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An Improved Approach for Studying Oscillation of Second Order Nonlinear Mixed Functional Differential Equations with Sublinear Neutral Terms

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Abstract
The aim of this paper is to find some new oscillation conditions for second order nonlinear mixed functional differential equations with sublinear neutral terms of the form such that

$$\big(b(\phi)\, w'(\phi)\big)' - p(\phi)\, \mathrm{f}\big(v(\xi(\phi))\big) = 0, \qquad \phi \ge \phi_0 > 0,$$

where $w(\phi) = v(\phi) + q_1(\phi)\, v^{\theta}(\chi(\phi)) - q_2(\phi)\, v^{\theta}(\mu(\phi))$, θ is the ratio of odd positive integers with $0 < \theta < 1$. Moreover, $\chi(\phi) \le \phi \le \mu(\phi)$ and $\xi(\phi)$ is a mixed type deviating argument. The presented results simplify, generalize, and improve existing results reported in the literature. Examples are given to illustrate the importance of the results.

Keywords: Oscillation, Nonlinear, Second order, Sublinear neutral terms.

1. Introduction

This paper is concentrated on the oscillatory behavior of solutions to the second order nonlinear mixed functional differential equations with sublinear neutral terms of the form

$$\big(b(\phi)\, w'(\phi)\big)' - p(\phi)\, \mathrm{f}\big(v(\xi(\phi))\big) = 0, \qquad \phi \ge \phi_0 > 0, \qquad (1.1)$$

where $w(\phi) = v(\phi) + q_1(\phi)\, v^{\theta}(\chi(\phi)) - q_2(\phi)\, v^{\theta}(\mu(\phi))$ and b, q_1, q_2, p, χ, μ, ξ are continuous real-valued functions on $[\phi_0, \infty)$.

The following hypotheses are taken for granted throughout the entire paper, unless otherwise stated.

H1: θ is the ratio of odd positive integers with $0 < \theta < 1$;

H2: b $\in C^1([\phi_0, \infty), R^+)$ and

$$\mathrm{I}(\phi) = \int_{\phi_0}^{\phi} \frac{1}{b(s)}\, ds \longrightarrow \infty \text{ as } \phi \longrightarrow \infty; \qquad (1.2)$$

H3: p, q_1, $q_2 \in C([\phi_0, \infty), [0, \infty))$ and q is a positive continuous real-valued function with $0 < q_2 \le$ q < 1;

H4: f $\in C(R, R)$ and, $\exists\, k > 0$ such that $\mathrm{f}(x) \ge k\, x^{\alpha}$, $\forall$ x $\ne 0$, where α is the ratio of odd positive integers;

H5: $\chi, \mu \in C([\phi_0, \infty), R), \chi(\phi)) \le \phi \le \mu(\phi)$, χ *and* μ are strictly increasing functions and $\lim_{\phi \to \infty} \chi(\phi) = \lim_{\phi \to \infty} \mu(\phi) = \infty$;

H6: $\xi \in C^1([\phi_0, \infty), R)$, $\xi'(\phi) > 0$ and $\lim_{\phi \to \infty} \xi(\phi) = \infty$.

It is noteworthy that the deviating argument $\xi(\phi)$ is considered to be of mixed type, implying that its delayed component

$D_{\xi} = \{\phi \in [\phi_0, \infty): \xi(\phi) < \phi\}$

and its advanced component

$A_{\xi} = \{\phi \in [\phi_0, \infty): \xi(\phi) > \phi\}$

are both unbounded subsets of $[\phi_0, \infty)$.

By a solution of (1.1), we mean a function $v(\phi) \in C([\phi_v, \infty), R)$, with w, $\mathrm{b}(\phi) w'(\phi) \in C^1([\phi_v, \infty), R)$ that satisfies the differential equation (1.1) on $[\phi_v, \infty)$ where $\phi_v \ge \phi_0$. Without further mention, we will assume throughout that the solutions that satisfy $\sup\{|v(\phi)|: \phi \ge T\} > 0, \forall\, T \ge \phi_v$.

A solution $v(\phi)$ of (1.1) is called oscillatory if it has arbitrarily large zeros on $[\phi_v, \infty)$, that is, $\forall\, \phi_1 \in [\phi_v, \infty)\ \exists\ \phi_2 \ge \phi_1$ such that $v(\phi_2) = 0$; if not, it is called nonoscillatory, that is, if it is eventually positive or eventually negative. If every solution to (1.1) is oscillatory, then (1.1) is called oscillatory.

The set W of all nonoscillatory solutions of (1.1) is the union

$$W = \cup_{j=1}^{j=4} W_j,$$

where

$W_1 : w(\phi) > 0$ and $w'(\phi) < 0$; $W_2 : w(\phi) > 0$ and $w'(\phi) > 0$;
$W_3 : w(\phi) < 0$ and $w'(\phi) < 0$; $W_4 : w(\phi) < 0$ and $w'(\phi) > 0$.

We consider the situation that W $= \emptyset$ for (1.1), i.e., every nontrivial solution to (1.1) is oscillatory.

2. Main results

Lemma 2.1 [7]. If X, Y > 0, then

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \quad \text{for } \lambda > 1, \tag{2.1}$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0, \quad \text{for } 0 < \lambda < 1, \tag{2.2}$$

where equalities hold if and only if X $=$ Y.

