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Forced Oscillation of Variable Order Nonlinear Fractional Differential Equations with Damping Term and Time Delay

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Abstract: In this paper, we give sufficient condition for the forced oscillation of all solutions to the variable order nonlinear fractional delay differential equation with damping term of the form
$\left(D_{0+}^{1+\alpha(t)}y\right)(t-\tau)+p(t-\tau)\left(D_{0+}^{\alpha(t)}y\right)(t-\tau)+q(t)f[y(t)]=g(t)$
$where\ 0<\alpha(t)<1$ based on Riemann - Liouville fractional order derivative.
Keywords: Non linear variable order fractional delay differential equations, Forced oscillatory property, Damping term, Time delay.

Introduction
Fractional calculus which deals with the integrals and derivatives of any arbitrary real or complex order has many applications in widespread fields of science and engineering. It also provides various tools to solve integral and differential equations. History of fractional calculus begins with the derivative of order $\alpha=\frac{1}{2}$ was mentioned by Leibniz. Many mathematicians like Leibniz, Liouville, Grünwald, Letnikov, Riemann, Able, Riesz, and Weyl have found many applications of integrals and derivatives of non-integer order, and the fractional integro differential equations in theoretical physics, mechanics and applied mathematics. We have books on fractional calculus and fractional order differential equations [1-6]. Many articles are investigated on fractional order differential equations about the numerical solutions [7-9], existence and uniqueness solutions [10-12], oscillation properties [13-19] and equations with delay [20, 21].
In [22] the authors discussed the oscillation of nonlinear fractional differential equations with damping term of the form:
$(D_{0+}^{1+\alpha}y)(t)+p(t)(D_{0+}^{\alpha}y)(t)+q(t)f\big(y(t)\big)=g(t)$
(1.1)
With the initial condition $I_{0+}^{1-\alpha}y=b,$ where b is a real number, $0<\alpha<1$ is a constant and D_{0+}^{α} is the Riemann-Liouville fractional order derivative with order α of y.

In [23] the authors discussed about the oscillatory properties of the solutions to nonlinear fractional order differential equations with time delay and damping term of the form:
$(D_{0+}^{1+\alpha}y)(t-\tau)+p(t-\tau)(D_{0+}^{\alpha}y)(t-\tau)+q(t)f\big(y(t)\big)=g(t)$
(1.2)
where $y(t)=\xi(t),$ is a continuous function with $\lim_{t\to 0^-}\xi(t)=0$ when $t\in[-\tau,0),\ I_{0+}^{1-\alpha}y=b,$ where b is a real number, $0<\alpha<1,\ \tau$ is a constant and D_{0+}^{α} is the Riemann-Liouville fractional order derivative with order α of y.

In this paper we study the forced oscillation of variable order non linear fractional delay differential equations with damping term and time delay of the form:
$\left(D_{0+}^{1+\alpha(t)}y\right)(t-\tau)+p(t-\tau)\left(D_{0+}^{\alpha(t)}y\right)(t-\tau)+q(t)f\big(y(t)\big)=g(t)$
(1.3)
where $y(t)=\xi(t),$ is a continuous function with $\lim_{t\to 0^-}\xi(t)=0$ when $t\in[-\tau,0),\ I_{0+}^{1-\alpha(t)}y=b,$ where b is a real number, $0<\alpha(t)<1,\ \tau$ is a constant and $D_{0+}^{\alpha(t)}$ is the Riemann-Liouville fractional order derivative with order $\alpha(t)$ of y.

Also we use the following assumption:
$p(t)\epsilon C(R^+,R), q(t)\epsilon C(R^+,R^+), g(t)\epsilon C(R^+,R), f\epsilon C(R,R)\ and\ \frac{f(u)}{u}$
$0\ for\ all\ u\neq 0$ (1.4)

2. Preliminaries

Definition 2.1: A solution of a differential equation is said to be oscillatory if it has arbitrarily many zeros. If all the solutions of an equation are oscillatory, then the differential equation is said to be oscillatory.

