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Reconstruction Conjectures in Graph Theory: A Study on its Formation and Application

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Abstract

The Reconstruction Conjecture, which states that every graph G on at least three vertices is reconstructible, is one of the most important unsolved problems in graph theory which has existed for many years. If a graph can be inferred from the collection of all of its one-vertex removed unlabeled sub graphs up to isomorphism, then it is reconstructible. In the 1980s, immense research was conducted and many meaningful results have been generated on the conundrum and its discrepancies. The last 30 years have witnessed a gradual slowing down of research work on this topic. Trees can be rebuilt, but the proof is laborious (P. J. Kelly 1957). A succinct argument was provided that uses a straightforward yet effective counting theorem and is credited to Greenwell and Hemminger (1973). This paper focuses on the formation and application of reconstruction Conjecture in graph theory. It provides an analysis of Graph theory and Lattice theory. The study discusses the methodology involved in the implementation of Graph Theory concerning Reconstruction conjectures and discusses the stages of reconstruction Conjectures. It provides a theoretical evaluation as well as a comparative analysis of reconstruction conjecturers in graph Theory. The purpose of this paper is to provide a solution to conjunctures.

Keywords: Graph reconstruction conjecture; graph asymmetry; unique sub graph; Graph auto orphism; Graph isomorphism; Connectedness, Recognizable property.

Introduction

An intriguing topic that has existed for more than seven decades is the Reconstruction Conjecture. It asserts that any graph with at least three vertices is determined by its deck up to isomorphism. Further, the multi-set of graphs created by removing one vertex from graph G in every conceivable way is known as the deck. The term "cards" refers to the components of a deck. If a graph can be determined by its deck up to isomorphism, it can be recreated. If each member of the class is reconstructible, the class as a whole can be said to be reconstructible. Any aspect of a graph, such as a graph invariant, that is controlled by the graph's deck is referred to as reconstructible. In this study, a graph's asymmetry—a sub graph that does not repeat—is used to rebuild the graph from its deck. Ulam first proposed the graph reconstruction conjecture in 1960. In his Ph.D. dissertation, Kelly showed that regular graphs, Eulerian graphs, disconnected graphs, and trees may be reconstructed. In graphs without any shared edges between cycles and graphs with only one common vertex per cycle, the outer planner generates separate graphs without end vertices. Reconstructible graphs come in a variety of well-known forms, including maximal planar graphs, essential blocks, and graphs with a specific degree's sequence. Another way to approach this problem is to search for reconstructible graph invariants. The fundamental finding in this area is Kelly's Lemma. According to this assertion, a graph's correct sub graphs can be rebuilt using information about how frequently they occur.

Analysis of Graph Theory: Graph Theory is a branch of mathematics that models paired relationships between points or objects. The Koinsber Bridge problem from 1735 inspired the Theory of Graphs. The Eulerian graph, named after Euler, is the result of research on the Koinsber Bridge. Euler built the structure to solve the Koinsber Bridge problem. Following this initial stage, other mathematicians contributed to the advancement of graph theory. A.F. Mobius contributed to the theory in 1840 by introducing the concept of complete and bipartite graphs. Kuratowski jumped on board by demonstrating that complete and bipartite graphs are planners. He accomplished this by utilizing recreational issues. Gustav Kirchhoff introduced a new concept (the tree concept) in 1845 as the theory advanced. In circuits or electrical networks, he used graph theory. After studying polyhedra cycles, Thomas P. Kirkman and William R. Hamilton invented the Hamiltonian Graph in 1952. The

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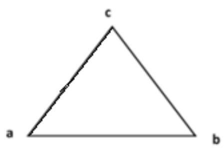
polyhedra cycles involved examining journeys that paid a single visit to specific destinations. H. Dudeney mentioned the puzzle problem in 1913. A century after its invention, Kenneth Appel and Wolfgang Haken solved the four-color problem. Analytical forms from differential calculus are another component that emerged after Caley studied it. It was planned to investigate the tree. This had an impact on theoretical chemistry, culminating in enumerative graph theory. Sylvester first used the term "Graph" in 1878. He drew a comparison between "Quantic Invariants" and algebra and molecular diagram covariants. Colorations were invented in 1941, and Ramsey was the brains behind the concept. There are numerous illustrations to define a graph developed by different researchers. In simpler terms, a graph is a mathematical model made up of points and lines. The points represent vertices, while the lines represent edges. Two points are linked to an edge.