We use the following notations for convenience:

$$G_1(\phi) := (1-\theta)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_1^{\frac{1}{1-\theta}}(\phi),$$

$$G_2(\phi) := (1-\theta)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_2^{\frac{1}{1-\theta}}(\phi),$$

and

$$G_3(\phi) := 1 - \left(\frac{G_1(\phi) - G_2(\phi)}{c_1\, I(\phi)}\right),$$

$$G_4(\phi) := 1 - \left(\frac{G_1(\phi) - G_2(\phi)}{c_2}\right),$$

in which the constants are $c_1 < 0$ and $c_2 > 0$.

We start with the preliminary lemma mentioned above, which is crucial to the proof of our main results.

Theorem 2.1. Assume q(ϕ) is such that

$$\lim_{\phi \to \infty} [G_1(\phi) - G_2(\phi)] = 0. \tag{2.3}$$

Furthermore, assume the following condition:

$$k \int_{\phi_0}^{\infty} \frac{1}{b(u)} \left(\int_u^{\infty} p(s)\, ds\right) du = \infty. \tag{2.4}$$

If $\exists$ $\{\phi_k\}$, $\{s_k\}$with ϕ_k, $s_k \longrightarrow \infty$ as $k \longrightarrow \infty$ such that $\phi_k \in D_{\xi}$ and $s_k \in A_{\xi}$ respectively,

$$\limsup_{k \longrightarrow \infty} \int_{\xi(\phi_k)}^{\phi_k} k\, n_1\, p(s) \left[I(\xi(\phi_k)) - I(\xi(s))\right] ds > 1, \tag{2.5}$$

and

$$\limsup_{k \longrightarrow \infty} \int_{s_k}^{\xi(s_k)} k\, n_2\, p(s) \left[I(\xi(s)) - I(\xi(s_k))\right] ds > 1, \tag{2.6}$$

$\forall$ k = 1, 2, 3, …, and $k > 0$, where n_1 and n_2 are constants, then (1.1) is oscillatory.

Proof. Assume on the contrary, that $v(\phi)$ is a nonoscillatory solution of (1.1). We can assume, without losing generality, that $v(\phi)$ is an eventually positive solution of (1.1) because, if $v(\phi)$ is a solution of (1.1), then $-\, v(\phi)$ is also a solution of (1.1). Hence, we get $v(\phi) > 0$, $v(\chi(\phi)) > 0$, $v(\mu(\phi)) > 0$, $v(\xi(\phi)) > 0$, for large sufficient ϕ. Then the following cases arise:

Case I. Suppose that $w(\phi) \in W_1$. According to the definition of $w(\phi)$, we have

$$\begin{aligned} v(\phi) &= w(\phi) - q_1(\phi)\, v^{\theta}(\chi(\phi)) + q_2(\phi)\, v^{\theta}(\mu(\phi)) \\ &= w(\phi) - \Big(q_1(\phi)\, v^{\theta}(\chi(\phi)) - q(\phi)\, v(\chi(\phi))\Big) + \Big(q_2(\phi)\, v^{\theta}(\mu(\phi)) - q(\phi)\, v(\mu(\phi))\Big). \end{aligned} \tag{2.7}$$

Using $0 < \lambda = \theta < 1$, $X = q_1^{\frac{1}{\theta}}(\phi)\, v(\chi(\phi))$ and $Y = \left(\frac{1}{\theta}\, q(\phi)\, q_1^{\frac{-1}{\theta}}(\phi)\right)^{\frac{1}{\theta-1}}$ in (2.2), we get

$$q_1(\phi)\, v^{\theta}(\chi(\phi)) - q(\phi)\, v(\chi(\phi)) \leq (1-\theta)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_1^{\frac{1}{1-\theta}}(\phi) := G_1(\phi). \tag{2.8}$$

Using $0 < \lambda = \theta < 1$, $X = q_2^{\frac{1}{\theta}}(\phi)\, v(\mu(\phi))$ and $Y = \left(\frac{1}{\theta}\, q(\phi)\, q_2^{\frac{-1}{\theta}}(\phi)\right)^{\frac{1}{\theta-1}}$ in (2.2), we get

$$q_2(\phi)\, v^{\theta}(\mu(\phi)) - q(\phi)\, v(\mu(\phi)) \leq (1-\theta)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_2^{\frac{1}{1-\theta}}(\phi) := G_2(\phi). \tag{2.9}$$

By applying (2.8) and (2.9) to (2.7), we get

$$v(\phi) \geq \left[1 - \left(\frac{G_1(\phi) - G_2(\phi)}{w(\phi)}\right)\right] w(\phi). \tag{2.10}$$

From (1.1),

$$\big(b(\phi)\, w'(\phi)\big)' = p(\phi)\, \mathrm{f}\Big(v\big(\xi(\phi)\big)\Big) \geq 0, \tag{2.11}$$

therefore b(ϕ)$w'(\phi)$ is increasing for $\phi \geq \phi_0$. It is simple to get

$$w(\phi) = \int_{\phi_0}^{\phi} \frac{b(s)\, w'(s)}{b(s)}\, ds \ge b(\phi_0)\, w'(\phi_0) \int_{\phi_0}^{\phi} \frac{1}{b(s)}\, ds = c_1\, I(\phi),$$

where $c_1 = \mathrm{b}(\phi_0)\, w'(\phi_0) < 0$ is a constant. Therefore

$$v(\phi) \ge \left[1 - \left(\frac{G_1(\phi) - G_2(\phi)}{c_1\, I(\phi)}\right)\right] w(\phi) := G_3(\phi)\, w(\phi).$$

(2.12)

Now, ∃ a constant $n_1 \ge 1$ such that

$$v\ (\phi) \ge n_1\ w(\phi).$$

(2.13)

When considering the fact that $\xi'(\phi) > 0$, it is simple to observe that $\phi_k \in D_\xi$ implies that $(\xi(\phi_k), \phi_k) \subset D_\xi$.