Definition 2.2: The variable order Riemann-Liouville integral of function $f(u)$ is given by

$$ {}_{RL}I_{0,t}^{-\alpha(t)}f(u)=\frac{1}{\Gamma(\alpha(t))}\int_0^t(u-\lambda)^{\alpha(t)-1}f(\lambda)d\lambda,\quad t>0,\alpha(t)>0 \qquad (2.1)$$

Definition 2.3: The variable order Riemann-Liouville derivative function $f(u)$ is given by

$$ {}_{RL}D_{0,t}^{\alpha(t)}f(u)=\frac{1}{\Gamma(n-\alpha(t))}\frac{d^n}{dt^n}\int_0^t(u-\lambda)^{n-\alpha(t)-1}f(\lambda)d\lambda,\ \ t>0,\alpha(t)>0 \qquad (2.2)$$

Definition 2.4: The variable order Caputo derivative of $f\ (u)$ is given by

$$ {}_{C}D_{0,t}^{\alpha(t)}f(u)=\frac{1}{\Gamma(n-\alpha(t))}\int_0^t(u-\lambda)^{n-\alpha(t)-1}f^{(n)}(\lambda)d\lambda,\ \ t>0,\alpha(t)>0 \qquad (2.3)$$

with $n=[\alpha]+1$ where $[\alpha]$ is the integer part of α and the function $f(u)\in C^n$, which have n-order continuous derivative.

Lemma 2.1: Let $\alpha(t)\geq 0$, then for every $y\in L_1[a,b]$, where $L_1[a,b]$ is the set of all real-valued functions whose absolute value is integrable in the interval $[a,b]$. we have

$$(D_{a+}^{\alpha(t)}I_{a+}^{\alpha(t)}y)(t)=y(t). \qquad (2.4)$$

Lemma 2.2: Let $\alpha(t)\geq 0, m\in N, and\ D=\frac{d}{dx}$, if the fractional order derivatives $\left(D_{a+}^{\alpha(t)}y\right)(x) and \left(D_{a+}^{\alpha(t)+m}y\right)(x)$ exists then

$$\left(D^m D_{a+}^{\alpha(t)}y\right)(x)=\left(D_{a+}^{\alpha(t)+m}y\right)(x). \qquad (2.5)$$

Lemma 2.3: Let $\alpha(t)>0\ and\ n=[\alpha]+1$, assume that y is such that $y(t)\in L_1[a,b], and\ I_{a+}^{n-\alpha(t)}y\in AC^m([a,b])$ be the fractional integral of order $n-\alpha$, then

$$(I_{a+}^{\alpha(t)}D_{a+}^{\alpha(t)}y)(t)=y(t-\tau)-\sum_{k=0}^{n-1}\frac{(t-\tau-\alpha(t))^{\alpha-k-1}}{\Gamma(\alpha(t)-k)}\lim_{z\to a^+}D^{n-k-1}\,I_{a+}^{n-\alpha(t)}y(z-\tau) \qquad (2.6)$$

In particular, for $\alpha(t)\in(0,1)$, we have

$$(I_{a+}^{\alpha(t)}D_{a+}^{\alpha(t)}y)(t)=y(t-\tau)-\frac{(t-\tau-\alpha)^{\alpha-1}}{\Gamma(\alpha(t))}\lim_{z\to a^+}I_{a+}^{n-\alpha(t)}y(z-\tau) \qquad (2.7)$$

where τ is a constant, $[\alpha]$ is the integer part of α, $AC^m([a,b])$ is the set of functions with absolutely continuous derivative of order $n-1\ in\ [a,b]$.

Proof: Since by our assumption on y, the limit on the right hand side exists and $D^{n-1}I_{a+}^{n-\alpha(t)}y$ is continuous. Also there exists some $\varphi\in L_1$ such that

$$D^{n-1}I_{a+}^{n-\alpha(t)}y=D^{n-1}I_{a+}^{n-\alpha(t)}y(a)+I_{a+}^{1}\varphi \qquad (2.8)$$

The solution of this classical differential equation of order $n-1$ for $I_{a+}^{n-\alpha(t)}$ is of the form

$$I_{a+}^{n-\alpha(t)}y(t-\tau)=\sum_{k=0}^{n-1}\frac{(t-\tau-\alpha(t))^k}{\Gamma(k+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau)+I_{a+}^{n}\varphi(t-\tau). \qquad (2.9)$$

