Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 45 No. 11
November 2024

Study of Hypergeometric Functions in Context of Nonelementary Integrals

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Abstract
The present paper is a study of relations between elementary functions, nonelementary functions, nonelementary integrals and hypergeometric functions in context of antiderivatives. Antiderivative is one of the important mathematical operator or machine, which produces new functions. That new function may or may not belong to the set from which the integrand might have been taken. The problem arises, when the new function doesn't belong to the previous set, the set of elementary functions, which opens the scope for the development of new functions. Till the new function is propounded, such integral, which doesn't belong to the previous set of elementary functions, are treated as nonelementary function or nonelementary integral. The introduction of special functions like hypergeometric function has ended the study of searching new elementary and nonelementary functions, which has great effect on expressing antiderivatives of those integrands, whose integrals are still not expressible in terms of elementary functions. Due to the less research on it, has almost terminated the development of new properties between such functions in context of elementary and nonelementary functions. In this paper an attempt has been made to propound some propositions between elementary and nonelementary functions with special functions based on sufficient examples in context of nonelementary integrals. The whole discussion is centred about five questions: are all elementary functions hypergeometric functions?, are all nonelementary functions hypergeometric functions?, are all hypergeometric functions elementary functions?, are all hypergeometric functions nonelementary functions?, and are all hypergeometric functions as output from Mathematica as antiderivative nonelementary functions? Studying on these queries we propounded eleven propositions, out of which one of the important finding is that the outcome of the input "Integrate[f[x], x]" from Mathematica in terms of hypergeometric functions is always nonelementary functions (or integrals) and every elementary and nonelementary functions cannot be expressed in terms of hypergeometric functions. We also found that every real (or complex) numbers can be expressed in terms of hypergeometric function. The paper ends with answers of the five questions and also opens a new scope of research for the computer software experts to explore new ideas of elementary and nonelementary functions in terms of antiderivatives.

Keywords: Elementary and Nonelementary Functions, Nonelementary Integrals, Hypergeometric Function.

1. Introduction

The function has an important contribution in the development of mathematics and in its allied sciences with their practical applications. A function is a relation between a set of inputs having one output each. In other words, a function is a relationship between inputs, where each input is related to exactly one output. The very important functions are being treated as special functions and have been given their own name like logarithmic, exponential, trigonometric, and hyperbolic functions and have been extended to cover the gamma, beta, zeta, spherical, parabolic functions, etc. (Hannah, 2013). Its concept enters whenever quantities are connected by a definite physical relationship. It was originally the idealization of how a varying quantity depends on another quantity. For example, the position of a planet is a function of time, the volume of a gas in a cylinder is a function of the temperature and of the pressure on the piston, the atmospheric pressure in a balloon is a function of the altitude above sea level, etc. The main body of modern mathematics centres about the concept of function. An expression $x^2 + 3x + 1$ has no definite numerical value until the value of x is assigned. We say that the value of this expression is a function of the value of x and denote it by $f(x) =$

$x^2 + 3x + 1$ (Hannah, 2013; Function (mathematics) - Wikipedia contributors, 2024).

Historically the concept of function was elaborated with the infinitesimal calculus at the end of the 17th century. 18th century onwards, mathematicians used it to express mathematical formulae to express the exact nature of the relationship and it was formalized at the end of the 19th century. Leibniz (1646-1716) first used the word "function" (Courant et al., 2021; Katz, 2019; Vygodsky, 1987; Function (mathematics) - Wikipedia contributors, 2024). Functions can be divided in many ways but in context of the present study and the nature of the antiderivative of functions, it will be divided into elementary and nonelementary functions (Function (mathematics) - Wikipedia contributors, 2024; Hannah, 2013).

