}}{(1-\alpha(t))(2-\alpha(t))(3-\alpha(t))} \quad (5.3)$$

Hence, $(D_0^{\alpha(t)}y)(t) = \frac{d}{dt}(I_0^{\alpha(t)}y)(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{2t^{2-\alpha(t)}}{(1-\alpha(t))(2-\alpha(t))} = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}$

This shows that $y(t) = t^2$ satisfies (5.1). Also, by (5.3) we have $\lim_{t\to0+}(I_0^{1-\alpha(t)}y)(t) = 0$.

This shows that $y(t) = t^2$ satisfies (5.2).

Hence we conclude that $y(t) = t^2$ is a non oscillatory solution of (5.1) (5.2).

6. Conclusion

In this paper, we established the oscillation criteria of variable order nonlinear fractional differential equation given in (1.1) (1.2) and we improved our result by providing the result for Caputo fractional order derivative.

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