A graph, popularly referred to as a linear graph and denoted as G, is composed of sets of objects (U and V), resulting in the equation $G = (U, V)$, $where\ U = u1, u2,$ is referred to as vertices and $V = u1, u2,$ .... is referred to as edges. The elements of $V = u1, u2,$ are referred to as edges. Each bk edge corresponds to a pair of points or vertices (Ur, Uk). These vertices Ur, Uk associated with edge $Vk$are referred to as terminal vertices of edge vk.

Consider the diagram below.

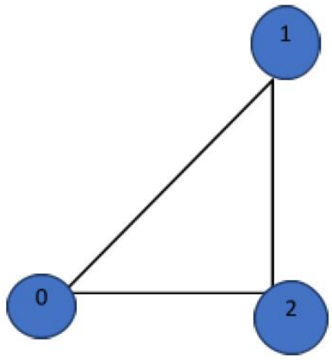


From the definition,

$G = \{U, V\}, where\ U = \{a,\ b, c\}, V = \{v1, v2, v3\},\ v1 = \{a, b\},\ v2 = \{a, c\}\ and,\ v3 = \{b, c\}$. The lines $v1$, $v2$, $and\ v3$, which are components of V, connect the vertices$V$ and thus equals $\{a, b\}$, $\{a, c\}$, $\{b, c\}$.

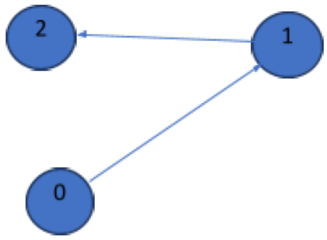
Classification of Graphs: Graphs are either classified as directed or undirected. It is said to be a directed graph if it contains an ordered pair of vertices. If the pairs are unordered, then it becomes an undirected graph.

Undirected graph:



Let G be a graph such that,

$G = \{U,\ V\},\ where\ U = \{0,\ 1,\ 2\}.\ and\ V = \{v1, v2, v3\},\ v1 = \{0, 1\},\ v2 = \{0, 2\}\ and\ v3 = \{1, 2\}$. If set V is altered such that $V = \{v1,\ v2\}\ where\ v1 = \{0, 1\}\ and\ v2 = \{1,\ 2\}$, a different structure with specific orientation is created. The graph will be reconstructed as below.

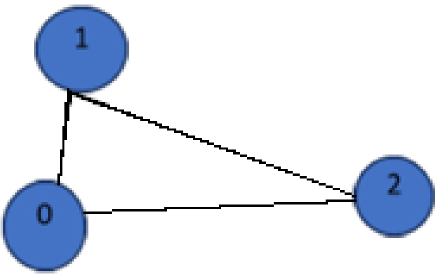


When the set V is altered, more graphs are reconstructed.

Implementation of Graph Theory: Various methodologies exist for the implementation of graph theory. The notable examples are the Adjacency Matrix Implementation and the Adjacency List Implementation.

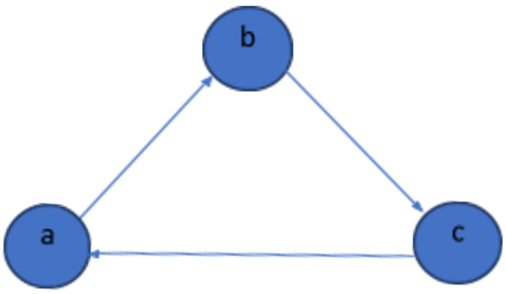
Adjacency Matrix Implementation: The idea of matrices is used in the adjacent matrix. A matrix is a collection of symbols or numbers that is typically displayed as a rectangular array. The entries or elements of the matrix are the numbers and symbols. Hence, nearby vertices serve as the matrix's elements or entries in the adjacent matrix implementation. If two vertices are supported by the same edge, they are regarded as being adjacent. This implementation counts the number of edges between two neighboring edges to describe a graph. Edges are displayed as values of an adjacent matrix, whereas nodes serve as the indices of a two-dimensional arrangement. Edges have non-zero values. Nodes are shown in rows and columns. The matrix contains either"0s" or "1s". A

1 denotes the presence of a path, while a zero denotes false, meaning that there is no path. The graph's vertices are the first row and column.
Consider a graph of the nature below:

