

Review of Spectral Analysis for Different Types of Signed Graphs

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Abstract: Spectral analysis is performed on the adjacency and Laplacian matrices associated with the different types of signed graphs. A signed graph Ω is a graph G in which each edge ' e ' carries a value $\sigma(e) \in \{+1, -1\}$ called its sign denoted specially as $\Omega = (G, \sigma)$. This review paper blends recent expansions in spectral analysis of different types of signed graphs such as splitting signed graphs, cartesian product of signed graphs, signed social networks, etc. Additionally, analyse the spectral properties of signed graphs on special sets and signed social networks. By examining key papers and integrate additional insights, this review offers the practical applications of signed graph theory with significant implications for network analysis and social interactions.

Keywords: Signed Graphs, Spectrum, Splitting signed graph, Social networks.

1. Introduction

Spectral graph theory is the branch of mathematics that studies the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvector of matrices associated with the graph [1]. The motive of the spectral graph theory is to investigate different types of matrices (adjacency matrices, laplacian matrices, incidence matrices) and compute their eigen values and eigen vectors correlated with signed graphs [2]. The graph is a very convenient and natural way of representing the relationships between objects; we represent objects by vertices and the relationship between them by lines [3]. Graphs are used as mathematical models, to simplify the graph-theoretic problems and then interpret the solution in terms of the original problem [4]. Signed graphs (**Figure 1**) with $+ve/-ve$ labels corresponding to solid lines /dotted lines (edges) are used to model complex systems in social networks, biology, and classical mathematical systems [5].

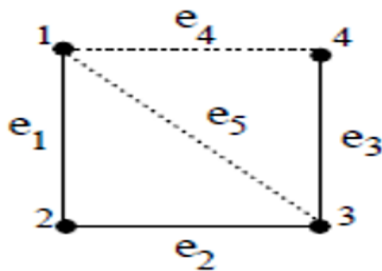


Figure 1: Signed Graph

In recent years, signed graphs are particularly helpful for modeling systems with dual

relationships, such as social networks with friendships and enmities, biological networks with co-operative and antagonistic interactions [6]. Some of the applications of signed graphs are road networks, the complex electricity or water supply network in a city, the railway or communication network in a country, the chemical bond structure of a molecule, and computer networks in an organization. The spectral properties of signed graphs can reveal important characteristics such as stability, community structure, and influence dynamics within these networks.

Spectral analysis of signed graphs is a vital area of exploration in graph theory and network analysis [7]. This review inspects five recent studies that suggest innovational techniques for spectral and Laplacian spectral analysis, contributing to theoretical expansions and practical applications. Spectral graph theory focuses on the properties of eigen values and eigen vectors of matrices associated with signed graphs [8]. These spectral properties provide critical insights into the structure and behaviour of signed graphs, facilitating developments in areas such as network analysis, optimization, and data science [9].

This review highlights the significant contributions of Deepa Sinha and her collaborators in the spectral analysis of signed graphs. Their work circumscribes innovative techniques and algorithmic approaches that upgrade our understanding and computational efficiency in this domain. The key papers reviewed include:

1.1 Embedding of sign-regular signed graphs and its spectral analysis: Sinha & Rao's Contribution

In 2022, Sinha et al. introduced the notion of sign-regular signed graphs with a detailed analysis of their structural properties. The authors focus on the spectral analysis of sign-regular signed graphs [10]. This study is essential for understanding the eigenvalue spectra of these graphs which have applications in areas such as social network analysis and molecular chemistry. The study discusses embedding techniques for these graphs, which are useful for visualizing and analyzing the spectral properties in higher-dimensional space [11].

A signed graph Ω is called sign-regular if $d^-(v_1)$ and $d^+(v_1)$ is the same for all $v_1 \in V$. This paper explores the relationship between characteristic polynomials of the signed graph and its embedded graph [12]. The embedding of a signed graph Ω means there exists a signed graph Ω' such that $\Omega \subseteq \Omega'$ (Figure 2) [13]. If $\Omega \subset \Omega'$ then embedding is called strict.

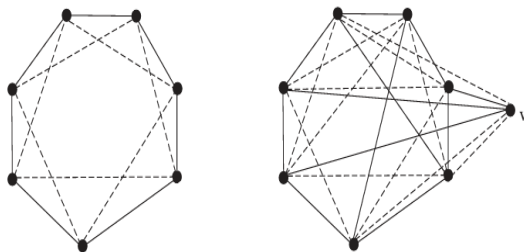


Figure 2: Embedding of signed graphs

This study explores various applications, particularly in social networks and molecular chemistry where interactions can be cooperative or antagonistic. the spectral properties helps in modelling analyzing such complex systems.