We call a function an elementary function if it can be expressed as an equation in the form y = f(x), where f(x) represents an expression formed by the combination of a finite collection of powers of x, constant, trigonometric functions, inverse trigonometric functions, hyperbolic functions, inverse hyperbolic functions, exponentials, and logarithms together through additions, subtractions, multiplications, divisions, powers, and compositions. For example, $f(x) = \sin x + x$, $g(x) = 5x^2 + 2x - e^x$, $h(x) = \tanh x$, $i(x) = \log x$ etc. are elementary functions. But all functions are not elementary function. A function which is not elementary is known as a nonelementary function. A popular example of a nonelementary function is a piecewise defined function

$$f(x) = \begin{cases} 0, & \text{if x is rational} \\ 1, & \text{if x is irrational} \end{cases}$$

on [0, 1]. The function f(x) = [x] is also a nonelementary function (Marchisotto et al., 1994; Elementary Function –- Wikipedia contributors, 2024; Cherry, 1985, 1986; Kasper, 1980; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972).

There is a unique property of the elementary function is that it can always be written in a closed form. It usually does not contain limit or integral. But a special care is needed in talking about elementary and nonelementary functions or closed form expressions and non-closed form expressions. There are expressions which are not in closed form but can be reduced into it after simplification or summation or using other operations. For example, the expression

$$f(x) = \sum_{n=0}^{\infty} \frac{x}{2^n} = \frac{x}{2^0} + \frac{x}{2^1} + \frac{x}{2^2} + \cdots + \frac{x}{2^n} + \cdots$$

is not in closed form because the summation contains infinite number of terms and elementary operations, however using the summation rule of a geometric series it can be expressed in the closed form as $f(x) = 2x$ (Marchisotto et al., 1994; Closed form expression – Wikipedia contributors, 2024; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972).

Some interesting facts exist for such functions. If a function is expressed in closed form expressions, then its derivative can also be expressed in closed form. But its integral (antiderivative) may or may not be expressed in closed form. An example of an elementary function whose antiderivative does not have a closed form expression is e^{-x^2}, whose one antiderivative is known as the error function, which is denoted and defined by

$$\text{erf}(x) = \frac{1}{\pi}\int_0^x e^{-t^2} dt.$$

Such integral is called a nonelementary integral. Many types of nonelementary functions arise in integration as antiderivatives. The integral $\int \frac{\sin x}{x} dx$ is also not an elementary function. Although many pioneer mathematicians contributed in the development of the concepts of elementary and nonelementary functions, a thorough study was introduced by Joseph Liouville in a series of papers from 1833 to 1841 and the algebraic treatment of elementary functions was started by Joseph Fels Ritt in 1930s (Marchisotto et al., 1994; Elementary Function – Wikipedia contributors, 2024; Cherry, 1985, 1986; Kasper, 1980; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972).

So a nonelementary integral (or antiderivative) of a given elementary function is an antiderivative that is not an elementary function and that cannot be expressed in closed form expression also. Some well known examples are the elliptic integral $\int \sqrt{1 - x^4}\, dx$, logarithmic integral $\int \frac{1}{\ln x} dx$, Gaussian integral $\int e^{-x^2} dx$, Fresnel integrals $\int \sin(x^2)\, dx$ and $\int \cos(x^2)\, dx$, Sine integral or Dirichlet integral $\int \frac{\sin x}{x} dx$, Cosine integral $\int \frac{\cos x}{x} dx$, Exponential integral $\int \frac{e^{-x}}{x} dx$, etc. (Calculus -

Wikipedia contributors, 2024; Closed-form expression - Wikipedia contributors, 2024; Differential calculus - Wikipedia contributors, 2024; Elementary function - Wikipedia contributors, 2024; Gale et al., 1923; Integral - Wikipedia contributors, 2024; Marchisotto et al., 1994; Singer et al., 1985; Trager, 2022; Corliss et al., 1989; Nijimbere, 2017, 2018, 2020a, 2020b; Sao, 2021; Sharma et al., 2020; Victor, 2017; Nonelementary integral – Wikipedia contributors, 2024). We also know that the integral of every continuous function exists and is itself a continuous function of the upper limit, and this fact has nothing to do with the question whether the integral can be expressed in terms of elementary functions or not. In this sense the process of integration forms a basis for the generation of new functions. These new functions are in general not elementary functions.

