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Transient and Steady State Analysis of M/D/1 Queue with Compulsory Vacation

S. Vanitha
Assistant Professor, Department of Mathematics,
Sri Sivasubramaniya Nadar College of Engineering, Tamilnadu, India.

Abstract: We analyse a single server vacation queue with Poisson arrivals, deterministic service of constant duration $d(> 0)$. As soon as the service of a customer is complete, the server will go for compulsory vacation. The vacation times are also assumed to be general. The supplementary variable technique is used to find explicitly the probability generating function of the number in the system and the mean number in the system.

Keywords: Poisson Arrivals, Probability Generating Function, Transient State, Steady State, Supplementary Variable Technique.

1. Introduction

Queueing theory is the mathematical study of waiting lines (or queues). A queue or waiting line is formed by a flow of customers from an infinite or finite population on account of lack of capability to serve them all at a time. Queueing theory enables mathematical analysis of several related processes, including arriving at the (back of the) queue, waiting in the queue (essentially a storage process) and being served by the server or servers at the front of the queue.

In recent years research on queueing systems with server vacations has acquired great importance. Queueing systems that allow servers to take vacation have a wide range of applications in many engineering systems such as production, manufacturing, communication networks and telecommunication systems. In fact, Queueing models with server vacations have been efficiently studied by many researchers in the last two decades and successfully applied in various practical problems. An excellent survey on the vacation queueing models have been documented in [1] [2], [3], [4], [5] and several others.

There are two basic vacation queueing models namely, multiple vacation queueing model and single vacation queueing model. In multiple vacation queueing models, the server keeps on taking sequential vacations until it finds some customers waiting in a queue at a vacation completion epoch; However, in single vacation queueing models, the server takes exactly one vacation between two sequential busy periods. These two types of vacation models were first introduced by [6].

An M/G/1 queue with vacation model is often referred as a tool of understanding congestion phenomena in local networks. Since the past two or three decades, it has emerged as an important area of study in real life problems such as telecommunication engineering, manufacturing and production industries, computers and communication networks etc. Several contributions have been made by dealing with queueing systems of M/G/1 type which include [7], [8], [9], [10], [11], [12] and [13].

Many researchers have developed several models involving single vacation policy but only few models have been developed with compulsory vacation. A single server queue with compulsory server vacations where the service was performed in batches of fixed size has been studied by [14]. In that paper, the Laplace transforms of the probability generating functions of different states of the system have been obtained, the corresponding steady state results have been derived and in a particular case the mean queue length has been obtained explicitly. [15] studied a M/G/1 queue with two stage heterogeneous service subject to compulsory vacation and random breakdowns. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also, the average number of customers in the queue and the average waiting time are derived for that queueing model.

A M/G/1 queue with two types of service and compulsory vacation under restricted admission policy was analyzed by [16]. For this model, the probability generating function for the number of customers in the queue at different server's state are obtained using supplementary variable technique. Some performance measures are also calculated for that model. [17] studied M/G/1 queue subject to compulsory vacation and three phase repairs. The supplementary variable technique is used to find explicitly the probability generating function of the number in the system and the mean number in the system.

There are many situations in real life where the service times are constant. The one common example we often notice is a cycle of a washing machine that takes a fixed length of time to complete one service. The M/D/1 queueing system is widely found in queueing literature (see [18], [19] and [20]). A M/D/1 queue with compulsory server vacation subject to random breakdowns and exponential repair has been studied by [21]. In that paper, the server takes compulsory vacation after the completion of each service and the service channel is subject to random breakdowns. Whenever the server channel breakdowns, it instantly undergoes a repair process and repair times are exponentially distributed. In the current work, we study a M/D/1 queue subject to compulsory server vacation without breakdowns and repairs. We calculate the time dependent solution using supplementary variable technique and the corresponding steady state results are derived explicitly.

The rest of the paper is organized as follows. The mathematical description of our model is in section 2 and definitions representing the model are given in section 3. The equations governing the model are there in section 4. The time dependent solutions have been obtained in section 5 using supplementary variable technique and the corresponding steady state results have been derived explicitly in section 6. The mean number in the system and the mean waiting time have been found in section 7.

2. Assumptions Underlying the Model

The following assumptions describe the mathematical model.

- Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(>0)$.
- The server provides deterministic (constant) service of length $d(>0)$ to each customer.
- After every service the server takes a compulsory vacation of random length.
- The vacation time follow general (arbitrary) distribution with distribution $B(v)$ and the density function $b(v)$.

