Equitable Colouring of Lexicographic Product of Semi-Total Point Graph with Certain Graphs

G. MURALI [#1], M. S. PAULRAJ [#2]
[#1]Assistant Professor, PG and Research Department of Mathematics,
Agurchand Manmull Jain College, University of Madras, Chennai, Tamilnadu, India.
[#2]Associate Professor, PG and Research Department of Mathematics,
Agurchand Manmull Jain College, University of Madras, Chennai, Tamilnadu, India.

Abstract
Equitable colouring of a graph G is a proper colouring of graph if the number of vertices with any two-colour classes vary by at most one and the equitable chromatic number is the minimum number of colour classes and is symbolized by $\chi_=(G)$. This paper attempts to establish the acceptance of equitable colouring to the lexicographic product of two graphs $T_2(G)$ and H, denoted by $T_2(G) \circ H$. First, G can be considered as the path and H as the path, cycle, complete and bipartite graph. Secondly, G as the cycle and H as the path, cycle, complete and bipartite graph.
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1. Introduction

All graphs taken for consideration here are finite, simple and undirected. Consider a graph G with $V(G)$ and $E(G)$ denote respectively the vertex set and edge set of G. One of the most awe instigating concepts in graph theory are the colouring problem having extensive applications. Meyer in 1973[12] developed an extension to proper coloring. An equitable coloring if a graph *G* is defined as if any two-colour sets of vertices of G vary by at most one. For any graph G, we have $\chi_=(G) \geq \chi(G)$. Felix Hausdorff in 1914 first introduced the notion of lexicographic product. In 1959, Harary [8], celebrated graph theorist named the lexicographic product as the composition. Dennis Geller and Saul Stahl [7] in 1975 established the chromatic number for lexicographic product of graphs. Feigenbaum and Schaffer [4] in 1986 shown that the problem of graph isomorphism is alike in intricacy to the lexicographic product of graphs.

For any two graphs G and H, the colouring of a lexicographic product $\chi(G \circ H)$ is equal to the b-fold chromatic number of G, where b is equal to the colouring of H [4]. A b-fold k-colouring of G is an assignment of b distinct colours to every vertex from a set of G and k colours, such that adjacent vertices do not have any colours in common. The b-fold chromatic number, $\chi_b(G)$, where the minimum number is k, where a b-fold k colouring exits [11]. In general, the product of graphs is non-commutative, but in some cases, namely in complete graphs and totally disconnected graphs, they commute [9]. Enthused by this concept, an effort is made to show the acceptance of equitable colouring for the lexicographic product of different kinds of graphs. An application to optimizing garbage collection, time tabling, job allotment and etc.

2. Preliminaries

Definition 2.1[12] A graph G is said to be equitably $k-$colourable if its vertices can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $\left||V_i| - |V_j|\right| \leq 1$, $i, j = 1,2, \ldots, k$. The smallest integer k for which G is equitably k- colourable is known as the equitable chromatic number of G and denoted by $\chi_=(G)$.

Definition 2.2[15] The Semi-Total point graph $T_2(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$. For $a, b \in V(T_2(G))$, a and b are adjacent if and only if the following conditions hold.
(i) $a, b \in V(G)$, a, b are adjacent vertices of G.
(ii) $a \in V(G)$ and $b \in E(G)$, b is incident with a in G.

Definition 2.3[7] The lexicographic product, $G \circ H$ of graphs G and H is a graph such that the vertex set of $G \circ H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (l, m) and (u, v) are adjacent in $G \circ H$ iff either l is adjacent with u in G or $l = u$ and m is adjacent with v in H.

Theorem 2.1 [5] For any graph G, $\chi_=(G) \leq \Delta(G) + 1$.

Theorem 2.2 [5] If G is a connected graph, different from C_{2n+1} and K_n $\forall\, n \geq 1$, then $\chi_=(G) \leq \Delta(G)$.

Theorem 2.3 [6] (The Equitable Δ-Colouring Conjecture-E ΔCC) A connected graph G is equitable $\Delta(G)$ − colourable if G is different from C_{2n+1}, K_n and $K_{2n+1,2n+1}$ $\forall\, n \geq 1$.

Theorem 2.4 [7] If G has k disjoint colour sets $\{S_i | 1 \leq i \leq k\}$ whose union is independent, then for any graph H, $\chi(G \circ H) \leq \chi(G)\chi(H) - [\chi(H)/k](k-1)$.

Theorem 2.5 [7] For any graph G, if $\chi(G) = 1$, then $\chi(G \circ H) = \chi(H)$; if $\chi(G) > 1$, then $\chi(G \circ H) \geq \chi(G) + 2\chi(H) - 2$.

3. Equitable Colouring of Lexicographic Product of Semi-Total Point graphs

Let $V(G) = \{u_i\,; 0 \leq i \leq m-1\}$, $V(T_2(G)) = \{(u_i),(e_i)\,; 0 \leq i \leq m-1\}$ and $V(H) = \{v_j\,; 0 \leq j \leq n-1\}$ are the set of vertices of G, $T_2(G)$ and H respectively. Let $V(T_2(G) \circ H) = \bigcup_{i=0}^{m-1}\{(z_{i,j}),(z^*_{i,j}); 0 \leq j \leq n-1\}$ (where $z_{i,j}$ be the vertices in the form of $u_i v_j$ and $z^*_{i,j}$ be the vertices in the form of $e_i v_j$) is the set of vertices of $T_2(G) \circ H$. In case H is bipartite graphs, $V(T_2(G) \circ H) = \bigcup_{i=0}^{m-1}\left\{(z_{i,j}),(z^*_{i,j}),(z_{i,j'}),\left(z^*_{i,j'}\right); 0 \leq j \leq p-1, 0 \leq j' \leq q-1\right\}$, where $z_{i,j'}$ be the vertices in the form of $u_i v_{j'}$, $z^*_{i,j'}$ be the vertices in the form of $e_i v_{j'}$ and also $V(H) = \{v_j : 1 \leq j \leq p\}$ and $V'(H) = \{v_{j'} : 1 \leq j' \leq q\}$.