From (1.1), we have

$$\big(b(\phi)\, w'(\phi)\big)' \ge k\ \mathrm{p}(\phi)\ v^{\alpha}\big(\xi(\phi)\big) \ge k\ \mathrm{p}(\phi)\ v\big(\xi(\phi)\big).$$

Now integrating (1.1) from $\xi(\phi_k)$ to ϕ_k, and using (2.13), we get

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \ge k \int_{\xi(\phi_k)}^{\phi_k} p(s)\ v\big(\xi(s)\big)\, ds \ge k\, n_1 \int_{\xi(\phi_k)}^{\phi_k} p(s)\ w\big(\xi(s)\big)\, ds.$$

(2.14)

For s ∈ $(\xi(\phi_k), \phi_k)$, we get

$$w\big(\xi(s)\big) \ge \int_{\xi(s)}^{\xi(\phi_k)} \frac{-\, b(u)\, w'(u)}{b(u)}\, du \ge -\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \int_{\xi(s)}^{\xi(\phi_k)} \frac{1}{b(u)}\, du = -\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \left[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\right],$$

consequently, in view of (2.14) implies

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \ge k\, n_1 \int_{\xi(\phi_k)}^{\phi_k} p(s)\, w\big(\xi(s)\big)\, ds,$$

that is

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \ge -\, k\, n_1\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\right] ds,$$

or, equivalent to

$$1 \ge \int_{\xi(\phi_k)}^{\phi_k} k\, n_1\, p(s) \left[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\right] ds.$$

Taking limit supremum as $k \longrightarrow \infty$, we get a contradiction to (2.5), and hence, $W_1 = \emptyset$, that is, (1.1) is oscillatory.

Case II. Suppose that $w(\phi) \in W_2$. Since $w(\phi)$ is increasing, ∃ a constant $c_2 > 0$ such that $w(\phi) \ge c_2$ for large sufficient ϕ, and so, (2.10) becomes

$$v(\phi) \ge \left[1 - \left(\frac{G_1(\phi) - G_2(\phi)}{c_2}\right)\right] w(\phi) := G_4(\phi)\, w(\phi).$$

(2.15)

Now, ∃ a positive constant $n_2 \in (0,1)$ such that

$$v(\phi) \ge n_2\ w(\phi).$$

(2.16)

There exists $\{s_k\}$ such that $s_k \in A_\xi$, and given that $\xi(\phi)$ is increasing, which implies that $\big(s_k, \xi(s_k)\big) \subset A_\xi$. From (1.1), we have

$$\big(b(\phi)\, w'(\phi)\big)' \ge k\ \mathrm{p}(\phi)\ v^{\alpha}\big(\xi(\phi)\big) \ge k\ \mathrm{p}(\phi)\ v\big(\xi(\phi)\big).$$

(2.17)

Now, integrating (2.17) from s_k to $\xi(s_k)$ and using $(\mathrm{b}(\phi)\, w'(\phi))' > 0$ and (2.16), we get

$$b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \ge k \int_{s_k}^{\xi(s_k)} p(s)\, v\big(\xi(s)\big)\, ds \ge k\, n_2 \int_{s_k}^{\xi(s_k)} p(s)\, w\big(\xi(s)\big)\, ds.$$

(2.18)

For s ∈ $\big(s_k, \xi(s_k)\big)$, we get

$$w\big(\xi(s)\big) \ge \int_{\xi(s_k)}^{\xi(s)} \frac{b(u)\, w'(u)}{b(u)}\, du \ge b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \int_{\xi(s_k)}^{\xi(s)} \frac{1}{b(u)}\, du = b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \left[I\big(\xi(s)\big) - I\big(\xi(s_k)\big)\right]$$

(2.19)

and so, taking (2.18) into account, we get

$$b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \ge k\, n_2 \int_{s_k}^{\xi(s_k)} p(s)\, w\big(\xi(s)\big)\, ds,$$

that is

$$b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \ge k\, n_2\, b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \int_{s_k}^{\xi(s_k)} p(s) \left[I\big(\xi(s)\big) - I\big(\xi(s_k)\big)\right] ds,$$

or, equivalent to

$$1 \ge \int_{s_k}^{\xi(s_k)} k\, n_2\, p(s) \left[I\big(\xi(s)\big) - I\big(\xi(s_k)\big)\right] ds.$$

Taking limit supremum as $k \longrightarrow \infty$, we get a contradiction to (2.6), and hence, $W_2 = \emptyset$, that is, (1.1) is oscillatory.