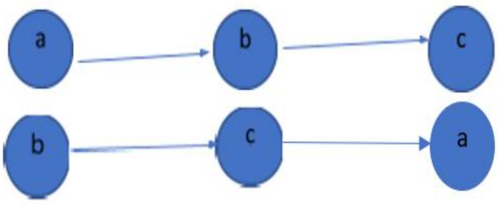


Three vertices (0, 1, 2) make up the graph. As a result, it can be represented as a matrix of 0s and 1s. The following is a depiction of the matrix.
0 1 2 1 0 1 2 1 0

Adjacency Linked List Implementation: With this approach, a graph is displayed as a list of linked schedules. Each element in the linked list that makes up the arrangement's index acts as a vertex for the other vertices that form a brink with the



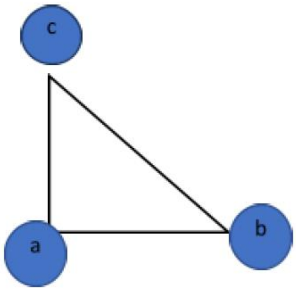
The above graph has been reconstructed by altering set V to obtain a directed graph. Using the adjacency list implementation results in the illustration below.



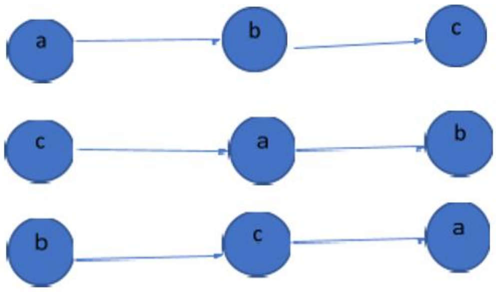
Research Questions

1. For a graph G denoted as (δ, θ) withδ≥3, are δ and θ reconstructible?
2. Given $G - v$ in the deck of$G$ = (δ − θ), are the degree of v and the degree of the
3. neighbor of v reconstructible?
4. Is connectivity of a graph G reconstructible?
5. Is connectedness a recognizable property for graphs

vertex. Take a look at the undirected graph below.



Using the adjacency list implementation result in the figure below:



Consider a directed graph as below:
of order ≥3?

6. Are disconnected graphs of order ≥3are reconstructible?
7. Are regular graphs reconstructible?

Research Objectives

1. To establish the constructability of δ and θ for a graph Gdenoted as (δ, θ) with δ≥3.
2. To determine the constructability of the degree of v and the degree of the neighbor of v,
3. given $G - v$ in the deck of G = (δ − θ).
4. To prove that the connectivity of graph G is reconstructible
5. To establish whether connectedness is a recognizable property for graphs of order ≥3
6. To determine if disconnected graphs of order ≥3are reconstructible
7. To define the reconstructibility of regular graphs.

Literature Review

Since the onset of Graph theory, various researchers have rendered their research contribution on this topic. This chapter presents an in-depth review of some empirical work done by researchers in this discipline. It reveals the limitations of the research work reviewed that offers an opportunity for researchers to indulge