An interesting story lies in the development and approaches of Calculus. Infinitesimal calculus was developed independently in the late 17th century by Isaac Newton (January 4, 1643 – March 31, 1727) and Gottfried Wilhelm Leibniz or Leibnitz (July 1, 1646 – November 14, 1716) (Isaac Newton - Wikipedia contributors, 2024; Gottfried Wilhelm Leibniz - Wikipedia contributors, 2024). An argument over priority led to the Leibniz and Newton calculus controversy continued until the death of Leibniz in 1716. Later both were given equal credit for the development of Calculus (History of calculus - Wikipedia contributors, 2024). Although they got equal credit for the development of the subject but their approaches were different. Newton followed the differentiation and integration in series of terms whereas Leibniz used the closed form expression to express derivative and integrals of functions. Later it was found that both approaches lead the same result (History of calculus - Wikipedia contributors, 2024).

Analyzing their approaches, we find that Newton's approach was in terms of nonelementary integrals whereas Leibniz's approach was in terms of elementary functions. But many such nonelementary functions were written in terms of elementary functions after the development of function theories and their mathematical notations. Those functions or integrals, which were not expressible in terms of elementary functions, were called nonelementary functions, which leaded to the development of many special functions (Yadav, 2015). Hypergeometric function is one such function whose series expansion is known as hypergeometric series.

The term "hypergeometric series" was first used by John Wallis in his 'Arithmetic Infinitorum' in 1655 to describe infinite series of the form

$$1 + a + a(a+1) + a(a+1)(a+2) + \cdots$$

Later hypergeometric series were studied by Leonhard Euler but the first full systematic treatment was given by Carl Friedrich Gauss in 1813 (Hypergeometric function - Wikipedia contributors, 2024). In 1836 this term was again used by Ernst Eduard Kummer (1810-1898) for the series

$$1 + \frac{\propto.\beta}{1.\gamma}x + \frac{\propto.(\propto+1).\beta.(\beta+1)}{1.2.\gamma.(\gamma+1)}x^2 + \frac{\propto.(\propto+1).(\propto+2).\beta.(\beta+1).(\beta+2)}{1.2.3.\gamma.(\gamma+1).(\gamma+2)}x^3 + \cdots$$

which is the now the present form of the hypergeometric series. The term 'hypergeometric' is generally applied for three situations: the hypergeometric equation - a linear second order differential equation, the hypergeometric series – a particular solution of the hypergeometric equation, and the hypergeometric function – the sum of the hypergeometric series. This type of function has been extensively studied by many mathematicians, including Johann Friedrich Pfaff, Kummer, Euler, Gauss, Srinivasa Ramanujan, etc. (Hannah, 2013; Hypergeometric function - Wikipedia contributors, 2024).

The hypergeometric function plays a central role in the realm of special mathematical functions, as all special functions can be expressed in terms of these functions. It is frequently encountered in pure mathematics, and its parametric nature provides a powerful tool for the solution of a wide range of applied problems (Hannah, 2013). The computer software Mathematica has played a major role in its applications in expressing new functions (due to antiderivatives) in terms of special, elementary and nonelementary functions.

Many computer algorithms have been developed for mathematics, and Mathematica is one of them, which investigates the antiderivative of function that whether it is elementary function or special nonelementary function or hypergeometric function. Petkovšek et al. (1996) aptly state and ask 'Can the hypergeometric series be summed in a simple closed form?' i.e., can we write it as a linear combination of a fixed number of hypergeometric terms or a ratio of gamma products. Chaundy (1943) names such hypergeometric series as 'reducible'. Maier (2006) points out that the hypergeometric series cannot be expressed in finite number of terms and to sum it up is still an unsolved problem.