The notion $i \equiv 02$ refers to $i \equiv 0(mod 2)$.

Theorem 3.1. Let G and H be any two graphs, where G is a semi-total point graph of path, $T_2(P_m)$ on $m \geq 2$ vertices, then the equitable colouring of the lexicographic product of G and H are

(i) $\chi_=(G \circ P_n) = 6\,; m \equiv 23$ and $n = 2k, k \geq 1$.

(ii) $\chi_=(G \circ C_n) = 6\,; m \equiv 26, m \geq 3$ and $n = 2k, k \geq 2$.

(iii) $\chi_=(G \circ K_{p,p}) = 6\,; m = 3k+2, k \geq 0$.

(iv) $\chi_=(G \circ K_n) = 3n\,; n \geq 2$.

Proof.

Define the map $\alpha : V(G \circ H) \rightarrow \{0,1,2,\ldots,l\}\ \forall\, l \in N$

The proof of the theorem is divided into four cases,

Case 1:

Let G be a semi-total point graph of path with $2m-1$ vertices and H be a path with n vertices then the number of vertices and edges of the lexicographic product of two graphs G and H are $n(2m-1)$ and $m(3n^2+2n-2) - 3n^2 - n + 1$ respectively, corresponding vertex set and edge set are given by

$$V(G \circ H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1} z^*_{i,j}\right)$$

$$E(G \circ H) = \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1}\bigcup_{k=0}^{n-1}(x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk})\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-2} x_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-2} x^*_{i,j}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{i+1,k})$, $x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{i+1,k})$, $\forall\, 0 \leq k \leq n-1$.

Also $x_{i,j}$ is the edge $(z_{i,j})(z_{i,j+1})$ and $x^*_{i,j}$ is the edge $(z^*_{i,j})(z^*_{i,j+1})$.

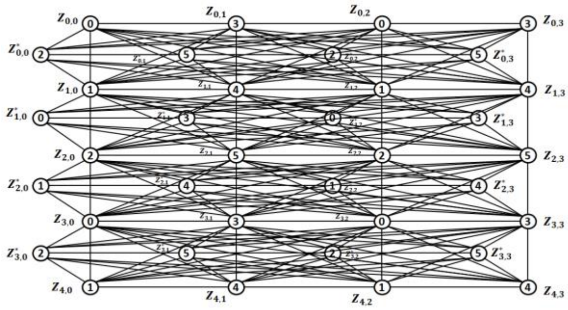

Fig.1 Example of lexicographic product $T_2(P_5) \circ P_4$

If $m \equiv 23$ and $n = 2k, k \geq 1$, then the colouring of vertices and partition the vertex set of V as below,

$$\alpha(z_{i,j}) = i(mod\ 3) + 3[j(mod\ 2)]$$

$$\alpha(z^*_{i,j}) = (i+2)(mod\ 3) + 3[j(mod\ 2)]$$

and

$$V_0 = \{z_{i\equiv 03, j\equiv 02}\,, z^*_{i\equiv 13, j\equiv 02}\}$$

$$V_1 = \{z_{i\equiv 13, j\equiv 02}\,, z^*_{i\equiv 23, j\equiv 02}\}$$

$$V_2 = \{z_{i\equiv 23, j\equiv 02}\,, z^*_{i\equiv 03, j\equiv 02}\}$$

$$V_3 = \{z_{i\equiv 03, j\equiv 12}\,, z^*_{i\equiv 13, j\equiv 12}\}$$

$$V_4 = \{z_{i\equiv 13, j\equiv 12}\,, z^*_{i\equiv 23, j\equiv 12}\}$$

and $V_5 = \{z_{i\equiv 23, j\equiv 12}\,, z^*_{i\equiv 03, j\equiv 12}\}$

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5|$ $= \frac{n(2m-1)}{6}$, it holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i,j). $\chi_=(G \circ H) \leq 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq \chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq 6$. Hence, $\chi_=(G \circ H) = 6$ for $m \equiv 23$ and $n = 2k, k \geq 1$.

Unfortunately, $\chi_=(T_2(P_m) \circ P_n)$ is not an equitably 6-colour for $m \not\equiv 23\ \forall\, n$.

Case 2:

Let G be a semi-total point graph of path with $2m-1$ vertices and H be a cycle with n vertices then the number of vertices of the lexicographic product of two graphs is same as the lexicographic product of semi-total point graph of path with path graph, but edges are difference. The number of edges is $n(3n+1)(m-$

$1) + mn$ respectively, corresponding vertex set and edge set are given by

$$V(G \circ H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1} z^*_{i,j}\right)$$

$$E(G \circ H) = \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1}\bigcup_{k=0}^{n-1} (x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk})\right)$$

$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} x_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1} x^*_{i,j}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{i+1,k})$, $x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x_{i,j}$ is the edge $(z_{i,j})(z_{i,(j+1)(mod\ n)})$, $x^*_{i,j}$ is the edge $(z^*_{i,j})(z^*_{i,(j+1)(mod\ n)})$ and $x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{i+1,k})$.

$\forall\ 0 \le k \le n-1$.