(Ameneh Farhadian, 2018).
In his paper “Simple Explanation for the Reconstruction of Graphs”, published by the Sharif University of Technology, proposed a general framework to explain the reconstruction conjecture that “all graphs on at least three vertices are determined up to isomorphism by their deck.” His paper either negates or approves the conjecture but offers an explanation of why a graph is reconstructible. He demonstrates that any irregular graph has a proper induced sub graph. The induced sub graph is peculiar because of either its structural orientation or how it is joined to the rest of the graph. He defines the non-regular graph as an anchor while the induced sub graph is a connectional anchor if it is distinguishable in the deck. He further demonstrates that if an orbit exists in a graph, with a minimum of three vertices whose exclusion leaves an anchor, then the graph is reconstructible.
Ameneh in his findings concluded that the emergence of an anchor results from insufficient graph symmetry to prevent it. As a result, all graphs have anchors except for those with sufficient symmetry. Anchor is therefore an efficient tool to prove the constructability of a graph. He compares the anchor of a graph with the anchor which fixes a boat or fixes some cards to enable comparison. He disagrees with the argument that only cards containing the anchor can be used to reconstruct graphs hence his method is inefficient. His explanation stems from the fact that, to find the anchor of a graph, graphs from all cards ought to be considered to find the unique ones. His technique, therefore, implicitly applies all cards to reconstruct a graph. He asserts in his findings, that graph anchors provide evidence for the reconstruction of small graphs. In his paper, he demonstrates that the anchor extension converts the reconstruction of any graph with anchor number k, where k<(n-2) to the reconstruction of either an anchor-free structure. He investigates the reconstruction of graphs with an (n-2) -vertex anchors. The results of the investigations were that, a symmetric graph with (n-2)- vertex anchor is reconstructible. The finding implies that every graph is reconstructible. According to his argument, it is sufficient to investigate graphs with an (n-2) -vertex anchor whose auto orphism is non-trivial.
Generally, a graph is reconstructible if the relative position of two shadow anchors is reconstructible from the deck. He studied a family of anchor-free graphs and graphs with (n-1) -vertex anchors. In this part, he concentrated on vertex-transitive graphs and asserts that these graphs are reconstructible due to regularity. To study the anchor-free graphs, he introduces the concept of connectional anchors. The limitation of this study is that it fails to show the applicability of this technique on every anchor-free graph.
(S.K. Gupta & Akash Khandelwal, 2018). The duo sought to prove the reconstruction conjecture of graphs that are isomorphic to the cube of a tree. They utilized the constructability of trees from their peripheral vertex deleted graphs. The results of their findings are anchored in the (a) characterization of the cube of a tree (b) recognizability of the cube of a tree (c) uniqueness of the tree as a cube root of a graph G, except when G is a complete graph, and (d) the re-constructability of trees from their peripheral vertex deleted sub graphs. In this paper, they concluded that trees and squares of trees were proven to be re-constructible. They proved the conjecture for graphs isomorphic to the cube of a tree. The approach however fails to prove conjectures of the higher powers of a tree. Its unique argument cannot hold when proving classes of graphs isomorphic to fourth powers and beyond.
(Kia et al, 2007). In their paper "The reconstruction conjecture and edge ideals", they proved that it is possible to reconstruct many algebraic characteristics of edge ideals from a graph G on n vertices. This is achieved by removing a vertex from G from a collection of sub graphs. The properties proven to e reconstructible were the Krull dimension, the Hilbert function, and all the graded Betti numbers i, j where j < n. The paper is however not exhaustive on the reconstructible of algebraic ideal properties.
(Amitesh 2015). In his master's thesis, presented a structural analysis of graphs to reconstruct algebra and category theory from some partial information. He postulates that "the basis of a reconstruction of graph G could either be the classical multiset of G's vertex-deleted sub graphs or the multiset of relative degree-sequence of all included sub graphs of G. The study considered two basic equivalence

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relations on the set of vertices of G (Card equivalence and Auto orphism equivalence). While card equivalence is closer to reconstruction conjecture, this paper found Card equivalence to be inconvenient to work with. He demonstrated that auto orphism equivalence has a more transparent structure and is closely related to card equivalence. This was demonstrated by a characterization theorem for card equivalence which illustrates how to align Card equivalence with Auto orphism Equivalence. To express degree sequence, Amitesh provides a generalization of observation by Nash-Williams. He proposed a new conjecture that “Every graph G is uniquely determined up to isomorphism by the multiset of relative degree-sequences of its induced sub graphs”. This conjecture demonstrates that the novel conjecture is probably even more challenging to demonstrate than the Reconstruction conjecture. The study also. The class of card-minimal graphs, the deck of which is a set, was also examined in the study. It also expanded the definition of pseudo-similarity to include pairs of vertices in any graph G. He, therefore, demonstrated the reconstructibility of every card-minimal graph G that does not contain pseudo-similar pairings of vertices.

Reconstruction Conjecture in Graph Theory

The reconstruction hypothesis states that the multi-set of vertex-deleted sub graphs of a graph predicts the graph if it contains at least three vertices. Stanislaw Ulam (1960) and Paul Kelly (1957) both separately proposed this problem. In 1957 Paul Kelly wrote his doctoral dissertation under the supervision of Stanislaw Ulam. His dissertation established the Reconstruction Conjecture, often known as the Reconstruction Hypothesis, to be accurate for trees. Ulam presented a description of the reconstruction conjecture three years later, but he was already aware of the concepts behind it in 1929. During his time in graduate school in Poland, Ulam had accumulated difficulties that had been posed to him by instructors and other graduate students for many years. This has made it challenging to pinpoint who is to blame for posing the still unresolved graph theory problem. The Kelly-Ulam hypothesis is the widely recognized conclusion to this disagreement. In this section, Ulma Conjecture is employed alongside reconstruction conjectures to address the issue of reconstruction Conjectures in graph theory. Various theorems will be illustrated and expounded in this section.