A major landmark in computerizing the search for closed form summations is Gosper's algorithm, developed by Gosper in 1970s. This algorithm decides whether a partial sum of a hypergeometric series can itself be expressed as a hypergeometric term, and gives its value if it does. It answers the question of whether a given sum involving factorials and binomial coefficients can be expressed in a closed form or not. This method has been used in various computer packages for investigating hypergeometric identities (Hannah, 2013). Gosper's algorithm has been discussed in his classical work of 1978 (Gopser, 1978). It is a decision procedure which returns the hypergeometric term or returns the response "No closed form hypergeometric anti-difference exists" (Hannah, 2013). Gosper's algorithm is run by computer packages such as the Gosper command in Maple's SumTools [Hypergeometric], and gopser.m in Mathematica (Hannah, 2013; Petkovsek et al., 1996).

In 1990 Zeilberger developed an algorithm for finding the recurrence relation for a hypergeometric term. He extended Gosper's algorithm to apply to a far wider range of cases (Zeilberger, 1990, 1991; Hanaah, 2013). While Gosper's procedure can establish whether or not a given hypergeometric term can be indefinitely summed, Zeilberger's method plays a similarly central role in the study of definite summation. But the Gosper-Zeilberger algorithm does not work for all sums (Hannah, 2013). In 1990, Wilf and Zeilberger published a short method for certifying combinatorial and hence hypergeometric identities, which is known as WZ method (Hannah, 2013; Wilf et al., 1990). Krattenthaler et al (2003) described an algorithm implemented in Mathematica, which uses a beta integral method to derive additional identities from existing ones (Hannah, 2013; Krattenthaler et al., 2003). Based on computer-based analysis Milgram (2004, 2006) found 66 new sums of hypergeometric functions (Hannah, 2013). But the result is still unsolved in terms of elementary and nonelementary functions. In this study for the problem of scarcity of some mathematical notations of many nonelementary functions originated from antiderivatives, the hypergeometric function would be discussed in context of the elementary and nonelementary functions.

2. Preliminary Ideas

Before going to study the relations of the hypergeometric functions with elementary and nonelementary functions (integrals), let us have some brief ideas about it:

2.1 General Hypergeometric Function: It is denoted and defined by

$$\mathrm{mFn}(\propto_1, \propto_2, \dots, \propto_m; \beta_1, \beta_2, \dots, \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\propto_1)_r.(\propto_2)_r, \dots, (\propto_m)_r}{(\beta_1)_r, (\beta_2)_r, \dots, (\beta_n)_r} \frac{x^r}{r!}$$

$$= 1 + \frac{\propto_1 \propto_2 \dots \propto_m}{\beta_1 \beta_2 \dots \beta_n} \frac{x^1}{1!} + \frac{\propto_1 (\propto_1 + 1) \propto_2 (\propto_2 + 1) \dots \propto_m (\propto_m + 1)}{\beta_1(\beta_1 + 1)\beta_2(\beta_2 + 1) \dots \beta_n(\beta_n + 1)} \frac{x^2}{2!} + \cdots$$

where $(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1)$ and $(\alpha)_0 = 1$ known as shifted factorial or rising factorial or Pochhammer symbol. The notation $\mathrm{mFn}(\propto_1, \propto_2, \dots, \propto_m; \beta_1, \beta_2, \dots, \beta_n; x)$ is called hypergeometric function and the series on the right-hand side is called the hypergeometric series. When one of the parameters $\propto_i$ is equal to –N, where N is a nonnegative integer, the hypergeometric function becomes a finite series and thus a polynomial in x. (Bailey, 1964; Du et al., 2002; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