Let $\beta(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x+dx]$ given that the elapsed vacation time is x, so that

$$\beta(\mathrm{x}) = \frac{b(x)}{1-B(x)}$$
(1)

and therefore

$$b(v) = \beta(v)e^{-\int_0^\infty \beta(x)dx}.$$
(2)

- On returning from vacation the server instantly starts serving the customer at the head of the queue.
- The customers are served according to the first come, first served rule.
- Various Stochastic Processes involved in the system are independent of each other.

3. Definitions, Notations and Equations Governing the System

We define

- $H_n(t)$: Probability that at time t, there are $n \geq 0$ customers in the system, including one in service if any and the server is present in the system which means the server is providing the service when $n>0$ and is idle when $n=0$.
- $V_n(x,t)$: Probability that at time t, the server is under vacation with elapsed vacation time x and there are $n \geq 0$ customers waiting in the system for service.

Consequently $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t, there are n customers in the system and the server is under vacation irrespective of the value of x.

The system is then governed by the following set of differential – difference equation

$$\frac{d}{dt}H_n(t) = -H_n(t) + \int_0^\infty V_n(x,t)\beta(x)dx\,, n = 0,1,\ldots, \quad (3)$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + (\lambda + \beta(x))V_n(x,t) = \lambda V_{n-1}(x,t), n = 1,2,\ldots, \quad (4)$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + (\lambda + \beta(x))V_0(x,t) = 0, \quad (5)$$

We assume that initially that there are j customers in the system and the server is working so that the initial conditions can be written as

$$H_n(t) = \delta_{nj} = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases} \text{ and } V_n(0) = 0, n \geq 0 \quad (6)$$

Equations (3) to (5) are to be solved subject to the following boundary condition.

$$V_n(0,t) = [H_0(t) + H_1(t)]K_n + \sum_{i=2}^{\infty} H_i(t)K_{n+1-i}, \quad n = 0,1,\ldots, \quad (7)$$

where $K_i, i = 0,1,\ldots$, is the probability of i arrivals during a service period of constant length d.

4. Generating Functions of the Queue Length : The Time Dependent Solution

In this section we define the transient solution for the above set of differential-difference equations. We define probability generating functions.

$$\left.\begin{aligned} V(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t), V(z,t) = \sum_{n=0}^{\infty} z^n V_n(t) \\ H(z,t) = \sum_{n=0}^{\infty} z^n H_n(t) \end{aligned}\right\} \quad (8)$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f(s)} = \int_0^\infty e^{-st} f(t)dt, \Re(s) > 0 \quad (9)$$

We take Laplace transforms of above equations and using initial conditions (6), we obtain

$$(s+1)\overline{H}_n(s) = \delta_{nj} + \int_0^\infty \overline{V}_n(x,s)\beta(x)dx, n = 0,1,\ldots, \quad (10)$$

$$\frac{\partial}{\partial x}\overline{V_n}(x,s) + (s + \lambda + \beta(x))\overline{V_n}(x,s) = \lambda\overline{V}_{n-1}(x,s), \quad n = 1,2,\ldots \quad (11)$$

$$\frac{\partial}{\partial x}\overline{V_0}(x,s) + (s + \lambda + \beta(x))\overline{V_0}(x,s) = 0, \quad (12)$$

$$\overline{V_n}(0,s) = [\overline{H}_0(s) + \overline{H}_1(s)]K_n + \sum_{i=2}^{\infty} \overline{H}_i(s)K_{n+1-i}, \quad n = 0,1,\ldots, \quad (13)$$

We define the following generating functions in terms of their Laplace transforms:

$$\left.\begin{aligned} V(x,z,s) = \sum_{n=0}^{\infty} z^n V_n(x,s), V(z,s) = \sum_{n=0}^{\infty} z^n V_n(s) \\ H(z,s) = \sum_{n=0}^{\infty} z^n H_n(s) \end{aligned}\right\} \quad (14)$$

$$K(z) = \sum_{n=0}^{\infty} K_n z^n = \sum_{n=0}^{\infty} \frac{e^{-\lambda d}(\lambda d)^n z^n}{n!} e^{-\lambda d(1-z)}, \quad |z| < 1. \quad (15)$$

Multiply both sides of equation (10) by $z^{n+1}, n = 0,1,\ldots$, and add for all n. Then

$$(s+1)\sum_{n=0}^{\infty} \overline{H}_n(s)z^{n+1} = \sum_{n=0}^{\infty} \delta_{nj} z^{n+1} + \int_0^\infty \sum_{n=0}^{\infty} z^{n+1}\, \overline{V}_n(x,s)\beta(x)dx, n = 0,1,\ldots \quad (16)$$

which on using equation (14) yields

$$(s+1)z\overline{H}(z,s) = z^{j+1} + z\int_0^\infty \overline{V}(x,z,s)\beta(x)dx. \quad (17)$$