By the definition of $D_{a+}^{\alpha(t)}y$, we have

$$I_{a+}^{\alpha(t)}D_{a+}^{\alpha(t)}y(t-\tau)=I_{a+}^{\alpha(t)}D_{a+}^{n}I_{a+}^{n-\alpha(t)}y(t-\tau)$$

$$=I_{a+}^{\alpha(t)}D_{a+}^{n}[\sum_{k=0}^{n-1}\frac{(t-\tau-\alpha)^k}{\Gamma(k+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau)+I_{a+}^{n}\varphi(t-\tau)]$$

$$=I_{a+}^{\alpha(t)}D_{a+}^{n}I_{a+}^{n}\varphi(t-\tau)+\sum_{k=0}^{n-1}\frac{I_{a+}^{\alpha(t)}D_{a+}^{n}[(t-\tau-\alpha(t))^k]}{\Gamma(k+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau)$$

$$=I_{a+}^{\alpha(t)}\varphi(t-\tau)+\sum_{k=0}^{n-1}\frac{(t-\tau-\alpha)^k}{\Gamma(k+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau) \qquad (2.10)$$

Applying the operator $D_{a+}^{n-\alpha(t)}$ to both sides of the above equation we get

$$y(t-\tau)=D_{a+}^{n-\alpha(t)}I_{a+}^{\alpha(t)}\varphi(t-\tau)+\sum_{k=0}^{n-1}\frac{D_{a+}^{n-\alpha(t)}[(t-\tau-\alpha(t))^k]}{\Gamma(k+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau) \qquad (2.11)$$

$$=I_{a+}^{\alpha(t)}\varphi(t-\tau)+\sum_{k=0}^{n-1}\frac{(t-\tau-\alpha(t))^{k-n-\alpha(t)}}{\Gamma(k-n+\alpha(t)+1)}\lim_{z\to a^+}D^k\,I_{a+}^{n-\alpha(t)}y(z-\tau) \qquad (2.12)$$

By replacing $k\ as\ n-k-1$ and solve for $I_{a+}^{n-\alpha(t)}$ then combining the result with $I_{a+}^{\alpha(t)}D_{a+}^{\alpha(t)}y(t-\tau)=I_{a+}^{\alpha(t)}y(t-\tau)$, we obtain

$$I_{a+}^{\alpha(t)}D_{a+}^{\alpha(t)}y(t-\tau)=I_{a+}^{\alpha(t)}y(t-\tau)$$

$= y(t-\tau) - \sum_{k=0}^{n-1} \frac{(t-\tau-\alpha(t))^{k-n-\alpha(t)}}{\Gamma(\alpha(t)-k)} \lim_{z\to a^+} D^{n-k-1} I_{a+}^{n-\alpha(t)} y(z-\tau)$. (2.13)

This gives the desired result.

3. Main Results

Theorem 3.1: Suppose that (1.4) and the following conditions hold:

$$\lim_{t\to\infty} inf \int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)} [M + \int_{t_0}^{\omega} g(s)V(s-\tau)\, ds]dw < 0$$

(3.1)

$$\lim_{t\to\infty} Sup \int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)} [M + \int_{t_0}^{\omega} g(s)V(s-\tau)\, ds]dw > 0$$

(3.2)

where $V(s) = exp \int_{t_0}^{s} p(v)dv, M$ is an arbitrary constant. Then each solution of the problem (1.3) is oscillatory.

Proof: Assume that contrary that $y(t)$ is a non oscillatory solution of (1.3).

Let us assume that there exists $T > 0, t_0 \geq T,$ such that $y(t) > 0\ for\ all\ t \geq t_0$. According to (1.3) and (1.4) the following inequality is satisfied:

$$[(D_{0+}^{\alpha(t)} y)(t-\tau)V(t-\tau)]' = (D_{0+}^{1+\alpha(t)} y)(t-\tau)V(t-\tau) + (D_{0+}^{\alpha(t)} y)(t-\tau)p(t-\tau)V(t-\tau)$$
$$= -q(t)f(y(t))V(t-\tau) + g(t)V(t-\tau)$$
$$< g(t)V(t-\tau)$$

By integrating from $t_0\ to\ t,$ we get

$$(D_{0+}^{\alpha(t)} y)(t-\tau)V(t-\tau) < (D_{0+}^{\alpha(t)} y)(t_0-\tau)V(t_0-\tau) + \int_{t_0}^{t} g(s)V(s-\tau)ds$$
$$= M + \int_{t_0}^{t} g(s)V(s-\tau)ds$$

Where $M = (D_{0+}^{\alpha(t)} y)(t_0-\tau)V(t_0-\tau)$.