1.2 Spectrum and Laplacian Spectrum of a Cartesian Product of Signed Graphs: Kaur, Kumar & Sinha's Contribution

In 2024, Kaur et al. presented a new method for calculating the spectrum and Laplacian spectrum of the Cartesian product of signed graphs [14].

The Spectrum and Laplacian spectrum of the signed graph is the set of eigenvalues along with their multiplicities of the corresponding adjacency matrix and Laplacian matrix [15].

Matrix representation of Signed Graph

Let Ω be a signed graph with ' n ' vertices. Then adjacency matrix of Ω is a ' $n \times n$ ' symmetric matrix (with all diagonal elements zero) denoted by, $A(\Omega) = (a_{ij})$, such that

$$(a_{ij}) = \begin{cases} 1, & \text{if } \sigma(v_i, v_j) \text{ is } +ve \\ -1, & \text{if } \sigma(v_i, v_j) \text{ is } -ve \\ 0, & \text{otherwise} \end{cases}$$

The degree matrix $D(\Omega) = (d_{ij})$ is a diagonal matrix of order ' $n \times n$ ' such that

$$(d_{ij}) = \begin{cases} d_i, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Where d_i represents the degree of a vertex v_i . The Laplacian matrix $L(\Omega)$ of order ' $n \times n$ ' such that $L(\Omega) = D(\Omega) - A(\Omega)$ [15].

The characteristic polynomial of the signed graph Ω is defined as $\phi(\Omega: \lambda) = \det(\lambda I - A(\Omega))$, where I is the identity matrix of order ' $n \times n$ '. And the roots of the characteristic polynomial $\phi(\Omega: \lambda) = 0$ denoted by $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are called eigenvalues of $A(\Omega)$ [8]. The spectrum of Ω is given by

$$Spec(\Omega) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \dots & \lambda_n \\ m_1 & m_2 & \dots & \dots & m_n \end{pmatrix}$$

Where $m_1, m_2, m_3, \dots, m_n$ are multiplicities of eigenvalues.

The energy of a signed graph is determined as the sum of absolute values of eigenvalues of the corresponding adjacency matrix; upper bound and lower bound can also be resolved for the respective energies [16].

The results in this article are used to construct a family of infinitely many cospectral and Laplacian cospectral graphs of some known classes of graphs. This approach leverages the structural properties of the cartesian product for large and complex networks.

1.3 Spectral Analysis for Signed Social Networks: Rao, Kaur, Somra, & Sinha's Contribution

In 2024, Rao et al. applied the spectral graph theory to analyze signed social networks, where the edges of the signed graph represent positive or negative relationships such as friendship or enmity [17]. The relation among vertices (people) in complex real-world networks changes over time. For any number of vertices, adding or deleting vertices ones can change the network's potential shown in Figure 3.

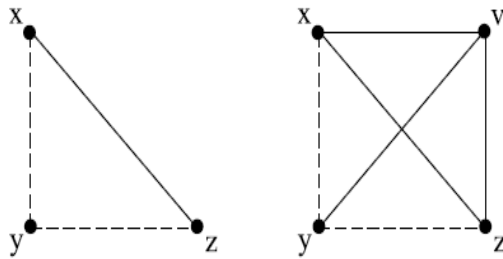


Figure 3: Signed graph and A graph obtained by adding a vertex 'v'

The Iterated Local Transitivity model is based on the principle of transitivity and local interaction among people and the same has been extended to the signed social networks [18].

The authors find the relation between the characteristic polynomials of the signed graph and the signed graph obtained by adding (deleting) vertices and edges, along with this, a recurrence relation is presented for the characteristic polynomial of the Iterated Local Transitivity (ILT) model for signed graphs [19].

The authors proposed ILT model for a signed social network and balanced, clustering and sign compatibility problems are taken up to boost ILT model to a new border in the signed graphs [20-22].

1.4 Spectral Analysis of Splitting Signed Graph: Kumar & Sinha's Contribution

In 2024, Kumar et al. conducted an inclusive spectral analysis of splitting signed graphs. An algorithm is generated for splitting signed graph $\Gamma(\Omega)$, splitting root signed graph from a given signed graph Ω [23].

Spectral analysis is done on the adjacency matrices and Laplacian matrices of the splitting signed graph $\Gamma(\Omega)$ to study its eigenvalues and eigenvectors along with this the energy of the original signed graph Ω and its splitting signed graph $\Gamma(\Omega)$ is determined [24].

The splitting signed graph ($\Gamma(\Omega)$) of a signed graph (Ω) is defined as for every vertex ' v_1 ' of the signed graph taking a new vertex v_1' . Then join v_1' to those vertices which are adjacent to ' v_1 ' in the given signed graph as shown in Figure 4.