If $m \equiv 26, m \ge 3$ and $n = 2k, k \ge 2$, then set the partition of V as below,

$$V_0 = \{z_{i\equiv 03, j\equiv 02}\,, z^*_{i\equiv 13, j\equiv 02}\}$$
$$V_1 = \{z_{i\equiv 13, j\equiv 02}\,, z^*_{i\equiv 23, j\equiv 02}\}$$
$$V_2 = \{z_{i\equiv 23, j\equiv 02}\,, z^*_{i\equiv 03, j\equiv 02}\}$$
$$V_3 = \{z_{i\equiv 03, j\equiv 12}\,, z^*_{i\equiv 13, j\equiv 12}\}$$
$$V_4 = \{z_{i\equiv 13, j\equiv 12}\,, z^*_{i\equiv 23, j\equiv 12}\}$$

And $V_5 = \{z_{i\equiv 23, j\equiv 12}\,, z^*_{i\equiv 03, j\equiv 12}\}$

$\forall\ 0 \le i \le m-1$ and $0 \le j \le n-1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5|$ = $\frac{n(2m-1)}{6}$, it holds inequality $\left||V_i| - |V_j|\right| \le 1$ for every pair (i,j). $\chi_=(G \circ H) \le 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \ge 6$, $\chi_=(G \circ H) \ge \chi(G \circ H) \ge 6$, $\chi_=(G \circ H) \ge 6$. Hence, $\chi_=(G \circ H) = 6$ for $m \equiv 26, m \ge 3$ and $n = 2k, k \ge 2$.

Unfortunately, $\chi_=(T_2(P_m) \circ C_n)$ is not an equitably 6-colour for $m \not\equiv 26\ \forall\ n$.

Case 3:

Let G be a semi-total point graph of path with $2m-1$ vertices and H be a complete bipartite with $2p$ vertices then the number of vertices and edges of the lexicographic product of G and H are $2p(2m-1)$ and $n(3n+1)(m-1) + mn$ and the vertex set and edge set are given by

$$V(G(H)) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1} (z_{i,j})(z_{i,j'})\right)$$

$$\cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1} (z^*_{i,j})(z^*_{i,j'})\right)$$

$$E(G \circ H)$$
$$= \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{k=0}^{p-1} (x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk})\right)$$
$$\cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{k'=0}^{p-1} (x_{i,jk'})\left(x^*_{i,jk'}\right)\left(x^{**}_{i,jk'}\right)\right)$$
$$\cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j'=0}^{p-1}\bigcup_{k=0}^{p-1} (x_{i,j'k})\left(x^*_{i,j'k}\right)\left(x^{**}_{i,j'k}\right)\right)$$
$$\cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j'=0}^{p-1}\bigcup_{k'=0}^{p-1} (x_{i,j'k'})\left(x^*_{i,j'k'}\right)\left(x^{**}_{i,j'k'}\right)\right)$$
$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{l'=0}^{p-1} x_{i,jl'}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{l'=0}^{p-1} x^*_{i,jl'}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{i+1,k})$, $x_{i,jk'}$ is the edge $(z_{i,j})(z_{i+1,k'})$, $x_{i,j'k}$ is the edge $(z_{i,j'})(z_{i+1,k})$, $x_{i,j'k'}$ is the edge $(z_{i,j'})(z_{i+1,k'})$.

$x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x^*_{i,jk'}$ is the edge $(z_{i,j})(z^*_{i,k'})$, $x^*_{i,j'k}$ is the edge $(z_{i,j'})(z^*_{i,k})$, $x^*_{i,j'k'}$ is the edge $(z_{i,j'})(z^*_{i,k'})$.

$x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{i+1,k})$, $x^{**}_{i,jk'}$ is the edge $(z^*_{i,j})(z_{i+1,k'})$, $x^{**}_{i,j'k}$ is the edge $(z^*_{i,j'})(z_{i+1,k})$,
$x^{**}_{i,j'k'}$ is the edge $(z^*_{i,j'})(z_{i+1,k'})$.

$\forall\ 0 \le k \le p-1$ & $0 \le k' \le p-1$.

Also $x_{i,jl'}$ is the edge $(z_{i,j})(z_{i,l'})$ and $x^*_{i,jl'}$ is the edge $(z^*_{i,j})(z^*_{i,l'})$, $\forall\ 0 \le l' \le p-1$.

If $m = 3k+2, k \ge 0$, then the colouring of vertices and partition the vertex set of V as below,

$$\alpha(z_{i,j}) = i(mod\ 3),$$
$$\alpha(z_{i,j'}) = i(mod\ 3) + 3$$
$$\alpha(z^*_{i,j}) = (i+2)(mod\ 3),$$
$$\alpha\left(z^*_{i,j'}\right) = (i+2)(mod\ 3) + 3$$

and

$$V_0 = \{z_{i\equiv 03, j}\,, z^*_{i\equiv 13, j}\}$$
$$V_1 = \{z_{i\equiv 13, j}\,, z^*_{i\equiv 23, j}\}$$
$$V_2 = \{z_{i\equiv 23, j}\,, z^*_{i\equiv 03, j}\}$$
$$V_3 = \{z_{i\equiv 03, j'}\,, z^*_{i\equiv 13, j'}\}$$
$$V_4 = \{z_{i\equiv 13, j'}\,, z^*_{i\equiv 23, j'}\}$$

and $V_5 = \{z_{i\equiv 23, j'}\,, z^*_{i\equiv 03, j'}\}$.

$\forall\ 0 \le i \le m-1, 0 \le j \le p-1$ & $0 \le j' \le p-1$

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5|$ = $\frac{p(2m-1)}{6}$, it holds inequality $\left||V_i| - |V_j|\right| \le 1$ for every pair (i,j). $\chi_=(G \circ H) \le 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \ge 6$, $\chi_=(G \circ H) \ge \chi(G \circ$

$H) \geq 6$, $\chi_{=}(G \circ H) \geq 6$. Hence, $\chi_{=}(G \circ H) = 6$ for $m = 3k + 2, k \geq 0$.

Unfortunately, $\chi_{=}\left(T_2(P_m) \circ K_{p,p}\right)$ is not an equitably 6-colour for $m \neq 3k + 2, k \geq 0$.

Case 4:

Let G be a semi-total point graph of path with $2m - 1$ vertices and H be a complete with n vertices then the number of vertices of the lexicographic product of two graphs is same as the lexicographic product of semi-total point graph of path with path graph, but edges are difference. The number of edges is $n\{3n(m-1) + 2(n-1)\}$ respectively, corresponding vertex set and edge set are given by

$$V(G \circ H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1} z^*_{i,j}\right)$$

$$E(G \circ H) = \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1}\bigcup_{k=0}^{n-1}(x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk})\right)$$

$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{k>j}^{n-1} x_{i,k}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{k>j}^{n-1} x^*_{i,k}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{i+1,k})$, $x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{i+1,k})$, $\forall\, 0 \leq k \leq n - 1$.