2.2 Confluent Hypergeometric Function: On putting m = n = 1 in above expression of hypergeometric function, we get the confluent hypergeometric function, which is denoted and defined by

$$1\mathrm{F}1(\propto, \beta, x) = \sum_{r=0}^{\infty} \frac{(\propto)_r}{(\beta)_r} \frac{x^r}{r!} = 1 + \frac{\propto}{\beta} \frac{x}{1!} + \frac{\propto(\propto+1)}{\beta(\beta+1)} \frac{x^2}{2!} + \cdots$$

It is also known as Kummer Function (Confluent hypergeometric function - Wikipedia contributors, 2024; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

2.3 Gauss Hypergeometric Function: It contains three parameters. On putting m = 2, n = 1 in general hypergeometric function, we get it and is denoted and defined by

$$F(\propto,\beta;\gamma;x) = 2F1(\propto,\beta;\gamma;x) = \sum_{r=0}^{\infty} \frac{(\propto)_r.(\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(\propto)_r.(\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$$

$$= 1 + \frac{\propto \beta}{\gamma} \frac{x}{1!} + \frac{\propto (\propto +1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \cdots$$

where x, a, b, c may be real or complex, $|x| < 1$ and c is not non-negative integers. It is also called a function of four complex variables rather than as a function of only one variable x (Hannah, 2013; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

In the present study, we will use both the notations $F(\propto,\beta;\gamma;x)$ and ${}_2F_1(\propto,\beta;\gamma;x)$ for it. For $\propto = \gamma,\ \beta = 1$, this series becomes the elementary geometric series. For either $\propto = 0$ or $\beta = 0$, this series reduces to unity. If any one or both of $\propto$ or β is (are) negative integer, it reduces to a hypergeometric polynomial of degree n in x, which terminates at the $(n+1)^{th}$ term, if either or both is (are) equal to –n, for natural number n as follows:

$$F(-n,\beta;\gamma;x) = 2F1(-n,\beta;\gamma;x)$$

$$= \sum_{r=0}^{n} \frac{(-1)^r.n!.(\beta)_r}{(\gamma)_r} \frac{x^r}{(n-r)!\,r!}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{(\beta)_r}{(\gamma)_r} x^r$$

Thus, the Binomial sums are also a form of terminating hypergeometric series (Hannah, 2013).

The Gauss hypergeometric function can be extended to form the general (or generalized) hypergeometric function, which contains any number of parameters in its numerator and denominator. According to Slater this extension was first done by Thomas Clausen (1801-1885) in 1828, using three numerator and two denominator parameters. Many other identities related to this were established by Louis Saalschutz (1835-1913), Alfred Dixon (1865-1936), Henry Watson (1827-1903), John Dougall (1867-1960) and Whipple. The theory was exhaustively covered by Bailey in a series of papers during 1920 to 1950. The general hypergeometric function is useful as all the special functions of mathematical physics can be expressed in terms of these functions (Hannah, 2013).

In the above definition of general hypergeometric function m numerator and n denominator parameters have been used. It is assumed that no denominator parameter is zero or a negative integer. If any numerator parameter is zero or negative integer, the function is a terminating hypergeometric polynomial. A dash is usually used to indicate when there is no parameter in either the numerator or the denominator. Based upon the above notations and functions, our study will be limited to the elementary and nonelementary functions and their representation in hypergeometric functions with some discussion on the relations between elementary and hypergeometric functions.

3. Discussion

In this section we will discuss some relations among real or complex numbers, elementary functions, nonelementary functions and nonelementary integrals with hypergeometric functions. For this our study will be focused on following five questions:

i. Are all elementary functions hypergeometric function?
ii. Are all hypergeometric functions elementary function?
iii. Are all nonelementary functions hypergeometric function?
iv. Are all hypergeometric functions nonelementary function?
v. Are all hypergeometric functions as output from Mathematica as antiderivative nonelementary function?