Case III. Suppose that $w(\phi) \in W_3$. Here $w(\phi)$ satisfies
either

$$\lim_{\phi \longrightarrow \infty} w(\phi) = -\infty \qquad (2.20)$$

or

$$\lim_{\phi \longrightarrow \infty} w(\phi) = k_1 < 0. \qquad (2.21)$$

We say that (2.20) is true. If not, from the definition of $w(\phi)$, we get

$$v(\phi) \geq \left(\frac{-w\big(\mu^{-1}(\phi)\big)}{q}\right)^{\frac{1}{\theta}}, \qquad \phi \geq \phi_1.$$

It is obvious that $v(\phi)$ is bounded and ∃ a constant M_1 *such that* $v(\phi) \geq M_1 > 0 \; \forall \; \phi \geq \phi_2 \geq \phi_1$.

Using (2.17), we get

$$\big(b(\phi)\, w'(\phi)\big)' \geq M_1 \;\; k \; \mathrm{p}(\phi), \qquad \phi \geq \phi_2. \qquad (2.22)$$

Integrating (2.22) from ϕ to u and then put u ⟶ ∞, we have

$$-\,b(\phi)\, w'(\phi) \geq M_1 \;\; k \int_{\phi}^{\infty} p(s)\, ds.$$

Now, integrating this from ϕ_2 to ϕ and then put ϕ ⟶ ∞, we obtain

$$\lim_{\phi \longrightarrow \infty} w(\phi) \leq -\,M_1 \;\; k \int_{\phi_2}^{\infty} \frac{1}{b(u)} \Big(\int_{u}^{\infty} p(s)\, ds\Big)\, du.$$

This contradicts with (2.21) from (2.4). Hence (2.20) is true and $W_3 = \emptyset$.

Case IV. Suppose that $w(\phi) \in W_4$. Since b(ϕ)w'(ϕ) is positive and increasing, ∃ a constant $M_2 > 0$ such that

$$\mathrm{b}(\phi)\; w'(\phi) \geq M_2 \;\forall\; \phi \geq \phi_1. \qquad (2.23)$$

Integrating (2.23) from ϕ_1 to ϕ and taking $\phi \longrightarrow \infty$ yields

$$\lim_{\phi \longrightarrow \infty} w(\phi) \geq w(\phi_1) + M_2 \int_{\phi_1}^{\infty} \frac{1}{b(s)}\, ds,$$

which is impossible due to (1.2). Thus $W_4 = \emptyset$, and completes the proof of the theorem. ■

Our next idea is the further improvement of Theorem 2.1. Let us state the necessary lemmas for doing this.

Lemma 2.2. Suppose there exists $\{\phi_k\}$, $\phi_k \longrightarrow \infty$ as k⟶ ∞, such that $\phi_k \in D_\xi$. Let ∃ a $\gamma > 0$ such that

$$k\, n_1 \big[I(\phi) - I\big(\xi(\phi)\big)\big]\, I(\phi)\, b(\phi)\, p(\phi) \geq \gamma, \quad on \;\; \big(\xi\big(\xi(\phi_k)\big), \xi(\phi_k)\big), \; \forall \; \mathrm{k} = 1, 2, 3, \ldots \qquad (2.24)$$

If $v(\phi)$ is a positive solution of (1.1) such that $w(\phi) \in W_1$, then $-I^{\gamma}(\phi)\, b(\phi)\, w'(\phi)$ is decreasing on $\big(\xi\big(\xi(\phi_k)\big), \xi(\phi_k)\big)$.

Proof. Since $-\mathrm{b}(\phi) w'(\phi)$ is decreasing, then it is simple to see that

$$w\big(\xi(\phi)\big) \geq \int_{\xi(\phi)}^{\phi} \frac{-\,b(u)\, w'(u)}{b(u)}\, du \geq -\,b(\phi)\, w'(\phi) \int_{\xi(\phi)}^{\phi} \frac{1}{b(u)}\, du = -\,b(\phi)\, w'(\phi) \big[I(\phi) - I\big(\xi(\phi)\big)\big].$$

From (2.13) and (2.17),

$$\big(b(\phi)\, w'(\phi)\big)' \geq k\, n_1\, p(\phi)\, b(\phi) \big(-w'(\phi)\big) \big[I(\phi) - I\big(\xi(\phi)\big)\big].$$

If we confine $\phi \in (\xi(\phi_k), \phi_k) \subset D_\xi$, k = 1, 2, 3, …, then from (2.24),

$$I(\phi) \big(b(\phi)\, w'(\phi)\big)' \geq \gamma \big(-w'(\phi)\big)$$

and hence

$$\big(-\,I^{\gamma}(\phi)\, b(\phi)\, w'(\phi)\big)' \leq -\,\gamma\, I^{\gamma-1}(\phi)\, I'(\phi) \big(b(\phi)\, w'(\phi)\big) - I^{\gamma}(\phi) \big(b(\phi)\, w'(\phi)\big)' \leq 0$$

completes the proof of the lemma. ■

Lemma 2.3. Suppose there exists $\{s_k\}$, $s_k \longrightarrow \infty$ as $k \longrightarrow \infty$, such that $s_k \in A_\xi$. Let ∃ a $\delta > 0$ such that

$$k\, n_2 \big[I\big(\xi(\phi)\big) - I(\phi)\big]\, I(\phi)\, b(\phi)\, p(\phi) \geq \delta, \; on \;\; \Big(\xi(s_k), \xi\big(\xi(s_k)\big)\Big), \quad \forall \;\; \mathrm{k} = 1, 2, 3, \ldots \qquad (2.25)$$

If $v(\phi)$ is a positive solution of (1.1) such that $w(\phi) \in W_2$, then $I^{-\delta}(\phi)\, b(\phi)\, w'(\phi)$ is increasing on $\Big(\xi(s_k), \xi\big(\xi(s_k)\big)\Big)$.