Also $x_{i,k}$ is the edge $(z_{i,j})(z_{i,k})$ and $x^*_{i,k}$ is the edge $(z^*_{i,j})(z^*_{i,k})$, $\forall\, 0 \leq j < k \leq n - 1$.

If $n \geq 2$, then the colouring of vertices and partition the vertex set of V as below,

$$\alpha(z_{i,j}) = i(mod\ 3) + 3j$$

$$\alpha(z^*_{i,j}) = (i + 2)(mod\ 3) + 3j$$

and

$$V_{3j} = \{z_{i\equiv 03,j}\,, z^*_{i\equiv 13,j}\}$$

$$V_{3j+1} = \{z_{i\equiv 13,j}\,, z^*_{i\equiv 23,j}\}$$

and $\quad V_{3j+2} = \{z_{i\equiv 23,j}\,, z^*_{i\equiv 03,j}\}$.

$\forall\ 0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$.

Clearly V_{3j}, V_{3j+1} and V_{3j+2} are independent of $V(G \circ H)$, also

(i) If $m \equiv 03$ then $|V_{3j}| = |V_{3j+2}| = \frac{2m}{3}$ and $|V_{3j+1}| = \frac{2m}{3} - 1$.

(ii) If $m \equiv 13$ then $|V_{3j+1}| = |V_{3j+2}| = \left\lfloor\frac{2m}{3}\right\rfloor$ and $|V_{3j}| = \left\lceil\frac{2m}{3}\right\rceil$.

(iii) If $m \equiv 23$ then $|V_{3j}| = |V_{3j+1}| = |V_{3j+2}| = \left\lfloor\frac{2m}{3}\right\rfloor$.

It holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i, j), $\chi_{=}(G \circ H) \leq 3n$, Since there exist cliques of order $3n$ in $V(G \circ H)$, $\chi(G \circ H) \geq 3n$, $\chi_{=}(G \circ H) \geq \chi(G \circ H) \geq 3n$, $\chi_{=}(G \circ H) \geq 3n$. Hence, $\chi_{=}(G \circ H) = 3n$ for $n \geq 2$.

Corollary 3.1.1

(i) For $m \geq 3$, $\chi_{=}(T_2(P_m) \circ P_2) = 6$.

(ii) For $n \geq 3$, $\chi_{=}(T_2(P_2) \circ P_n) = 6$.

Proof.

Define the map $\alpha :\ V(G \circ H) \ \rightarrow \{0,1,2, \ldots, 5\}$

Partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03,j\equiv 02}\,, z^*_{i\equiv 13,j\equiv 02}\}$$

$$V_1 = \{z_{i\equiv 13,j\equiv 02}\,, z^*_{i\equiv 23,j\equiv 02}\}$$

$$V_2 = \{z_{i\equiv 23,j\equiv 02}\,, z^*_{i\equiv 03,j\equiv 02}\}$$

$$V_3 = \{z_{i\equiv 03,j\equiv 12}\,, z^*_{i\equiv 13,j\equiv 12}\}$$

$$V_4 = \{z_{i\equiv 13,j\equiv 12}\,, z^*_{i\equiv 23,j\equiv 12}\}$$

and $\quad V_5 = \{z_{i\equiv 23,j\equiv 12}\,, z^*_{i\equiv 03,j\equiv 12}\}$.

$\forall\ 0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also

Claim (i): For $m \geq 3$,

(a) If $m \equiv 03$ then $|V_0| = |V_3| = \left\lceil\frac{2m-1}{3}\right\rceil$ and $|V_1| = |V_2| = |V_4| = |V_5| = \left\lfloor\frac{2m-1}{3}\right\rfloor$.

(b) If $m \equiv 13$ then $|V_0| = |V_1| = |V_3| = |V_4| = \left\lceil\frac{2m-1}{3}\right\rceil$ and $|V_2| = |V_5| = \left\lfloor\frac{2m-1}{3}\right\rfloor$

(c) If $m \equiv 23$ then $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = \frac{2m-1}{3}$.

Claim (ii): For $n \geq 3$,

(a) If $n \equiv 02$ then $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = \frac{n}{2}$.

(b) If $n \equiv 12$ then $|V_0| = |V_1| = |V_2| = \left\lceil\frac{n}{2}\right\rceil$ and $|V_3| = |V_4| = |V_5| = \left\lfloor\frac{n}{2}\right\rfloor$.

It holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i, j). $\chi_{=}(G \circ H) \leq 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \geq 6$, $\chi_{=}(G \circ H) \geq \chi(G \circ H) \geq 6$, $\chi_{=}(G \circ H) \geq 6$. Hence, $\chi_{=}(G \circ H) = 6$.

Corollary 3.1.2 For $k \geq 2$,

$$\chi_{=}(T_2(P_2) \circ C_n) = \begin{cases} 6\,;\ n = 2k \\ 9\,;\ n = 2k - 1 \end{cases}$$

Proof.

Define the map $\alpha :\ V(G \circ H) \ \rightarrow \{0,1,2, \ldots, l\}\ \forall\, l \in N$

Case 1:

If $n = 2k$, $k \geq 2$ then partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03,j\equiv 02}\,, z^*_{i\equiv 13,j\equiv 02}\}$$

$$V_1 = \{z_{i\equiv 13,j\equiv 02}\,, z^*_{i\equiv 23,j\equiv 02}\}$$

$$V_2 = \{z_{i\equiv 23,j\equiv 02}\,, z^*_{i\equiv 03,j\equiv 02}\}$$

$$V_3 = \{z_{i\equiv 03,j\equiv 12}\,, z^*_{i\equiv 13,j\equiv 12}\}$$

$$V_4 = \{z_{i\equiv 13,j\equiv 12}\,, z^*_{i\equiv 23,j\equiv 12}\}$$

and $\quad V_5 = \{z_{i\equiv 23,j\equiv 12}\,, z^*_{i\equiv 03,j\equiv 12}\}$.