Now we multiply (11) by z^n, sum for $n = 1,2,\ldots$ and add the result to (12). We then have

$$\frac{\partial}{\partial x}\overline{V}(x,z,s) + (s + \lambda - \lambda z + \beta(x))\overline{V}(x,z,s) = 0. \quad (18)$$

We again multiply (13) by z^{n+1}, sum for all $n = 0,1,\ldots$, and use (14). We thus have

$$z\overline{V}(0,z,s) = \overline{H}(z,s)K(z) + (z-1)\,K(z)\,\overline{H_0}(s). \quad (19)$$

Replacing $K(z) = e^{-\lambda d(1-z)}$ into (19) and simplifying, we have

$$Z\overline{V}(0,z,s) = \overline{H}(z,s)e^{-\lambda d(1-z)} + (z-1)\,e^{-\lambda d(1-z)}\overline{H_0}(s). \quad (20)$$

Integrating (18) between 0 and x,we get

$$\overline{V}(x,z,s) = \overline{V}(0,z,s)e^{-(s+\lambda-\lambda z)x - \int_0^\infty \beta(t)dt}. \quad (21)$$

We again integrate (21) by parts with respect to x and obtain

$\overline{V}(z,s)=\overline{V}(0,z,s)\left[\frac{1-\overline{B}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]$.
(22)
where
Next, we use (20) in (22),

$$\overline{V}(z,s)=\left[\frac{1-\overline{B}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]\left[\frac{e^{-\lambda d(1-z)}}{z}\right]\left[\overline{H}(z,s)+(z-1)\overline{H_0}(s)\right], \quad (23)$$

By virtue of (22), we obtain

$\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx=\overline{V}(0,z,s)\overline{B}(s+\lambda-\lambda z)$.
(24)
Using (22) in (15), We have
$(s+1)z\overline{H}(z,s)=z^{j+1}+z\overline{V}(0,z,s)\overline{B}(s+\lambda-\lambda z)$,(25)
Further we use (20) in (25) and obtain

$$(s+1)z\overline{H}(z,s)=z^{j+1}+\begin{bmatrix}\overline{H}(z,s)e^{-\lambda d(1-z)}+\\(z-1)e^{-\lambda d(1-z)}\overline{H_0}(s)\end{bmatrix}\overline{B}(s+\lambda-\lambda z).$$

(26)
On simplification (26) yields
$\overline{H}(z,s)=\left[\frac{z^{j+1}+\overline{B}(s+\lambda-\lambda z)(z-1)e^{-\lambda d(1-z)}\overline{H_0}(s)}{(s+1)z-\overline{B}(s+\lambda-\lambda z)e^{-\lambda d(1-z)}}\right]$.
(27)
Then we use equation (27) in equation (23), we get

$$\overline{V}(z,s)=\left[\frac{\left[\frac{1-\overline{B}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]e^{-\lambda d(1-z)}\left[z^{j+1}+(s+1)(z-1)\overline{H_0}(s)\right]}{(s+1)z-\overline{B}(s+\lambda-\lambda z)e^{-\lambda d(1-z)}}\right]. \quad (28)$$

Now to determine the only unknown constant $\overline{H_0}(s)$ which appears in the right side of equations (27) and (28), we note that it is easy to see that the denominator of right side of (27) and (28) has one zero inside the unit circle $|z|=1$. This zero is sufficient to determine the unknown $\overline{H_0}(s)$ enabling us completely to determine all desired probability generating functions.

5. The Steady State Results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument ***t*** wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property.

$$\lim_{n\to 0} s\overline{f}(s)=\lim_{t\to\infty} f(t)$$

(29)

In order to determine $\overline{H}(z,s)$ and $\overline{V}(z,s)$ completely, we have yet to determine the unknown $\overline{H_0}(s)$. For that purpose, we shall use the normalizing condition

$H(1)+V(1)=1$.
(30)

Thus multiplying both sides of equations (27) and (28) by s, taking limit as $s\to 0$, applying property (29) and simplifying we have