The proof is similar to that of Lemma 2.2.

Theorem 2.2. Suppose (2.4) holds and ∃ q(ϕ) such that (2.3) holds. Furthermore, suppose ∃ $\{\phi_k\}$, $\{s_k\}$ with ϕ_k, $s_k \longrightarrow \infty$ as $k \longrightarrow \infty$ such that $\phi_k \in D_\xi$ and $s_k \in A_\xi$.

If

$$\limsup_{k \to \infty} k\, n_1\, I^{\gamma}\big(\xi(\phi_k)\big) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds > 1 \qquad (2.26)$$

and

$$\limsup_{k \to \infty} k\, n_2\, I^{-\delta}\big(\xi(s_k)\big) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right] ds > 1 \qquad (2.27)$$

where γ and δ are defined by (2.24) and (2.25) respectively, n_1 and n_2 are constants, then (1.1) is oscillatory.

Proof. Assume on the contrary, that $v(\phi)$ is an eventually positive solution of (1.1). Then the following cases arise:

Case I. Suppose that $w(\phi) \in W_1$. By lemma 2.2, the function $-I^{\gamma}(\phi) b(\phi) w'(\phi)$ is decreasing on $\big(\xi(\xi(\phi_k)), \xi(\phi_k)\big)$. Thus, for s $\in (\xi(\phi_k), \phi_k)$, we obtain

$$w\big(\xi(s)\big) \geq \int_{\xi(s)}^{\xi(\phi_k)} \frac{-b(u)\, I^{\gamma}(u)\, w'(u)}{b(u)\, I^{\gamma}(u)}\, du$$

$$\geq -b\big(\xi(\phi_k)\big)\, I^{\gamma}\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \int_{\xi(s)}^{\xi(\phi_k)} \frac{1}{b(u)\, I^{\gamma}(u)}\, du$$

$$\geq -b\big(\xi(\phi_k)\big)\, I^{\gamma}\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right].$$

Using the above inequality in (2.14), we get

$$-b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \geq k\, n_1 \Big(-b\big(\xi(\phi_k)\big) I^{\gamma}\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big)\Big) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds,$$

that is,

$$1 \geq k\, n_1\, I^{\gamma}\big(\xi(\phi_k)\big) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds,$$

which contradicts the condition (2.26), and hence $W_1 = \emptyset$.

Case II. Suppose that $w(\phi) \in W_2$. By lemma 2.3, the function $I^{-\delta}(\phi)\, b(\phi)\, w'(\phi)$ is increasing on $\Big(\xi(s_k), \xi\big(\xi(s_k)\big)\Big)$. Thus, for s $\in \big(s_k, \xi(s_k)\big)$, we obtain

$$w\big(\xi(s)\big) \geq \int_{\xi(s_k)}^{\xi(s)} \frac{b(u)\, I^{-\delta}(u)\, w'(u)}{b(u)\, I^{-\delta}(u)}\, du$$

$$\geq b\big(\xi(s_k)\big)\, I^{-\delta}\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \int_{\xi(s_k)}^{\xi(s)} \frac{1}{b(u)\, I^{-\delta}(u)}\, du$$

$$\geq b\big(\xi(s_k)\big)\, I^{-\delta}\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right].$$

Using the last inequality in (2.18), we get

$$b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \geq k\, n_2 \Big(b\big(\xi(s_k)\big)\, I^{-\delta}\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big)\Big) \int_{s_k}^{\xi(s_k)}$$

,

that is,

$$1 \geq k\, n_2\, I^{-\delta}\big(\xi(s_k)\big) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right] ds,$$

which contradicts the condition (2.27), and hence $W_2 = \emptyset$.

The remaining cases i.e. Case III and Case IV are identical to those in Theorem 2.1, and completes the proof of the theorem. ■

3. Examples

In this section, we use a few examples to show how our main result is applied.

Example 3.1. Consider the second order differential equation of the form

$$\left(v(\phi) + \phi^{\frac{1}{3}}\, v^{\frac{1}{5}}\left(\frac{\phi}{2}\right) - \frac{1}{\phi^2}\, v^{\frac{1}{5}}(3\phi)\right)'' - \frac{c}{\phi^2}\, v\left(\phi\left(1 + \frac{1}{3}\cos(\ln\phi)\right)\right) = 0, \ \ \phi > 0, \ \ c > 0. \qquad (3.1)$$

This is a special form of (1.1), where b$(\phi) = 1$, $w(\phi) = v(\phi) + \phi^{\frac{1}{3}}\, v^{\frac{1}{5}}\left(\frac{\phi}{2}\right) - \frac{1}{\phi^2}\, v^{\frac{1}{5}}(3\phi)$, p$(\phi) = \frac{c}{\phi^2}$ with c is a constant, $\xi(\phi) = \phi\left(1 + \frac{1}{3}\cos(\ln\phi)\right)$, $\alpha = 1, k = 1$, f$\Big(v\big(\xi(\phi)\big)\Big) = v\big(\xi(\phi)\big)$, $\phi_0 = 0$, $0 < \theta = \frac{1}{5} < 1$, $q_1(\phi) = \phi^{\frac{1}{3}}$, $q_2(\phi) = \frac{1}{\phi^2}$, $\chi(\phi) = \frac{\phi}{2}$, $\mu(\phi) = 3\,\phi$. Clearly, the deviating argument $\xi(\phi)$ is of mixed type.