$\forall\ 0 \le i \le m-1$ and $0 \le j \le n-1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(T_2(P_2) \circ C_n)$, also $|V_0| = |V_3| = \left\lceil \frac{2m-1}{3} \right\rceil$ and $|V_1| = |V_2| = |V_4| = |V_5| = \left\lfloor \frac{2m-1}{3} \right\rfloor$, it holds inequality $\left||V_i| - |V_j|\right| \le 1$ for every pair (i,j). $\chi_=(T_2(P_2) \circ C_n) \le 6$, Since there exist cliques of order 6 in $V(T_2(P_2) \circ C_n)$, $\chi(T_2(P_2) \circ C_n) \ge 6$, $\chi_=(T_2(P_2) \circ C_n) \ge \chi(T_2(P_2) \circ C_n) \ge 6$, $\chi_=(T_2(P_2) \circ C_n) \ge 6$. Hence, $\chi_=(T_2(P_2) \circ C_n) = 6$ for $n = 2k, k \ge 2$.

Case 2:

If $n = 2k-1$, $k \ge 2$, then the colouring of vertices and partition the vertex set of V as below,

$$\alpha(z_{i,j}) = i(mod\ 3) + 3[j(mod\ 3)]$$
$$\alpha(z^*_{i,j}) = (i+2)(mod\ 3) + 3[j(mod\ 3)]$$

and

$$V_0 = \{z_{i\equiv 03, j\equiv 03}, z^*_{i\equiv 13, j\equiv 03}\}$$
$$V_1 = \{z_{i\equiv 13, j\equiv 03}, z^*_{i\equiv 23, j\equiv 03}\}$$
$$V_2 = \{z_{i\equiv 23, j\equiv 03}, z^*_{i\equiv 03, j\equiv 03}\}$$
$$V_3 = \{z_{i\equiv 03, j\equiv 13}, z^*_{i\equiv 13, j\equiv 13}\}$$
$$V_4 = \{z_{i\equiv 13, j\equiv 13}, z^*_{i\equiv 23, j\equiv 13}\}$$
$$V_5 = \{z_{i\equiv 23, j\equiv 13}, z^*_{i\equiv 03, j\equiv 13}\}$$
$$V_6 = \{z_{i\equiv 03, j\equiv 23}, z^*_{i\equiv 13, j\equiv 23}\}$$
$$V_7 = \{z_{i\equiv 13, j\equiv 23}, z^*_{i\equiv 23, j\equiv 23}\}$$

and $V_8 = \{z_{i\equiv 23, j\equiv 23}, z^*_{i\equiv 03, j\equiv 23}\}$.

$\forall\ 0 \le i \le m-1$ and $0 \le j \le n-1$.

Clearly $V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7$ and V_8 are independent of $V(T_2(P_2) \circ C_n)$, also

(a) If $n = 3(2k-3)$ then $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = |V_6| = |V_7| = |V_8| = (2k-3)$.

(b) If $n = (6k-7)$ then $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = \left\lceil \frac{6k-7}{3} \right\rceil$ and $|V_6| = |V_7| = |V_8| = \left\lfloor \frac{6k-7}{3} \right\rfloor$.

(c) If $n = (6k-5)$ then $|V_0| = |V_1| = |V_2| = \left\lceil \frac{6k-5}{3} \right\rceil$ and $|V_3| = |V_4| = |V_5| = |V_6| = |V_7| = |V_8| = \left\lfloor \frac{6k-5}{3} \right\rfloor$.

It holds inequality $\left||V_i| - |V_j|\right| \le 1$ for every pair (i,j). $\chi_=(T_2(P_2) \circ C_n) \le 9$, Since there exist cliques of order 9 in $V(T_2(P_2) \circ C_n)$, $\chi(T_2(P_2) \circ C_n) \ge 9$, $\chi_=(T_2(P_2) \circ C_n) \ge \chi(T_2(P_2) \circ C_n) \ge 9$, $\chi_=(T_2(P_2) \circ C_n) \ge 9$. Hence, $\chi_=(T_2(P_2) \circ C_n) = 9$ for $n = 2k-1, k \ge 2$.

Corollary 3.1.3 For $p \ge 1$,

$\chi_=(T_2(P_2) \circ K_{p,p+1}) = 6$.

Proof.

Define the map $\alpha : V(G \circ H) \to \{0,1,2,\dots,5\}$

If $p \ge 1$, then partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03, j}, z^*_{i\equiv 13, j}\}$$
$$V_1 = \{z_{i\equiv 13, j}, z^*_{i\equiv 23, j}\}$$
$$V_2 = \{z_{i\equiv 23, j}, z^*_{i\equiv 03, j}\}$$
$$V_3 = \{z_{i\equiv 03, j'}, z^*_{i\equiv 13, j'}\}$$
$$V_4 = \{z_{i\equiv 13, j'}, z^*_{i\equiv 23, j'}\}$$

and $V_5 = \{z_{i\equiv 23, j'}, z^*_{i\equiv 03, j'}\}$.

$\forall\ 0 \le i \le m-1, 0 \le j \le p-1$ & $0 \le j' \le p-1$

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(T_2(P_2) \circ K_{p,p+1})$, also $|V_0| = |V_1| = |V_2| = p$ and $|V_3| = |V_4| = |V_5| = p+1$, it holds inequality $\left||V_i| - |V_j|\right| \le 1$ for every pair (i,j). $\chi_=(T_2(P_2) \circ K_{p,p+1}) \le 6$, Since there exist cliques of order 6 in $V(T_2(P_2) \circ K_{p,p+1})$, $\chi(T_2(P_2) \circ K_{p,p+1}) \ge 6$, $\chi_=(T_2(P_2) \circ K_{p,p+1}) \ge \chi(T_2(P_2) \circ K_{p,p+1}) \ge 6$, $\chi_=(T_2(P_2) \circ K_{p,p+1}) \ge 6$. Hence, $\chi_=(T_2(P_2) \circ K_{p,p+1}) = 6$.