After going through many references we find that there are some elementary functions, which can be expressed in terms of hypergeometric functions. But all hypergeometric functions cannot be expressed in terms of elementary functions. Those hypergeometric functions which are not expressible in closed form can undoubtedly be called as nonelementary functions (Closed-form expression- Wikipedia contributors, 2024; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024). Based on many mathematical formulae and expressions, some propositions have been propounded on hypergeometric functions in context of elementary and nonelementary functions as follows:

Proposition-I: Every real number or complex number can be expressed in terms of a hypergeometric function.

Proof: We have discussed in the preliminary section that for $\propto = 0$ or $\beta = 0$, the hypergeometric series reduces to unity. Therefore, we can represent any number (real or complex) z as z = z.1 and then 1 as $1 = F(0, \beta; \gamma; x) = F(\propto, 0; \gamma; x)$ i.e. $z = z.1 = z.F(0, \beta; \gamma; x) = z.F(\propto, 0; \gamma; x)$ (Du et al., 2002; Pearson, 2009).

Proposition-II: Every polynomial of finite degree can be expressed as a hypergeometric function.

Proof: We have discussed a well established fact about hypergeometric function in the preliminary section that when the parameter $\propto_i = -N$, for a nonnegative integer N, the hypergeometric function becomes a finite series and thus a polynomial in x (Du et al., 2002; Pearson, 2009; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

In addition to this, we had also discussed that if any one or both of $\propto$ or β is (are) negative integer, the hypergeometric series reduces to a hypergeometric polynomial of degree n in x, which terminates at the (n+1)th term, if either or both is (are) equal to –n, for natural number n as follows:

$$F(-n, \beta; \gamma; x) = 2F1(-n, \beta; \gamma; x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{(\beta)_r}{(\gamma)_r} x^r$$

Thus, the Binomial sums are a form of terminating hypergeometric series making a polynomial (Hannah, 2013; Pearson, 2009).

We also know that

$$mFn(\propto_1, \propto_2, \dots, \propto_m; \beta_1, \beta_2, \dots, \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\propto_1)_r.(\propto_2)_r, \dots, (\propto_m)_r}{(\beta_1)_r, (\beta_2)_r, \dots, (\beta_n)_r} \frac{x^r}{r!}$$

Putting different values of m and n, we can observe many elementary functions or polynomials in terms of hypergeometric functions as:

$$2F1(-n, 1; 1; -x) = (1 + x)^n$$
$$2F1(n, b; b; x) = (1 - x)^{-n}$$
$$2F1\left(\frac{1}{2}, \frac{-1}{2}; \frac{1}{2}; x^2\right) = \sqrt{1 - x^2}$$
$$2F1(1, b; b; x) = \frac{1}{(1 - x)}$$

etc. (Du et al., 2002; Pearson, 2009). Obviously the hypergeometric function is a series containing all types of polynomials even beyond the imagination of general polynomials, we study in general. So, we can state that every hypergeometric function in not a polynomial but we can convert it into a polynomial.

Proposition-III: All trigonometric and inverse trigonometric functions are hypergeometric functions but not vice versa.

Proof: We Know that

$$x.0F1\left(;\frac{3}{2};\frac{-x^2}{4}\right) = \sin x$$
$$0F1\left(;\frac{1}{2};\frac{-x^2}{4}\right) = \cos x$$
$$2F1\left(\frac{1}{2}, 1; 1; \sin^2 x\right) = \sec x$$
$$x.2F1\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right) = \tan^{-1} x$$
$$x.2F1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \sin^{-1} x$$

Similarly, we can compute others using the following formulae

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\operatorname{cosec} x = \frac{1}{\sin x}$$
$$\cos^{-1} x = \sin^{-1}\sqrt{1 - x^2}$$
$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$
$$\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$
$$\sec^{-1} x = \frac{\pi}{2} - \operatorname{cosec}^{-1} x$$

etc. after taking into consideration their respective domain (Du et al., 2002; Pearson, 2009). But obviously all hypergeometric functions are not trigonometric or inverse trigonometric functions, as it has been created beyond the region of elementary functions after the failure of expressing new functions in terms of elementary functions. For example, $2F1(1, b; b; x)$ is not a trigonometric or inverse trigonometric function.