By Lemma 2.3 and the inequality (3.3) we obtain

$$y(t-\tau) < \frac{I_{0+}^{1-\alpha(t)}(-\tau)}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} + I_{0+}^{\alpha(t)}[\frac{M}{V(t-\tau)} + \frac{1}{V(t-\tau)}\int_{t_0}^{t} g(s)V(s-\tau)ds]$$
$$= \frac{b}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} + \frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau} \frac{(t-\tau-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega.$$

Replacing $t-\tau\ as\ t$ we obtain:

$$y(t) < \frac{b}{\Gamma(\alpha(t))} t^{\alpha(t)-1} + \frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau} \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega.$$

As $t \to +\infty$, we get

$$\lim_{t\to+\infty} \inf y(t) \leq \lim_{t\to+\infty} inf \frac{b}{\Gamma(\alpha(t))} t^{\alpha(t)-1} + \lim_{t\to+\infty} inf \frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau} \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega < 0$$

Which is the contradiction for out assumption that $y(t) > 0$.

On the other hand, let us assume that there exists $T > 0, t_0 \geq T,$ such that $y(t) < 0\ for\ all\ t \geq t_0$. According to (1.3) and (1.4) the following inequality is satisfied:

$$[(D_{0+}^{\alpha(t)} y)(t-\tau)V(t-\tau)]' = (D_{0+}^{1+\alpha(t)} y)(t-\tau)V(t-\tau) + (D_{0+}^{\alpha(t)} y)(t-\tau)p(t-\tau)V(t-\tau)$$
$$= -q(t)f(y(t))V(t-\tau) + g(t)V(t-\tau)$$
$$> g(t)V(t-\tau)$$

By integrating from $t_0\ to\ t,$ we get

$$(D_{0+}^{\alpha(t)} y)(t-\tau)V(t-\tau) > (D_{0+}^{\alpha(t)} y)(t_0-\tau)V(t_0-\tau) + \int_{t_0}^{t} g(s)V(s-\tau)ds$$
$$= M + \int_{t_0}^{t} g(s)V(s-\tau)ds$$

Where $M = (D_{0+}^{\alpha(t)} y)(t_0-\tau)V(t_0-\tau)$.

By Lemma 2.3 and the inequality (3.3) we obtain

$$y(t-\tau) > \frac{I_{0+}^{1-\alpha(t)}(-\tau)}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} + I_{0+}^{\alpha(t)}[\frac{M}{V(t-\tau)} + \frac{1}{V(t-\tau)}\int_{t_0}^{t} g(s)V(s-\tau)ds]$$
$$= \frac{b}{\Gamma(\alpha(t))}(t-\tau)^{\alpha(t)-1} + \frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau} \frac{(t-\tau-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega.$$

Replacing $t-\tau$ as t we obtain:

$$y(t) > \frac{b}{\Gamma(\alpha(t))}t^{\alpha(t)-1} + \frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau}\frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega.$$

As $t \to +\infty$, we get

$$\lim_{t\to+\infty}\sup y(t) \geq \lim_{t\to+\infty}\sup\frac{b}{\Gamma(\alpha(t))}t^{\alpha(t)-1} + \lim_{t\to+\infty}\sup\frac{1}{\Gamma(\alpha(t))}\int_{t_0}^{t-\tau}\frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}(M + \int_{t_0}^{t} g(s)V(s-\tau)ds)d\omega > 0$$

Which is the contradiction for out assumption that $y(t) < 0$.

This completes the proof.

4. Example

Consider the variable order nonlinear fractional delay differential equation with damping term of the form: $\left(D_{0+}^{\frac{3}{2}}y\right)(t-2) - \left(D_{0+}^{\frac{1}{2}}y\right)(t-2) + t^2ye^y = e^{2t}sint$

(4.1)

in which $\tau = 2, \alpha(t) = \frac{t}{2}$ with $t = 2n-1, n \in N, p(t-\tau) = -1, q(t) = t^2, f(y(t)) = ye^y$ and $g(t) = e^{2t}sint.$

Let $\alpha(t) = \frac{1}{2}$. Then:

$V(s) = exp\int_{t_0}^{s} p(v)dv = exp\int_{t_0}^{s} -1dv = \exp(-s+t_0) = e^{t_0-s}$.