Theorem 3.2. Let G and H be any two graphs, where G is a semi-total point graph of cycle, $T_2(C_m)$ on $m \ge 3$ vertices, the equitable colouring of the lexicographic product of G and H are

(i) $\chi_=(G \circ P_n) = 6$; $m = 3k$ and $n = 2k, k \ge 1$

(ii) $\chi_=(G \circ C_n) = \begin{cases} 6 ;\ n = 2k+2 \\ 9 ;\ n = 3 \end{cases}$; $m = 3k, k \ge 1$.

(iii) $\chi_=(G \circ K_n) = 3n$; $m = 3k, k \ge 1$ and $n \ge 2$.

(iv) $\chi_=(G \circ K_{p,p}) = 6$; $p \ge 1$.

Proof.

Define the map $\alpha : V(G \circ H) \to \{0,1,2,\dots,l\}\ \forall\ l \in N$

The proof of the theorem is divided into four cases,

Case 1:

Let G be a semi-total point graph of cycle with $2m$ vertices and H be a path with n vertices then the number of vertices and edges of the lexicographic product of two graphs G and H are $2mn$ and $m(3n^2 + 2n - 2)$ respectively, corresponding vertex set and edge set are given by

$$V(G \circ H) = \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} z_{i,j} \right) \cup \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} z^*_{i,j} \right)$$

$$E(G \circ H) = \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} \bigcup_{k=0}^{n-1} (x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk}) \right) \cup \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} (x_{i,j})\,(x^*_{i,j}) \right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{(i+1)(mod\ m),k})$, $x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{(i+1)(mod\ m),k})$, $\forall\ 0 \le k \le n-1$.

Also $x_{i,j}$ is the edge $(z_{i,j})(z_{i,j+1})$ and $x^*_{i,j}$ is the edge $(z^*_{i,j})(z^*_{i,j+1})$.

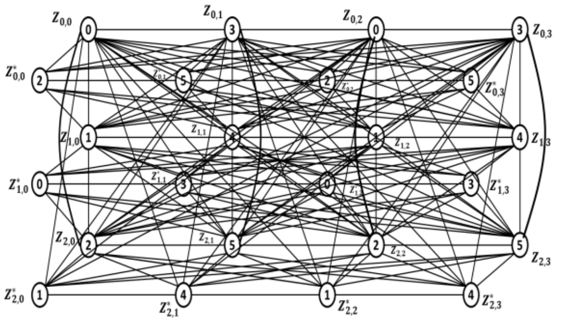

Fig.1 Example of lexicographic product $T_2(C_3) \circ P_4$

If $m = 3k$ and $n = 2k, k \geq 1$ then partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03, j\equiv 02}, z^*_{i\equiv 13, j\equiv 02}\}$$
$$V_1 = \{z_{i\equiv 13, j\equiv 02}, z^*_{i\equiv 23, j\equiv 02}\}$$
$$V_2 = \{z_{i\equiv 23, j\equiv 02}, z^*_{i\equiv 03, j\equiv 02}\}$$
$$V_3 = \{z_{i\equiv 03, j\equiv 12}, z^*_{i\equiv 13, j\equiv 12}\}$$
$$V_4 = \{z_{i\equiv 13, j\equiv 12}, z^*_{i\equiv 23, j\equiv 12}\}$$

and $V_5 = \{z_{i\equiv 23, j\equiv 12}, z^*_{i\equiv 03, j\equiv 12}\}$.

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5|$ $= \frac{mn}{3}$, it holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i,j). $\chi_=(G \circ H) \leq 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq \chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq 6$. Hence, $\chi_=(G \circ H) = 6$ for $m = 3k$ and $n = 2k, k \geq 1$.

Unfortunately, $\chi_=(T_2(C_m) \circ P_n)$ is not an equitably 6-colour for $m = 3k$ and $n \neq 2k, k \geq 1$ also $\forall n, m \neq 3k, k \geq 1$.

Case 2:

Let G be a semi-total point graph of cycle with $2m$ vertices and H be a cycle with n vertices then the number of vertices of the lexicographic product of two graphs is same as the lexicographic product of semi-total point graph of cycle with path graph, but edges are difference. The number of edges is $mn(3n+2)$ respectively, corresponding vertex set and edge set are given by

$$V(G \circ H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z^*_{i,j}\right)$$

$$E(G \circ H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\bigcup_{k=0}^{n-1} (x_{i,jk})\,(x^*_{i,jk})(x^{**}_{i,jk})\right)$$

$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} (x_{i,j})\,(x^*_{i,j})\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{(i+1)(mod\ m),k})$, $x^*_{i,jk}$ is the edge $(z_{i,j})(z^*_{i,k})$, $x^{**}_{i,jk}$ is the edge $(z^*_{i,j})(z_{(i+1)(mod\ m),k})$, $x_{i,j}$ is the edge $(z_{i,j})(z_{i,(j+1)(mod\ n)})$ and $x^*_{i,j}$ is the edge $(z^*_{i,j})(z^*_{i,(j+1)(mod\ n)})$, $\forall\ 0 \leq k \leq n-1$.

Claim (a):

If $n = 2k+2$ and $m = 3k, k \geq 1$ then partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03, j\equiv 02}, z^*_{i\equiv 13, j\equiv 02}\}$$
$$V_1 = \{z_{i\equiv 13, j\equiv 02}, z^*_{i\equiv 23, j\equiv 02}\}$$
$$V_2 = \{z_{i\equiv 23, j\equiv 02}, z^*_{i\equiv 03, j\equiv 02}\}$$
$$V_3 = \{z_{i\equiv 03, j\equiv 12}, z^*_{i\equiv 13, j\equiv 12}\}$$
$$V_4 = \{z_{i\equiv 13, j\equiv 12}, z^*_{i\equiv 23, j\equiv 12}\}$$

and $V_5 = \{z_{i\equiv 23, j\equiv 12}, z^*_{i\equiv 03, j\equiv 12}\}$.