Theorem 1.0: Ulam’s Conjecture

Ulam's Conjecture is arguably the most important unresolved topic in Graph Theory. The problem is attributed to P.J. Kelly and S.M. Ulam. Kelly's Ph.D. Thesis, prepared under S.M. Ulam in 1942, addressed this issue. It was introduced by Ulam as a set theory issue in "A Collection of Mathematical Problems," his well-known book. Originally, Ulam’s problem was stated as follows: “If E and F are two sets, each having m elements, such that a distance function with values 1 or 2 is defined for every pair of unique points and μ(ρ, ρ) = 0. If, for every subset of n − 1 points of$E$, there exists an isometric system of $m - 1$ points of F and the number of distinct subsets isometric to any given subset of m-1 points is the same in E and in F, then E and F are isometric “Kelly (1957), solved the graph theoretic form of this issue for trees and unconnected graphs, and validated it for graphs with up to six vertices.

The theorem is stated as: -

Let,

$X\, and\, Y\, be\, graphs\, with\, V(X) = v1, v2,\, \ldots\ldots vn\, and$
$for\, n \geq 3.\ If\, X - v1 \cong Y - ui,\ for\, all\, then\, X \cong Y.$

The case for all graphs supporting this assertion has advanced significantly. We've made excellent strides and discovered a lot about the kinds of graphs that can and cannot be reconstructed. There are classes of graphs that we are aware of, such as regular graphs, for which the Reconstruction Conjecture is unfalsifiable. Competitions are a type of graph that is one of many where the Reconstruction Conjecture is never valid.

The discovery of a number of qualities that may be drawn about any graph from its vertex-deleted sub graphs is another accomplishment to date. These characteristics are based on the information that each graph maintains when we look at those sub graphs, not on the ability to rebuild the graph itself. This might perhaps provide more information regarding the theory overall, according to several graph theorists. Many mathematicians started to consider various strategies as they worked on a potential proof for the Reconstruction Conjecture. One strategy that was somewhat popular was to tie the

Reconstruction Conjecture to a graph's edges rather than its vertices. People came to understand that, logically, more data about the original graph would be kept when only one edge of a graph was deleted.

While the Theorem has been restated differently by various theorists, Frank Harary formulated the Reconstruction conjecture. It is so far the most recent version of the problem. Frank Harary restated it as follows: -

Theorem 1.1: Reconstruction Conjecture as stated by Frank Harray. If G is a simple graph with n ≥ 3 vertices and if the n sub graphs G - vi are given, then the entire graph G can be reconstructed, uniquely up to isomorphism, from these vertex-deleted sub graphs.

Although they initially appear to be somewhat different, the Kelly-Ulam version—which just discusses the presence of an isomorphism—and the Harary version—which deals with figuring out the structure of the graph G—are logically comparable. Therefore, it is appropriate to work toward a solution of any statement of the problem that is logically equal to any one of these while working on an overall proof of this problem. This implies that very graph on at least three vertices is uniquely determined up to isomorphism by the collection of its one vertex-deleted sub graphs. Graphs that follow the aforementioned hypothesis are considered to be reconstructible.

Theorem 1.2. For a graph G denoted as (δ, θ) with δ≥3, then δ and θ are reconstructible.

Proof: It is inconsequential to determine the number δ, which must be greater than the order of any subgraph $G - v$. To determine θ, the subgraphs are labeled by G_i, i= 1, 2, ……. θ and suppose $G_i \cong G - vi$, where $vi \in V(G)$. Let θ_i denote the size of G_i. Consider an arbitrary edge e of G say $e = vjvk$. Then e belongs to δ − 2 of the subgraphs G_i and G_k.

Hence,

$$\sum_{i=1}^{\delta} \theta_i$$

Counts each edge δ − 2 times resulting in

$$\sum_{i=1}^{\delta} \theta_i = (\delta - 2)\theta$$

Consequently,

$$\frac{\sum_{i-1}^{\delta} \theta_i}{\delta - 2}$$

Corollary 1.3. The degree of v and the degree of the neighbor of v are reconstructible given $G - v$ in the deck of G = (δ − θ).

Proof: Let d' be the degree of the sequence of $G - v$. The vector differences $d - d'$ are nonzero entries and occur in positions corresponding to the neighbor of v in G, and their degrees can be read off from d. Then the degree of v is given by θ − $|E(G - v)|$. Since θ is reconstructible, the degree of v=θ − $|E(G - v)|$ is reconstructible.