Proposition-IV: All hyperbolic and inverse hyperbolic functions are hypergeometric functions but not vice versa.

Proof: We Know that

$$\sinh x = x.0F1\left(-;\frac{3}{2};\frac{x^2}{4}\right)$$

$$\cosh x = 0F1\left(-;\frac{1}{2};\frac{x^2}{4}\right)$$
$$\sinh^{-1}x = x.\,2F1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-x^2\right)$$
$$\tanh^{-1}x = x.\,2F1\left(\frac{1}{2},1;\frac{3}{2};x^2\right)$$
$$\lim_{a,b\to\infty} 2F1\left(a,b;\frac{1}{2};\frac{x^2}{4ab}\right) = \cosh x$$

etc. (Du et al., 2002). Similarly, we can compute others using the following formulae

$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x}$$
$$\mathrm{cosech}\, x = \frac{1}{\sinh x}$$
$$\mathrm{sech}\, x = \frac{1}{\cosh x}$$
$$\cosh^{-1}x = -i\cos^{-1}x$$

etc. (Pearson, 2009). Obviously all hypergeometric functions are not hyperbolic or inverse hyperbolic functions. For example, $2F1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right)$ is not a hyperbolic or inverse hyperbolic function.

Proposition-V: The logarithmic functions are hypergeometric functions but not vice versa.

Proof: We know that

$$x.\,2F1(1,1;2;-x) = \log(1+x)$$
$$-x.\,2F1(1,1;2;x) = \log(1-x)$$
$$-2x.\,2F1\left(1,\frac{1}{2};\frac{3}{2};x^2\right) = \log\frac{(1-x)}{(1+x)}$$
$$-2\left(\frac{1-x}{1+x}\right).\,2F1\left(1,\frac{1}{2};\frac{3}{2};\left(\frac{1-x}{1+x}\right)^2\right) = \log x$$

etc. (Du et al., 2002). Similarly we can express others. Obviously all hypergeometric functions are not logarithmic functions. For example, $2F1(0,1;2;x)$ is not a logarithmic function.

Proposition-VI: The exponential functions are hypergeometric functions but not vice versa.

Proof: We have discussed in the preliminary section that for $\propto = \gamma$, $\beta = 1$, the hypergeometric series reduces to the elementary geometric series, which is exponential series. Also, we know that

$$0F0(\,;\,;x) = e^x = 1F1(-;-;x) = F(a;a;x)$$

(Du et al., 2002). Simplifying the variable x by different variable (ax + b), we can find different hypergeometric functions for different exponential functions. Obviously all hypergeometric functions are not exponential function. For example, $2F1(1,1;1;x)$ is not an exponential function.

Proposition-VII**:** All elementary functions are hypergeometric functions for some particular values of m and n, but its converse is not true.

Proof: In above six propositions, we have proved that all most all types of standard elementary functions are hypergeometric functions i.e. we have proved that most non-algebraic functions such as exponential, logarithmic, trigonometric functions and hyperbolic functions are hypergeometric functions. All the non-algebraic mathematical functions in a standard library of computer software such as math.h in C and C++ are hypergeometric. In fact, most common elementary functions are hypergeometric functions (Du et al., 2002). All of the classic special functions can be expressed in terms of the powerful hypergeometric function (Hannah, 2013). Some of the examples have been given in above propositions. Here we call elementary functions, the most commonly used mathematical functions like sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, sinh, cosh, tanh, $\sinh^{-1}$, $\cosh^{-1}$, $\tanh^{-1}$, exponential and logarithms. The elementary functions are very often approximated by polynomials. It is natural to try to approximate the elementary functions by polynomials and rational functions (Muller, 1997, 2006).