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G \circ H)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5|$ $= \frac{mn}{3}$, it holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i,j). $\chi_=(G \circ H) \leq 6$, Since there exist cliques of order 6 in $V(G \circ H)$, $\chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq \chi(G \circ H) \geq 6$, $\chi_=(G \circ H) \geq 6$. Hence, $\chi_=(G \circ H) = 6$ for $m = 3k$ and $n = 2k+2, k \geq 1$.

Unfortunately, $\chi_=(T_2(C_m) \circ C_n)$ is not an equitably 6-colour for $m = 3k$ and $n \neq 2k+2, k \geq 1$ also $\forall\ n, m \neq 3k, k \geq 1$.

Claim (b):

If $n = 3$ and $m = 3k, k \geq 1$ then partition the vertex set of V as below,

$$V_0 = \{z_{i\equiv 03, j\equiv 03}, z^*_{i\equiv 13, j\equiv 03}\}$$
$$V_1 = \{z_{i\equiv 13, j\equiv 03}, z^*_{i\equiv 23, j\equiv 03}\}$$
$$V_2 = \{z_{i\equiv 23, j\equiv 03}, z^*_{i\equiv 03, j\equiv 03}\}$$
$$V_3 = \{z_{i\equiv 03, j\equiv 13}, z^*_{i\equiv 13, j\equiv 13}\}$$
$$V_4 = \{z_{i\equiv 13, j\equiv 13}, z^*_{i\equiv 23, j\equiv 13}\}$$
$$V_5 = \{z_{i\equiv 23, j\equiv 13}, z^*_{i\equiv 03, j\equiv 13}\}$$
$$V_6 = \{z_{i\equiv 03, j\equiv 23}, z^*_{i\equiv 13, j\equiv 23}\}$$
$$V_7 = \{z_{i\equiv 13, j\equiv 23}, z^*_{i\equiv 23, j\equiv 23}\}$$

and $V_8 = \{z_{i\equiv 23, j\equiv 23}, z^*_{i\equiv 03, j\equiv 23}\}$.

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Clearly $V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7$ and V_8 are independent of $V(T_2(C_{3k}) \circ C_3)$, also $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = \frac{2m}{3}$, it holds inequality $\left||V_i| - |V_j|\right| \leq 1$ for every pair (i,j). $\chi_=(T_2(C_{3k}) \circ C_3) \leq 9$, Since there exist cliques of order 9 in $V(T_2(C_{3k}) \circ C_3)$, $\chi(T_2(C_{3k}) \circ C_3) \geq 9$, $\chi_=(T_2(C_{3k}) \circ C_3) \geq \chi(T_2(C_{3k}) \circ C_3) \geq 9$, $\chi_=(T_2(C_{3k}) \circ C_3) \geq 9$. Hence, $\chi_=(T_2(C_{3k}) \circ C_3) = 9$ for $k \geq 1$.

Case 3:

Let G be a semi-total point graph of cycle with $2m$ vertices and H be a complete with n vertices then the number of vertices of the lexicographic product of two graphs is same as the lexicographic product of semi-total point graph of cycle with path graph, but edges are difference. The number of edges is $n\{3n(m-1)+2(n-1)\}$ respectively, corresponding vertex set and edge set are given by

$$V(G\circ H)=\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}\right)\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1} z_{i,j}^{*}\right)$$

$$E(G\circ H)=\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\bigcup_{k=0}^{n-1}(x_{i,jk})\,(x_{i,jk}^{*})(x_{i,jk}^{**})\right)$$

$$\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{k>j}^{n-1} x_{i,k}\right)\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{k>j}^{n-1} x_{i,k}^{*}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{(i+1)(mod\ m),k})$, $x_{i,jk}^{*}$ is the edge $(z_{i,j})(z_{i,k}^{*})$, $x_{i,jk}^{**}$ is the edge $(z_{i,j}^{*})(z_{(i+1)(mod\ m),k})$, $\forall\ 0\le k\le n-1$.

Also $x_{i,k}$ is the edge $(z_{i,j})(z_{i,k})$ and $x_{i,k}^{*}$ is the edge $(z_{i,j}^{*})(z_{i,k}^{*})$, $\forall\ 0\le j<k\le n-1$.

If $m=3k, k\ge 1$ and $n\ge 2$, then the colouring of vertices and partition the vertex set of V as below,

$$\alpha(z_{i,j})=\begin{cases} j & ; i\equiv 03\\ j+n & ; i\equiv 13\\ j+2n & ; i\equiv 23\end{cases}$$

$$\alpha(z_{i,j}^{*})=\begin{cases} j+2n & ; i\equiv 03\\ j & ; i\equiv 13\\ j+n & ; i\equiv 23\end{cases}$$

and

$$V_j=\{z_{i\equiv 03,j}\,,z_{i\equiv 13,j}^{*}\}$$

$$V_{j+n}=\{z_{i\equiv 13,j}\,,z_{i\equiv 23,j}^{*}\}$$

and $V_{j+2n}=\{z_{i\equiv 23,j}\,,z_{i\equiv 03,j}^{*}\}$.

$\forall\ 0\le i\le m-1$ and $0\le j\le n-1$.

Clearly V_j, V_{j+n} and V_{j+2n} are independent of $V(G\circ H)$, also $|V_j|=|V_{j+n}|=|V_{j+2n}|=\frac{2m}{3}$, it holds inequality $\big||V_i|-|V_j|\big|\le 1$ for every pair (i,j), $\chi_{=}(G\circ H)\le 3n$, Since there exist cliques of order $3n$ in $V(G\circ H)$, $\chi(G\circ H)\ge 3n$, $\chi_{=}(G\circ H)\ge\chi(G\circ H)\ge 3n$, $\chi_{=}(G\circ H)\ge 3n$. Hence, $\chi_{=}(G\circ H)=3n$ for $m=3k, k\ge 1$ and $n\ge 2$.