Theorem 1.4: The connectivity of graph G is reconstructible.

Proof: Let the connectivity of a graph de denoted by $K(G)$Then graph, G is said to be disconnected if and only if $K(G) = 0$. If G is connected and $K(G) = k$ (≥1), then there exists a set of k-vertices vi, $v2$, ………., vk in $V(G)$ such that $G - (vi, v2, \ldots, vk)$ is disconnected and the deletion of a set of vertices less than k does not alter the graph G. It then follows that $K \; G - v = k - 1$, $i = 1, 2$, ………. k. $K(G - v) \geq k - 1$ for $v \in V(G) = \{v1, v2, vk$. Consequently, the R.H.S of the equation is known from the deck of G. For connected graph G, $K(G)$ − $K(G - v) + 1$, $K(G)$ is reconstructible.

Theorem 1.5: For graphs of order≥3, connectedness is a recognizable property. If G is a graph with $V(G) = v1, v2, \ldots v\delta$, δ≥3, the G is connected iff, at least two of the subgraphs $G - vi$ are connected.

Proof: Consider G to be a connected graph. Let $v1$ and $v2$ be at least two vertices that are not cut vertices. Then $G - \boldsymbol{v}$ and in $G - v2$ are connected. Similarly, assuming there exist two vertices vi and $v2$ where$vi, v2 \in V(G)$**.** Let both $G - v1$ and $G - v2$ be connected, then in $G - v1$ and G, $v2$ is connected to vi, i≥3. Further, $v1$ is connected to each vi, i≥3 both in G and in $G - v2$. This implies that all pairs of vertices of G are connected, hence G is connected. Because connectedness is a recognizable property, it is possible to infer from the sub graphs $G - v$, $v \in V(G)$Whether or not, the graph G of order ≥3 is connected.

Theorem 1.6: Disconnected graphs of order ≥ 3 are reconstructible.

Proof: From Theorem 1.5, since connectedness is a recognizable property, a general assumption is made that graph G is disconnected with $V(G) = v1$, $v2$, ……$v\delta$, δ≥3. For i = 1, 2, 3….., δ, let $G = G - vi$. G is reconstructible if it contains an isolated vertex. Let G contain no isolated vertex. In graph $D(G)$, let Tbe the component with the maximum number of vertices, then$T$ is a component of G. Consider a vertex of T that is not cut vertex denoted by v_0. Consider all the graphs in $D(G)$ with the least number of components isomorphic to T. Let $G - v$ be the graph with the greatest number of components isomorphic to $T = v_o$. Then$G$ can only be formed from $G - v$by replacing $T - vo$ with T.

Theorem 1.7: Regular graphs are reconstructible.

Proof: The degree sequence may be reconstructed from the deck of G. As a result, given D (G), it is possible to establish if G is regular, and if so, whether its degree r is reconstructible. Consequently, we may assume that G is a regular graph without losing generality, hence $V(G) = v1$, $v2$, …….. $V\delta$, δ≥3. Picking any $G - vi$ in the deck and adding a new vertex vi and joining all the vertices of degree $d - 1\, in\, G - v$, we reconstruct a regular graph of degree r in $G - vi$. This is implying that G is uniquely reconstructible.

Theorem 1.8 (Kelly's Lemma) Let R and S be graphs of order β_1, and β such that β_1<β. Then the number s(R, S) is recognizable from the subgraph $S - v$, $v \in V(S)$.

Proof: Each sub graph of $S$is isomorphic to R if it occurs in exactly $\beta - \beta_1$ subgraphs $S - v$, $v \in V(S)$. Hence,

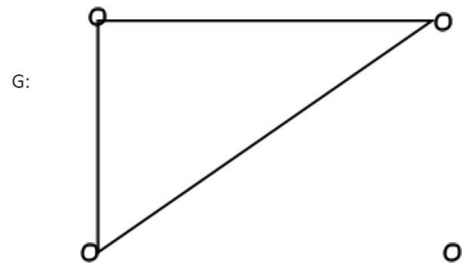
$$(\beta - \beta_1)_{s(R,S)} = \sum_{v \in V(S)} s(R, S - v)$$