If we choose $\phi_k = e^{\pi + 2k\pi}$, k = 1, 2, 3, …, then $\phi_k \in D_{\xi}$ and moreover $\xi(\phi_k) = \frac{2}{3}\, e^{\pi + 2k\pi}$.

Condition (2.5) takes the form

$$\limsup_{k \to \infty} n_1 \int_{\xi(\phi_k)}^{\phi_k} \frac{c}{s^2} \left[\frac{2}{3}\, e^{\pi + 2k\pi} - s\left(1 + \frac{1}{3}\cos(\ln s)\right)\right] ds$$

$$= \limsup_{k \to \infty} c\, n_1 \int_{\xi(\phi_k)}^{\phi_k} \left[\frac{2}{3}\, e^{\pi + 2k\pi}\, \frac{1}{s^2} - \frac{1}{s} - \left(\frac{1}{3}\right)\left(\frac{1}{s}\right) \cos(\ln s)\right] ds$$

$$= \limsup_{k \to \infty} c\, n_1 \left[-\frac{2}{3}\, e^{\pi+2k\pi} \left(\frac{1}{s}\right)_{\xi(\phi_k)}^{\phi_k} - (\ln s\,)_{\xi(\phi_k)}^{\phi_k} - \frac{1}{3}\, (\sin(\ln s\,))\,)_{\xi(\phi_k)}^{\phi_k} \right]$$

$$= c\, n_1 \left[\frac{1}{3} + \ln\frac{2}{3} - \frac{1}{3} \sin\left(\ln\frac{2}{3}\right) \right]$$

$$> 1,$$

which (by Theorem 2.1) guarantees that $W_1 = \emptyset$ $\left(i.e. for\ c > \frac{1}{n_1}\ 16.84911682 \right)$.

On the other hand, if we choose $s_k = e^{2k\pi}$, k $= 1, 2, 3, \ldots$, then $s_k \in A_\xi$ and moreover $\xi(s_k) = \frac{4}{3}\, e^{2k\pi}$.

Condition (2.6) takes the form

$$\limsup_{k \to \infty} n_2 \int_{s_k}^{\xi(s_k)} \frac{c}{s^2} \left[s\left(1 + \frac{1}{3} \cos(\ln s\,)\right) - \frac{4}{3}\, e^{2k\pi} \right] ds$$

$$= \limsup_{k \to \infty} c\, n_2 \int_{s_k}^{\xi(s_k)} \left[\frac{1}{s} + \left(\frac{1}{3}\right)\left(\frac{1}{s}\right) \cos(\ln s\,) - \frac{4}{3}\, e^{2k\pi}\, \frac{1}{s^2} \right] ds$$

$$= \limsup_{k \to \infty} c\, n_2 \left[(\ln s\,)_{s_k}^{\xi(s_k)} + \frac{1}{3}\, (\sin(\ln s))_{s_k}^{\xi(s_k)} + \frac{4}{3}\, e^{2k\pi} \left(\frac{1}{s}\right)_{s_k}^{\xi(s_k)} \right]$$

$$= c\, n_2 \left[\ln\frac{4}{3} + \frac{1}{3} \sin\left(\ln\frac{4}{3}\right) - \frac{1}{3} \right]$$

$$= c\, n_2 \left[-\frac{1}{3} + \ln\frac{4}{3} + \frac{1}{3} \sin\left(\ln\frac{4}{3}\right) \right]$$

$$> 1,$$

which ensures that $W_2 = \emptyset$ $\left(i.e. for\ c > \frac{1}{n_2}\ 20.43923418 \right)$.s

Moreover, we can verify that

$$c \int_0^\infty \int_u^\infty \frac{1}{s^2}\, ds du = \infty.$$

that means, (2.4) is also satisfied. Based on the two criteria, we can see that the condition c > $\frac{1}{n_2}$ 20.43923418 suggests that (3.1) oscillates.

Example 3.2. We consider again the differential equation (3.1).