Unfortunately, $\chi_{=}(T_2(C_m)\circ K_n)$ is not an equitably $3n$-colour for $m\ne 3k, k\ge 1$ and $n\ge 2$.

Case 4:

Let G be a semi-total point graph of cycle with $2m$ vertices and H be a complete bipartite with $2p$ vertices then the number of vertices and edges of the lexicographic product of two graphs G and H are $4mp$ and $p^2(14m-13)$ respectively, corresponding vertex set and edge set are given by

$$V(G(H))=\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}(z_{i,j})(z_{i,j'})\right)$$

$$\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}(z_{i,j}^{*})(z_{i,j'}^{*})\right)$$

$$E(G\circ H)$$

$$=\left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{k=0}^{p-1}(x_{i,jk})\,(x_{i,jk}^{*})(x_{i,jk}^{**})\right)$$

$$\cup\left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{k'=0}^{p-1}(x_{i,jk'})\left(x_{i,jk'}^{*}\right)\left(x_{i,jk'}^{**}\right)\right)$$

$$\cup\left(\bigcup_{i=0}^{m-2}\bigcup_{j'=0}^{p-1}\bigcup_{k=0}^{p-1}(x_{i,j'k})\left(x_{i,j'k}^{*}\right)\left(x_{i,j'k}^{**}\right)\right)$$

$$\cup\left(\bigcup_{i=0}^{m-2}\bigcup_{j'=0}^{p-1}\bigcup_{k'=0}^{p-1}(x_{i,j'k'})\left(x_{i,j'k'}^{*}\right)\left(x_{i,j'k'}^{**}\right)\right)$$

$$\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{l'=0}^{p-1} x_{i,jl'}\right)\cup\left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{l'=0}^{p-1} x_{i,jl'}^{*}\right)$$

Where, $x_{i,jk}$ is the edge $(z_{i,j})(z_{(i+1)(mod\ m),k})$, $x_{i,jk'}$ is the edge $(z_{i,j})(z_{i+1,k'})$, $x_{i,j'k}$ is the edge $(z_{i,j'})(z_{(i+1)(mod\ m),k})$, $x_{i,j'k'}$ is the edge $(z_{i,j'})(z_{i+1,k'})$,

$x_{i,jk}^{*}$ is the edge $(z_{i,j})(z_{i,k}^{*})$, $x_{i,jk'}^{*}$ is the edge $(z_{i,j})(z_{i,k'}^{*})$, $x_{i,j'k}^{*}$ is the edge $(z_{i,j'})(z_{i,k}^{*})$, $x_{i,j'k'}^{*}$ is the edge $(z_{i,j'})(z_{i,k'}^{*})$,

$x_{i,jk}^{**}$ is the edge $(z_{i,j}^{*})(z_{(i+1)(mod\ m),k})$, $x_{i,jk'}^{**}$ is the edge $(z_{i,j}^{*})(z_{i+1,k'})$, $x_{i,j'k}^{**}$ is the edge $(z_{i,j'}^{*})(z_{(i+1)(mod\ m),k})$, $x_{i,j'k'}^{**}$ is the edge $(z_{i,j'}^{*})(z_{i+1,k'})$, $\forall\ 0\le k\le p-1$ & $0\le k'\le p-1$.

Also $x_{i,jl'}$ is the edge $(z_{i,j})(z_{i,l'})$ and $x_{i,jl'}^{*}$ is the edge $(z_{i,j}^{*})(z_{i,l'}^{*})$, $\forall\ 0\le l'\le p-1$.

If $p\ge 1$, then partition the vertex set of V as below,

$$V_0=\{z_{i\equiv 03,j}\,,z_{i\equiv 13,j}^{*}\}$$

$$V_1=\{z_{i\equiv 13,j}\,,z_{i\equiv 23,j}^{*}\}$$

$$V_2=\{z_{i\equiv 23,j}\,,z_{i\equiv 03,j}^{*}\}$$

$$V_3=\{z_{i\equiv 03,j'}\,,z_{i\equiv 13,j'}^{*}\}$$

$$V_4=\{z_{i\equiv 13,j'}\,,z_{i\equiv 23,j'}^{*}\}$$

and $V_5=\{z_{i\equiv 23,j'}\,,z_{i\equiv 03,j'}^{*}\}$.

$\forall\ 0\le i\le m-1$, $0\le j\le p-1$ & $0\le j'\le p-1$

Clearly V_0, V_1, V_2, V_3, V_4 and V_5 are independent of $V(G\circ H)$, also $|V_0|=|V_1|=|V_2|=|V_3|=|V_4|=|V_5|$ $=\frac{2mp}{3}$, it holds inequality $\big||V_i|-|V_j|\big|\le 1$ for every pair (i,j). $\chi_{=}(G\circ H)\le 6$, Since there exist cliques of order

6 in $V(G \circ H)$, $\chi(G \circ H) \geq 6$, $\chi_{=}(G \circ H) \geq \chi(G \circ H) \geq 6$, $\chi_{=}(G \circ H) \geq 6$. Hence, $\chi_{=}(G \circ H) = 6$.

Remark 1: [13] This paper proved that the equitable colouring of a lexicographic product of semi total point of path graph with path, cycle, complete and complete bipartite graphs are commutative, also the equitable colouring of a lexicographic product of semi total point of cycle graph with path, cycle, complete and complete bipartite graphs are commutative that is, $\chi_{=}(G \circ H) = \chi_{=}(H \circ G)$.

Conclusion

This paper, the Equitable colouring of lexicographic product of G and H has been showed. In a similar way, Equitable colouring of the lexicographic product of other graphs can be verified. The proof of the theorems are recognized by different cases, each cases being discussed elaborately.

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