The numerator of the R.H.S is recognizable and β_1, and β are known. Hence,

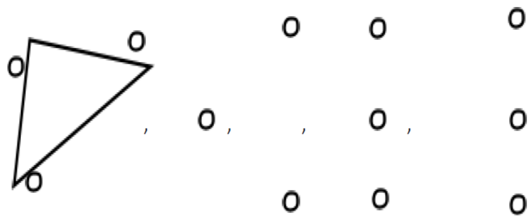
$$s(R, S) = \frac{\sum_{v \in V(S)}}{\beta - \beta_1}$$

Theoretical Evaluation: Given a graph G, a vertex-deleted subgraph of G is referred to as a card of G. The deck of G is the collection of all G cards and is symbolized by $D\ (G)$ The graphs in the deck are unlabeled, and if G includes isomorphic vertex-deleted subgraphs, such subgraphs are duplicated in $D\ (G)$ in proportion to the number of isomorphic subgraphs in G. As a result, $D\ (G)$ is a multiset of isomorphism type graphs rather than a set.

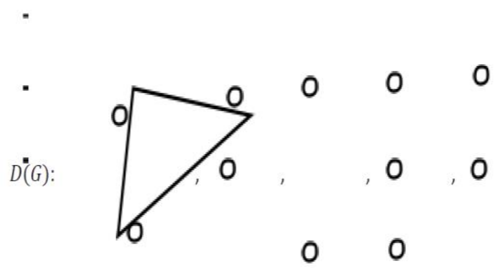
Consider the figure below: -



Deleting the vertices results in sub graphs as shown below: -



The above are sub graphs of the graph G formed by deleting vertices of the Graph. In isolation, each subgraph is a card of graph G, and collectively, they form a deck of G denoted by $D(G)$. Since the subgraphs are isomorphic, then $D(G)$ is a multiset of isomorphic type of graphs expressed as:



From the above illustration, any Graph with a deck $D(J) = D(G)$ is a reconstruction of G. G will be said to be reconstructible if every reconstruction of G is isomorphic to G and vice vasa. if J is a graph with two isolated vertices, then plainly J is a reconstruction of G but it is not isomorphic. Graph $D(J)$ is not reconstructible. It is therefore expressed as $G \cong K_2$.

Given a class F of graphs, and a property p defined of class F such that $p(G) = p(J)$ whenever $G \in F$ and J is a reconstruction of G, then p is said to be a recognizable property of G. A class of graphs F is said to be recognizable if, for all graphs G in F, any reconstruction of G must be in F. If, for all reconstruction $J\ of\ G$, $\vartheta(J) = \vartheta(G)$, the parameter $\vartheta = \vartheta(G)$ is said to be reconstructible. This implies that if $\vartheta(G)$ can be uniquely determined from its deck of G, then $\vartheta(G)$ is reconstructible. Similarly, if every graph in F is reconstructible, the class F of graphs is reconstructible. If for all graphs G in F, any graph in F that is a reconstruction of G isomorphic to G, then F is said to be weakly reconstructible.

If we can extract the unique graph from the vertex-deleted sub graphs up to isomorphism, then a graph G is said to be reconstructible. We also point out that a graph G is said to be labeled if each of its vertices has a unique label assigned to it in a one-to-one relationship. We presume that our graph is not labeled for the Reconstruction Conjecture's purposes. The conjecture would be trivially true if we were to examine labeled graphs since we could quickly determine the relationships between any two vertices by examining the vertex-deleted sub graph in which neither of these vertices has been deleted.

Conclusion

This paper carefully examined the reconstruction conjecture dating back to the origin and the subsequent restatement of Ulam's conjecture. Theorems reviewed revealed the following: - Every graph on at least three vertices is uniquely determined up to isomorphism by the collection of its one vertex-deleted sub graph. Every graph on at least three vertices is uniquely determined up to isomorphism by the collection of its one vertex-deleted sub graph. For a graph G denoted as (δ, θ) with δ≥3, then δ and θ are reconstructible. The degree of v and the degree of the neighbor of v are reconstructible given $G - v$ in the deck of G = (δ − θ). The connectivity of graph G is reconstructible and Disconnected graphs of order ≥ 3 are reconstructible. Regular graphs are reconstructible if the neighbor of v is reconstructible given $G - v$ in the deck of G = (δ − θ). The connectivity of graph G is reconstructible. Disconnected graphs of order ≥ 3 are reconstructible as well as Regular graphs. While this paper elaborates on reconstruction conjectures concerning various theorems, it does not shed light on reconstruction conjectures of the order of less than three. Further research is invited to include analysis of conjecture of order ≤ 3.

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