By setting $t_0 = \frac{\pi}{4}$, we get $V(s) = e^{\frac{\pi}{4}-s}$ (4.2)

Also, $\int_{t_0}^{\omega} g(s)V(s-\tau)ds = \int_{\frac{\pi}{4}}^{\omega} e^{2s}sins\, e^{\frac{\pi}{4}-s+2}ds = \int_{\frac{\pi}{4}}^{\omega} sins\, e^{\frac{\pi}{4}+s+2}ds$ (4.3)

Now Consider,

$$\int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}[M + \int_{t_0}^{\omega} g(s)V(s-\tau)\,ds]d\omega$$

$= \int_0^t \frac{(t-\omega)^{-1/2}}{e^{\frac{\pi}{4}-\omega+2}}[M + \int_{\frac{\pi}{4}}^{\omega} sins\, e^{\frac{\pi}{4}+s+2}ds]d\omega$ (by using (4.3))

$= \int_0^t (t-\omega)^{-1/2}e^{-\frac{\pi}{4}+\omega-2}[M + \int_{\frac{\pi}{4}}^{\omega} sins\, e^{\frac{\pi}{4}+s+2}ds]d\omega$

$= \int_0^t (t-\omega)^{-1/2}[Me^{-\frac{\pi}{4}+\omega-2} + \int_{\frac{\pi}{4}}^{\omega} e^{s+\omega}sins\, ds]d\omega$

$= \int_0^t (t-\omega)^{-1/2}[Me^{-\frac{\pi}{4}+\omega-2} + \frac{\sqrt{2}}{2}e^{2\omega}\sin(\omega - \frac{\pi}{4})]d\omega$

By setting $t-\omega = s^2$, the above value

$$= \int_{\sqrt{t}}^{0} s^{-1}[Me^{-\frac{\pi}{4}+t-s^2-2} + \frac{\sqrt{2}}{2}e^{2(t-s^2)}\sin\left(t-s^2-\frac{\pi}{4}\right)](-2s)ds$$

$$= \int_0^{\sqrt{t}}[2Me^{-\frac{\pi}{4}+t-s^2-2} + \sqrt{2}e^{2(t-s^2)}\sin\left(t-\frac{\pi}{4}-s^2\right)]ds$$

$$= 2Me^{-\frac{\pi}{4}+t-2}\int_0^{\sqrt{t}} e^{-s^2}ds + \sqrt{2}e^{2t}\int_0^{\sqrt{t}} e^{-2s^2}[\sin\left(t-\frac{\pi}{4}\right)cos\, s^2 - \cos\left(t-\frac{\pi}{4}\right)sin\, s^2]ds$$

$= 2Me^{-\frac{\pi}{4}+t-2}\int_0^{\sqrt{t}} e^{-s^2}ds + \sqrt{2}e^{2t}sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}cos\, s^2ds - \sqrt{2}e^{2t}cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}sin\, s^2ds$ (4.4)

Let $t \to +\infty$. Also $\left|e^{-2s^2}cos\, s^2\right| \leq e^{2s^2}$ and $\lim_{t\to+\infty}\int_0^{\sqrt{t}} e^{-2s^2}ds = \frac{\sqrt{2\pi}}{4}$, also $\lim_{t\to+\infty}\int_0^{\sqrt{t}} e^{-2s^2}cos\, s^2ds$ is convergent.

Let $\lim_{t\to+\infty}\int_0^{\sqrt{t}} e^{-2s^2}cos\, s^2ds = P$ and $\lim_{t\to+\infty}\int_0^{\sqrt{t}} e^{-2s^2}sin\, s^2ds = Q.$

Let us select the sequence $\{t_k\} = \left\{\frac{3\pi}{2} + \frac{\pi}{4} + 2k\pi - tan^{-1}(\frac{-Q}{P})\right\}$ for which $\lim_{t\to+\infty} e^{t_k} = +\infty$

calculating the value of

$\lim_{k\to\infty} e^{t_k}[2Me^{-\frac{\pi}{4}+t_k-2}\int_0^{\sqrt{t_k}} e^{-s^2}ds + \sqrt{2}e^{t_k}sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{t_k-2s^2}cos\, s^2ds - \sqrt{2}e^{t_k}cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{t_k-2s^2}sin\, s^2ds$ (4.5)