$H(z)=\left[\frac{z^{j+1}+\overline{B}(\lambda-\lambda z)(z-1)e^{-\lambda d(1-z)}H_0}{z-\overline{B}(\lambda-\lambda z)e^{-\lambda d(1-z)}}\right]$.
(31)

$$V(z)=\left[\frac{\left[\frac{1-\overline{B}(\lambda-\lambda z)}{\lambda-\lambda z}\right]\left[\frac{e^{-\lambda d(1-z)}}{z}\right]\left[z^{j+1}+z(z-1)H_0\right]}{z-\overline{B}(\lambda-\lambda z)e^{-\lambda d(1-z)}}\right]. \quad (32)$$

To determine the only H_0 which appears in the numerators of the right-hand side of (31) and (32), we shall use the normalizing condition

$$V(1)+H(1)=1 \quad (33)$$

However, since $H(z)$and $V(z)$ are indeterminate of the form $\frac{0}{0}$ at $z=1$. We apply LHopital's rule and obtain

$H(1)=\left[\frac{H_0}{1-\lambda E(v)-\lambda d}\right]$,
(34)
$V(1)=\left[\frac{E(v)H_0}{1-\lambda E(v)-\lambda d}\right]$,
(35)
where $E(v)$ is the mean vacation time of the server.

Using (34) and (35) in the normalizing condition (33), we obtain

$H_0=\left[\frac{1-\lambda E(v)-\lambda d}{1+E(v)}\right]$,
(36)
Equation (36) yields the condition

$$\lambda E(v)+\lambda d<1. \quad (37)$$

which is the stability condition under which the steady state condition shall exist.

$\rho=H(1)-H_0=\frac{\lambda E(v)+\lambda d}{1+E(v)}$.
(38)

Thus on substituting the value of H_0 from (36) into (34) and (35), we have now completely obtained all steady state probability generating functions.

6. The Mean Number in the system

Let $L_q(z)$ denote the mean number of customers in the queue. Then, we have $V(1) = \frac{d}{dz} P_q(z)$ at $z = 1$ where $P_q(z) = H(z) + V(z)$ is obtained by adding equations (31) and (32). Since $P_q(z)$ is indeterminate of the form $\frac{0}{0}$ at $z = 1$, we let

$$P_q(z) = \frac{N_q(z)}{D_q(z)} \quad (39)$$

where $N_q(z)$ and $D_q(z)$ respectively denote the numerator and the denominator of the right of equation (39). Also

$$N_q(z) = z^{j+1}(\lambda - \lambda z) + \overline{B(}\lambda - \lambda z)(\lambda - \lambda z) e^{-\lambda d(1-z)} H_0\,(z-1) + e^{-\lambda d(1-z)} H_0\,(z-1)\left[1 - \overline{B(}\lambda - \lambda z)\right] + z^j e^{-\lambda d(1-z)}\left[1 - \overline{B(}\lambda - \lambda z)\right], \quad (40)$$

$$D_q(z) = (\lambda - \lambda z)z - \overline{B(}\lambda - \lambda z)(\lambda - \lambda z)e^{-\lambda d(1-z)}. \quad (41)$$

Now $P_q(z)$is indeterminate of the form $\frac{0}{0}$. Also

$$L_q = \frac{d}{dz} P_q(z) = P_q'(1) = \lim_{z \to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2\left(D'(z)\right)^2}, \quad (42)$$

We carry out the required derivatives at $z = 1$, using the fact $\overline{B}(0) = 1, -\overline{B}'(0) = E(v), \overline{B}''(0) = E(v^2)$. After a lot of algebraic simplifications, we obtain

$$N'(z) = H_0 + H_0 E(v), \quad (43)$$

$$N''(z) = 2\lambda E(v)H_0 + 2\lambda d H_0 + \overline{B}''(0)\lambda H_0 + 2\lambda E(v)H_0 d - 2H_0\lambda + 2\overline{B}'(0)\lambda H_0, \quad (44)$$

$$D'(z) = [1 - \lambda E(v) - \lambda d], \quad (45)$$

$$D''(z) = \begin{bmatrix} -\lambda^2\overline{B}''(0) - 2\lambda^2 E(v)d - \lambda^2 d^2 - 2\lambda \\ -2\overline{B}'(0)\lambda^2 + 2\lambda^2 d \end{bmatrix}. \quad (46)$$

Using equations (43) to (46) into (42), we have obtained L_q in closed form, where H_0 has been found in equation (36).

Further, we find the average system size L using Little's formula. Thus, we have

$$L = L_q + \rho\,, \quad (47)$$

where L_q has been found in equation (42) and ρ is obtained from equation (38).

7. The Mean Waiting Time

Let W_q and W denote the mean waiting time in the queue and the system respectively. Then using Little's formula we obtain

$$W_q = \frac{L_q}{\lambda}\,, \quad (48)$$

$$W = \frac{L}{\lambda} \quad (49)$$

where L_q and L have been found in equations (42) and (47).