We know that trigonometric function and their inverses, hyperbolic function and their inverses, exponential function, logarithmic function, polynomial of finite degree and their compositions are all elementary functions. So, to prove them hypergeometric, it is sufficient to prove that they can be expressed in terms of general hypergeometric functions for particular values of m and n as have been done in earlier propositions. Thus we can say that all elementary functions are hypergeometric functions. But obviously its converse is not true as has been discussed in earlier propositions.

Proposition-VIII: All hypergeometric functions, which are not elementary functions, are nonelementary functions.

Proof: We have proved in the previous proposition that all hypergeometric functions are not elementary functions and so they are nonelementary functions in such case. For example, following hypergeometric functions are nonelementary functions:

$$2F1[\frac{1}{2}-\frac{i}{2},1,\frac{3}{2}-\frac{i}{2},-e^{2ix}]$$
$$2F1[-\frac{i}{2},1,1-\frac{i}{2},-e^{2ix}]$$
$$2F1\left[1,1-\frac{i}{2},2-\frac{i}{2},e^{2ix}\right], \text{etc.}$$

This is why in the study of indefinite integrals, those antiderivatives which are expressed in terms of hypergeometric functions and are not expressible in terms of elementary functions, are called nonelementary integrals or nonelementary functions (Closed-form expression- Wikipedia contributors, 2024; Sao, 2021; Sharma, et al., 2020; Hypergeometric function – Wikipedia contributors, 2024; Yadav, 2015).

Proposition-IX: All elementary functions are hypergeometric functions for some particular values of m and n, but its converse is not true.

Proof: We have discussed the relations between standard elementary functions and their hypergeometric equivalent functions, which prove that all elementary functions are hypergeometric function. Their converses are not true as have been discussed for each one previously in above propositions, so the converse is also not true.

Proposition-X: All nonelementary functions are not necessarily hypergeometric functions.

Proof: We know that the greatest integer function [x] and piecewise function, Mobius function etc. are not elementary functions. Obviously we cannot express them in terms of hypergeometric functions. So the nonelementary functions are not necessarily hypergeometric function.

Proposition-XI: All hypergeometric functions originated as output from Wolfram Mathematica in finding the antiderivative of an elementary function f(x) with respect to x using the Mathematica code "In[i]: Integrate[f[x], x]" is always nonelementary integrals.

Proof: As we have discussed earlier in ten propositions that if a function is elementary, there is no need to express them in other special functions like in terms of hypergeometric function. Different computing software like Mathematica also does the same thing. Using any mathematical operations, generally it doesn't produce any result or output in terms of special functions, if that is an elementary function or can be expressed in elementary functions. It gives output in terms of hypergeometric function only, when it is not possible to express them in elementary functions. Therefore we can call all the output in terms of hypergeometric functions from Mathematica as nonelementary functions (or integrals). So without any hesitation, when we are getting such special function in integrating a function, we can call them nonelementary integrals.

Therefore we can now answer the five questions respectively as (i) all elementary functions are hypergeometric function, (ii) all hypergeometric functions are not elementary function, (iii) all nonelementary functions are not hypergeometric function, (iv) all hypergeometric functions are not nonelementary function and (v) all hypergeometric functions as output from Mathematica as antiderivative are nonelementary function (integral).

4. Conclusion

All elementary functions like polynomial, exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic, inverse hyperbolic function, and real or complex numbers etc. are hypergeometric functions but their converses need not be true. All hypergeometric functions as output from Mathematica are nonelementary functions and when the output comes as an integral or antiderivative of an elementary function, in terms of hypergeometric functions, means they are nonelementary integrals.

5. Future Scope of Research

Equations involving hypergeometric functions are of great interest for mathematicians and scientists, and especially for researchers. Hypergeometric functions provide a rich field for ongoing research, which continues to produce new results. Therefore there is a lot of scope is available for further research in this area.

6. Acknowledgment

There is no hesitation to accept that the properties and their proof might appear somewhere but we have not found any reference except some discussion in Du et al (2002).

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