Let us first consider the following limit

$$\lim_{k\to\infty}[sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2 ds - cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2 ds]$$

$$= P\lim_{k\to\infty}\sin\left[\frac{3\pi}{2}+2k\pi-tan^{-1}\left(\frac{-Q}{P}\right)\right] - Q\lim_{k\to\infty}\cos\left[\frac{3\pi}{2}+2k\pi - tan^{-1}\left(\frac{-Q}{P}\right)\right]$$

$$= P\lim_{k\to\infty}\sin\left[\frac{3\pi}{2}-tan^{-1}\left(\frac{-Q}{P}\right)\right] - Q\lim_{k\to\infty}\cos\left[\frac{3\pi}{2}-tan^{-1}\left(\frac{-Q}{P}\right)\right]$$

$$= \sqrt{P^2+Q^2}\sin\left[\frac{3\pi}{2}-tan^{-1}\left(\frac{-Q}{P}\right)+tan^{-1}\left(\frac{-Q}{P}\right)\right]$$

$$= \sqrt{P^2+Q^2}\sin\frac{3\pi}{2}$$

$$= -\sqrt{P^2+Q^2} \quad (4.6)$$

Next we calculate the

$$\lim_{k\to\infty} 2Me^{-\frac{\pi}{4}-2}\int_0^{\sqrt{t_k}} e^{-s^2}ds = 2Me^{-\frac{\pi}{4}-2}\frac{\sqrt{\pi}}{2} = \sqrt{\pi}Me^{-\frac{\pi}{4}-2} \quad (4.7)$$

Also $\lim_{t\to+\infty} \mathrm{e}^{t_k} = +\infty$

$$(4.8)$$

Using (4.6), (4.7) and (4.8) in (4.5) we obtain

$$\lim_{k\to\infty} \mathrm{e}^{t_k}[\,2Me^{-\frac{\pi}{4}+t_k-2}\int_0^{\sqrt{t_k}} e^{-s^2}ds + \sqrt{2}e^{t_k}sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{t_k-2s^2}\cos s^2 ds - \sqrt{2}e^{t_k}cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{t_k-2s^2}\sin s^2 ds$$

$$= (+\infty)\left(\sqrt{\pi}Me^{-\frac{\pi}{4}-2}\right) + (+\infty)(-\sqrt{P^2+Q^2})$$

$$= -\infty$$

Hence from (4.4) we obtain

$$\lim_{t\to+\infty} inf\int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}[M + \int_{t_0}^{\omega} g(s)V(s-\tau)\,ds]d\omega$$

$$\le \lim_{k\to\infty}[2Me^{-\frac{\pi}{4}+t_k-2}\int_0^{\sqrt{t_k}} e^{-s^2}ds + \sqrt{2}e^{2t_k}sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2 ds - \sqrt{2}e^{2t_k}cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2 ds\,]$$

$$= -\infty < 0.$$

Thus $\lim_{t\to+\infty} inf\int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}[M + \int_{t_0}^{\omega} g(s)V(s-\tau)\,ds]d\omega < 0.$ (4.9)

Similarly by selecting the sequence $\{t_j\} = \left\{\frac{\pi}{2}+\frac{\pi}{4}+2j\pi - tan^{-1}(\frac{-Q}{P})\right\}$ we get

$$\lim_{t\to+\infty} sup\int_0^t \frac{(t-\omega)^{\alpha(t)-1}}{V(\omega-\tau)}[M + \int_{t_0}^{\omega} g(s)V(s-\tau)\,ds]d\omega > 0. \quad (4.10)$$

Hence by Theorem (3.1) all the solutions of equation (4.1) are oscillatory.

5. Conclusion

This paper provides the sufficient condition for the forced oscillation of variable order nonlinear fractional differential equation with damping term and time delay along with illustrated example. The conclusion is that if the conditions (3.1) and (3.2) hold, then each solution of the problem (1.3) oscillates. In our future research, we would like to acquire the desired result for the problem with multiple delays.

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