8. Conclusion

We have studied a single server M/D/1 queueing system with compulsory server vacation where the server provides deterministic service to all arriving customers and takes compulsory vacation after completion of each service. We have found the time-dependent probability generating functions in terms of their Laplace transforms using supplementary variable technique and have derived explicitly the corresponding steady state results. Further we find explicit expressions for the mean queue length and mean waiting time.

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References

[1] L. Kleinrock, Queueing Systems Vol. I, Theory, John Wiley and Sons, New York, 1976a.

[2] L. Kleinrock, Queueing Systems Vol. II, Computer Applications, John Wiley and Sons, New York, 1976b.

[3] J. W. Cohen, The Single Server Queue. 2nd edn. North – Holland, Amsterdam, 1982.

[4] S. S. Lavenberg, A Perspective on Queueing Models of Computer Performance in Queueing Theory and its Applications, Liber Amicorium for J.W.Cohen; CWI Monograph 7, North – Holland, Amsterdam, 1988.

[5] H. Takagi, Queueing Analysis: Vacation and Priority Systems, Vol. 1. North Holland, Amsterdam, 1991.

[6] Y. Levy and U.Yechiali, "Utilization of Idle Time in an M/G/1 Queueing System", Manag. Sci., vol 22, no. 2, pp. 202 – 211. Oct. 1975,

[7] D. Bertsimas and X. Papaconstantinou, "On the steady state solution of the $M/C_2(a,b)/ s$ queueing system", Transp. Sci., vol. 22, no. 2, pp.125 – 138, May 1988.

[8] K. C. Madan, "On an M/G/1 queue providing first essential service with customer's choice for general or deterministic second optional service", AIP Conference Proceedings, vol. 2253, no. 1, Aug. 2020.

[9] K. C. Madan, "On an $M^X/G/1$ queue with a random set up time, random breakdowns and delayed deterministic repairs", J. Math. Comput. Sci., vol. 11, no. 5, pp. 6568-6580, 2021.

[10] K. C. Madan and G. Choudhury, "An $M^X/G/1$ queue with Bernoulli vacation schedule under Restricted Admissibility policy", Sankhya, vol. 66, no. 1, pp. 175–193, Feb. 2004.

[11] K. C. Madan and K. Hadjar, "Time dependent and steady state solution of an $M^X/G/1$ queueing system with server's long and short vacations", J. Math. Comput. Sci., vol. 6, pp. 486 – 506, Apr. 2016.

[12] G. Choudhury and Chandi Ram Kalita, "An M/G/1 Queue with Two Types of General Heteronomous Service and Optional Repeated Service Subject to Server's Breakdown and Delayed Repair", Qual. Technol. Quant. Manag., vol. 15, pp. 1 – 33, May 2017.

[13] S. Vanitha, "M/G/1 Feedback Queue with Two Stage Heterogenous Service and Deterministic Server Vacations", International Journal of Applied Engineering Research, vol. 13, pp. 15899 – 15907, 2018.

[14] K. C. Madan, "An M/G/l Queueing System with Compulsory Server Vacations",Trabajos de Investigacion Operativa, vol. 7, pp. 105-115, Dec. 1992.

[15] V. Thangaraj and S. Vanitha, "M/G/1 queue with two stage heterogeneous service compulsory vacation and random breakdowns, Int. J. Contemp. Math. Sciences, vol. 5, no. 7, pp. 307 – 322, Jan. 2010.

[16] R. Kalyanaraman and V. Suvitha, "A single server Bernoulli vacation queue with two types of services with two types of services and with restricted admissibility", International Journal of Mathematical Modelling & Computations, vol. 2, pp. 261 – 276, 2016.

[17] S. Vanitha, "Analysis of M/G/1 Queue with Optional Services and Deterministic Repair", International Journal of Advanced Research in Engineering and Technology, vol. 11, no. 11, pp.1208 – 1217, 2020.

[18] B. D. Bunday, Basic Queueing Theory, Edward Arnold, Australia, 1986.

[19] B. R. K. Kashyap and M. L. Chaudhry,An Introduction to Queueing Theory, A & A Publications, Kingston, Ontarrio, Canada, 1988.

[20] U. N. Bhat, Elements of Applied Stochastic Processes, John Wiley and Sons. Inc.,1972.

[21] S. Vanitha, "M/D/1 Queue with Compulsory Vacation and Random Breakdowns", International Journal of Advanced Research in Engineering and Technology, vol. 11, no. 12, pp. 560 –567, 2020.