At first, by theorem 2.2, we shall show that $W_1 = \emptyset$ for c $\geq \frac{1}{n_1}$ 11.68030228. So, we set c $= \frac{1}{n_1}$ 11.68030228. Again taking $\phi_k = e^{\pi+2k\pi}$, k $= 1, 2, 3, \ldots$, then $\xi(\phi_k) = \frac{2}{3}\, e^{\pi+2k\pi}$ and $\xi(\xi(\phi_k)) = \left(\frac{2}{3} - \frac{2}{9} \cos\left(\ln\frac{2}{3}\right)\right) e^{\pi+2k\pi}$. In view of Lemma 2.2, the condition (2.24) reduces to

$$-\frac{1}{3}\, n_1\, c \cos(\ln\phi) \geq \gamma, \quad on \quad (\xi(\xi(\phi_k)), \xi(\phi_k)\,), \ k = 1, 2, 3, \ldots.$$

Since $-\frac{1}{3}\, n_1\, c \cos(\ln\phi)$ is increasing function on $(\xi(\xi(\phi_k)), \xi(\phi_k)\,)$, we have

$$\gamma = -\frac{1}{3}\, n_1\, c \cos\left(\ln\left(\xi(\xi(\phi_k))\right)\right) = \frac{1}{3}\, n_1\, c \cos\left(\ln\left(\frac{2}{3} - \frac{2}{9} \cos\left(\ln\left(\frac{2}{3}\right)\right)\right)\right) = 2.791910726$$

so that γ is the same on each interval $(\xi(\xi(\phi_k)), \xi(\phi_k)\,)$.

Now, we verify the condition (2.26).

$$\limsup_{k \to \infty} n_1\, \xi^\gamma(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds$$

$$= \limsup_{k \to \infty} \frac{n_1 c}{1-\gamma} \left(\frac{2}{3}\, e^{\pi+2k\pi}\right)^\gamma \int_{\xi(\phi_k)}^{\phi_k} \frac{1}{s^2} \left[\left(\frac{2}{3}\, e^{\pi+2k\pi}\right)^{1-\gamma} - \left(s\left(1 + \frac{1}{3} \cos(\ln s\,)\right)\right)^{1-\gamma} \right] ds$$

$$= \limsup_{k \to \infty} \frac{n_1 c}{1-\gamma} \left[\frac{1}{3} - \left(\frac{2}{3}\, e^{\pi+2k\pi}\right)^\gamma \int_{\xi(\phi_k)}^{\phi_k} s^{-1-\gamma} \left(1 + \frac{1}{3} \cos(\ln s\,)\right)^{1-\gamma} ds \right].$$

Substituting s $= e^{\pi+2k\pi}\, \phi$, the above equation, we get

$$\limsup_{k \to \infty} n_1\, \xi^\gamma(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds$$

$$= \frac{n_1 c}{1-\gamma} \left[\frac{1}{3} - \left(\frac{2}{3}\right)^\gamma \int_{\frac{2}{3}}^{1} \phi^{-1-\gamma} \left(1 + \frac{1}{3} \cos(\ln(e^{\pi+2k\pi}\, \phi)\,)\right)^{1-\gamma} d\phi \right]$$

$$= \frac{n_1 c}{1-\gamma} \left[\frac{1}{3} - \left(\frac{2}{3}\right)^\gamma \int_{\frac{2}{3}}^{1} \phi^{-1-\gamma} \left(1 - \frac{1}{3} \cos(\ln\phi\,)\right)^{1-\gamma} d\phi \right].$$

Using Matlab for calculation, we get

$$\int_{\frac{2}{3}}^{1} \phi^{-1-\gamma} \left(1 - \frac{1}{3} \cos(\ln\phi\,)\right)^{1-\gamma} d\phi = 1.50985$$

with $\gamma = 2.791910726$

and finally, we get

$$\limsup_{k \to \infty} n_1\, \xi^\gamma(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds$$

$= 1.000000004 > 1$

which by Theorem 2.2 guarantees that $W_1 = \emptyset$.

At second, by theorem 2.2, we shall show that $W_2 = \emptyset$ for c $\geq \frac{1}{n_2}$ 13.6964. So, we set c = $\frac{1}{n_2}$ 13.6964. Again taking $s_k = e^{2k\pi}$, k = 1, 2, 3, …, then $\xi(s_k) = \frac{4}{3}\, e^{2k\pi}$ and $\xi(\xi(s_k)) = \left(\frac{4}{3}+\frac{4}{9}\cos\left(\ln\frac{4}{3}\right)\right) e^{2k\pi}$. In view of Lemma 2.3, the condition (2.25) reduces to

$$\frac{1}{3}\, n_2\, c\, \cos(\ln\phi) \geq \delta, \quad on \quad \left(\xi(s_k), \xi(\xi(s_k))\right), \; k = 1, 2, 3, \ldots.$$

Since $\frac{1}{3}\, n_2\, c \cos(\ln\phi)$ is decreasing function on $\left(\xi(s_k), \xi(\xi(s_k))\right)$, we have

$$\delta = \frac{1}{3}\, n_2\, c\, \cos\left(\ln\left(\xi(\xi(s_k))\right)\right) = \frac{1}{3}\, n_2\, c \cos\left(\ln\left(\frac{4}{3}+\frac{4}{9}\cos\left(\ln\left(\frac{4}{3}\right)\right)\right)\right) = 3.855850576$$

so that δ is the same on each interval $\left(\xi(s_k), \xi(\xi(s_k))\right)$.