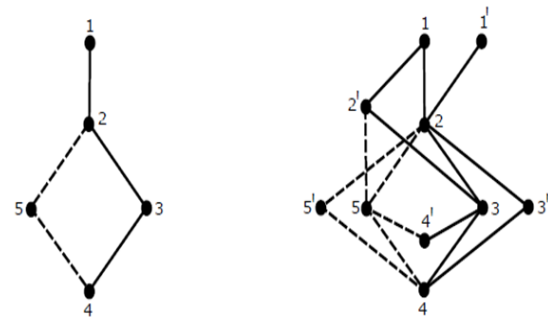


Figure 4: Signed Graph and its Splitting Signed Graph

The authors determined the properties of splitting signed graphs and offered extensive spectral analysis results and their applications in various domains [25]. The complexity of finding splitting signed graph with adjacency matrix is $O(n^2)$, Where ' n ' be the number of vertices.

1.5 Spectral Analysis of a Graph on the Special Set: Rao, Sinha & Kumar's Contribution

In 2024, Rao et al. introduced a new family of interconnection networks that are Cayley graphs ($\Omega^-(Z_n)$) with vertex set $Z_n = \{0, 1, 2, \dots, n-1\}$ on the special set S' [26]. The special set S' is defined as

$$S' = \{a \in Z_n : \exists b \in Z_n, ba = a, b \neq a, b \neq 1\}$$

Where Z_n be the integer modulo ' n ' with binary operations addition modulo ' n ' and multiplication modulo ' n '. The concept of associating graphs with rings is introduced by Istvan Beck [27]. There after algebraic structures like unitary Cayley graphs, unitary addition Cayley graphs, absorption Cayley graphs, etc came into existence.

The Unitary Cayley graph X_n has vertex set $Z_n = \{0, 1, 2, \dots, n-1\}$ and two vertices x and y are adjacent if $\gcd(x-y, n) = 1$.

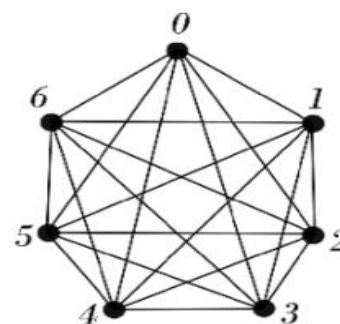


Figure 5: Unitary Cayley graph X_7

The authors defined the relationship between G and G_e^* , where G_e^* is the derived graph after

removing ' r ' edges from G (r is known fixed value) and spectrum of absorption Cayley graph, unitary addition Cayley graph [28].

The paper explores the structural properties of Cayley graph $\Omega^-(Z_n)$ such as planarity and connectedness [29]. In the last decade, signed graph spectra's importance in computer science has been broadly accepted. This research entails algebra, and linear algebra to analyze Cayley graphs. The edge set of a Cayley graph is described by using an adjacency matrix and inspecting its eigen-values and eigen-vectors. Cayley graph spectra have many applications in various fields, such as data processing, pattern identification, computer graphics, and image processing. The largest eigen-value of a Cayley graph is very useful in simulating virus transmission in computer networks.

2. Applications and Practical Implications:

The strategies developed in these papers have wide exertion in network theory, social network analysis, and other fields requiring efficient spectral analysis of signed graphs. The advancement of spectral signed graphs develops innovative techniques and algorithms, enhancing both the theoretical framework and practical applications. These spectral properties provide critical insights into the structure and behavior of signed graphs, facilitating developments in areas such as network analysis, optimization, Stability analysis, Graph coloring and data science.

3. Research Gaps

Existing research synthesizes recent advancements in the spectral analysis of signed graphs. The study focuses on novel techniques and algorithms for understanding the spectral properties of different types of signed graphs like Embedded signed graphs, Social signed networks, Splitting signed graphs, etc. In the same manner, the spectra and energy of a signed complement graph (a new type of signed graph) can be found. The advancement of spectral signed complement graphs develops innovative techniques, enhancing both the theoretical and practical applications. Some graphs can be found which have the same energy but different spectra. Comparison can be done in the spectra, energies of signed graphs, and signed

complement graphs which have broad applications in various domains including network theory and algebra.

4. Conclusion

The spectrum is a finite sequence of numerical invariants. After calculating the spectra of the signed graph encryption-decryption can be done through the spectrum (sequence of numbers) instead of the signed graph. The Spectra and energy of signed graphs can be used in various applications: a graph-based predictive model for PPI using an optimal signed variational graph autoencoder, computer networks, data science, etc. The strength of the molecules is sharply associated with the spectrum of signed graphs and corresponding eigenvectors.

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