Now, we verify (2.27).

$$\limsup_{k\to\infty} n_2\, \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds$$
$$= \limsup_{k\to\infty} \frac{n_2\, c}{1+\delta} \left(\frac{4}{3}\, e^{2k\pi}\right)^{-\delta} \int_{s_k}^{\xi(s_k)} \frac{1}{s^2}\left[\left(s\left(1+\frac{1}{3}\cos(\ln s)\right)\right)^{1+\delta} - \left(\frac{4}{3}\, e^{2k\pi}\right)^{1+\delta}\right] ds$$
$$= \limsup_{k\to\infty} \frac{n_2\, c}{1+\delta} \left[\left(\frac{4}{3}\, e^{2k\pi}\right)^{-\delta} \int_{s_k}^{\xi(s_k)} s^{\delta-1}\left(1+\frac{1}{3}\cos(\ln s)\right)^{1+\delta} ds - \frac{1}{3}\right].$$

Using s $= e^{2k\pi}\,\phi$ in the last integral, we get

$$\limsup_{k\to\infty} n_2\, \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds$$
$$= \frac{n_2\, c}{1+\delta}\left[\left(\frac{4}{3}\right)^{-\delta} \int_1^{\frac{4}{3}} \phi^{\delta-1}\left(1+\frac{1}{3}\cos\left(\ln\left(e^{2k\pi}\,\phi\right)\right)\right)^{1+\delta} d\phi - \frac{1}{3}\right]$$
$$= \frac{n_2\, c}{1+\delta}\left[-\frac{1}{3} + \left(\frac{4}{3}\right)^{-\delta} \int_1^{\frac{4}{3}} \phi^{\delta-1}\left(1+\frac{1}{3}\cos(\ln\phi)\right)^{1+\delta} d\phi\right].$$

Using Matlab for calculation, we get

$$\int_1^{\frac{4}{3}} \phi^{\delta-1}\left(1+\frac{1}{3}\cos(\ln\phi)\right)^{1+\delta} d\phi = 2.0857$$

with $\delta = 3.855850576$

and finally, we get

$$\limsup_{k\to\infty} n_2\, \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds = 1.000006901 > 1$$

which by Theorem 2.2 guarantees that $W_2 = \emptyset$. Hence, based on two criteria, we can see that the condition c $> \frac{1}{n_2}$ 13.6964 suggests that (3.1) oscillates, while Theorem 2.1 requires c > $\frac{1}{n_2}$ 20.43923418.

4. Conclusion

This paper studies a class of second order nonlinear mixed functional differential equations with sublinear neutral terms and determines some criteria for oscillation. Also, we obtained stronger conditions for equation 3.1 to be oscillatory and hence, a further improvement of Theorem 2.1 is Theorem 2.2.

References

[1] Agarwal RP, Bohner M, Li TX, Zhang CH. Oscillation of second-order differential equations with sublinear neutral term. Carpathian Journal of Mathematics 2014; 30 (1): 1-6.

[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic Publishers, The Netherlands, 2002, 672pp.

[3] Agarwal RP, Zhang CH, Li TX. Some remarks on oscillation of second order neutral differential equations. Applied Mathematics and Computation 2016; 274: 178-181.

[4] B. Baculíková, Oscillation of second order half-linear differential equations with deviating arguments of mixed type, Applied Mathematics Letters 119 (2021) 107228.

[5] Bohner M, Sudha B, Tangavelu K, Thandapani E. Oscillation criteria for second-order differential equations with superlinear neutral term. Nonlinear Studies 2019; 26 (2): 425-434.

[6] Graef, J.R., Özdemir, O., Kaymaz, A., Tunç, E.: Oscillation of damped second-order linear mixedneutral differential equations. Monatsh. Math. 194, 85–104 (2021)

[7] Hardy GH, Littlewood IE, Polya G. Inequalities. Cambridge University Press, Cambridge,1959.

[8] Lin XY, Tang XH. Oscillation of solutions of neutral differential equations with a superlinear neutral term. Applied Mathematics Letters 2007; 20: 1016-1022.

[9] Moaaz O, Awrejcewicz J, Muhib A. Establishing new criteria for oscillation of odd-order nonlinear differential equations. Mathematics 2020; 8 (6): 1-15. doi: 10.3390/math8060937

[10] Muhib A, Elabbasy EM, Moaaz O. New oscillation criteria for differential equations with subinear and superlinear neutral terms. Turkish Journal of Mathematics 2021; 45: 919-928.

[11] Orhan Özdemir, Ayla Kılıç, Oscillation of Second-Order Functional Differential Equations with Superlinear Neutral Terms, Bulletin of the Malaysian Mathematical Sciences Society volume 45, pages 83–99 (2022).

[12] Rama Renu, Sridevi Ravindran, Oscillation criterion of first order non-linear delay differentialequation with several deviating arguments, Turkish online Journal of Qualitative inquiry, Volume 12, Issue 7, July 2021:3004-3015.

[13] Rama Renu, Sridevi Ravindran, An improved oscillation results of first order non-linear delaydifferential equation with several non-monotone arguments, International Journal of Special Education vol.37, No.3S, 2022.

[14] Said R. Grace, John R. Graef, Tongxing Li, Ercan Tunc, Oscillatory Behaviour Of Second-Order Nonlinear Differential Equations With Mixed Neutral Terms, Tatra Mt. Math. Publ. 79 (2021), 119–134.

[15] Shan SHI, Zhenlai HAN, Oscillation of second order mixed functional differential equations with sublinear and superlinear neutral terms, Turk J Math (2022) 46: 3045 -3056.

[16] E. Tunç and O. Özdemir, Oscillatory behavior of second-order damped differential equations with a superlinear neutral term, Opuscula Math. 40, 629–639 (2020).