Oscillation Conditions for a System of First Order Non-Linear Advanced Differential Equations

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Abstract:

In this paper, we provide sufficient conditions for the oscillation of every solution of non-linear advanced differential equations of the form

$$\dot{x}(t) - \sum_{i=1}^{n} q_i g_i(x(\tau_i(t))) = 0$$

where the coefficients $q_i \in R$ and $\tau_i \in R^+$ for i = 1, 2,…n. Our results essentially improve the conditions in the literature.

Keywords: Non-Linear advanced differential equation, Oscillation.

1. Introduction

In this paper, we consider the system of first-order non-linear advanced differential equations of the form

$$\dot{x}(t) - \sum_{i=1}^{n} q_i g_i(x(\tau_i(t))) = 0 \tag{1.1}$$

where the coefficients $q_i \in R$ and $\tau_i \in R^+$ for i = 1,2,…n are functions of positive real numbers such that

$$\tau_i(t) \geq t \;,\; t \geq t_0 \text{ and } \lim_{t\to\infty} \tau_i(t) = \infty, \text{ for } 1 \leq i \leq n \;,\; \tau(t) = \min_{1 \leq i \leq n} \tau_i(t) \tag{1.2}$$

and

$g_i \in C(R,R)$ and $x g_i(x) > 0$ for $x \neq 0$, $1 \leq i \leq n$.

By a solution of (1.1) , we mean a function that is continuously differentiable in $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1.1) is satisfied for $t \geq T_0$.

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, (1.1) is called nonoscillatory. An equation is oscillatory if all its solutions oscillate.

In 1978, Ladde [3] and in 1982 Ladas and Stavroulakis [4] proved that if

$$\liminf_{t\to\infty} \int_t^{\tau(t)} \sum_{i=1}^{n} q_i(s)\,ds > \frac{1}{e}, \tag{1.3}$$

then all solutions of (1.1) oscillate.

To study the situation when the argument $\tau_i(t)$ is not necessarily monotone $1 \leq i \leq n$, set $h_i(t) = \inf_{s \geq t}\{\tau_i(s)\}$, $h(t) = \min_{1 \leq i \leq n}\{h_i(t)\}$, $t \geq 0$. (1.4)

Clearly , the function $h_i(t)$ is non-decreasing and $\tau_i(t) \geq h_i(t) \geq h(t) > t$ for all $t \geq 0$ and $1 \leq i \leq n$.

Assume further that the function g_i in equation (1.1) hold the following condition for $1 \leq i \leq n$.

$$\limsup_{|x|\to\infty} \frac{x}{g_i(x)} = M_i \;,\; 0 \leq M_i < \infty. \tag{1.5}$$

Using the previous notation, we define $\beta = \liminf_{t\to\infty} \int_t^{\tau(t)} \sum_{i=1}^{n} q_i(s)\, ds$ (1.6)

and

$$D(\omega) = \begin{cases} 0, & \text{if } \omega > 1/e \\ \dfrac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0,1/e] \end{cases} \tag{1.7}$$

Throughout this chapter, we assume the following hypotheses hold:

(H1) $q_i(t)$, $h_i(t) \in C([t_0, \infty), R)$, $h_i(t)$ is non- monotone or nondecreasing.

(H2) $h_i(t) \geq t$ for $t \geq t_0$ and $\lim_{t\to\infty} h_i(t) = \infty$ for $1 \leq i \leq n$.

(H3) $h_i \in C$ (R,R) and $x g_i(x) > 0$ for $x \neq 0$ for $1 \leq i \leq n$.

2. Preliminary Results

The following Lemmas and Theorems are used to prove the main result.

Lemma 2.1: [2, Lemma 1]

Assume that h(t) is defined by (1.4), then

$$\liminf_{t\to\infty} \int_t^{\tau(t)} \sum_{i=1}^{n} q_i(s)\, ds = \liminf_{t\to\infty} \int_t^{h(t)} \sum_{i=1}^{n} q_i(s)\, ds \tag{2.1}$$

Lemma 2.2: [2 , Lemma 2]

Assume that $\beta \in (0, 1/e]$ and x(t) is an eventually positive solution of (1.1). Then

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq D(\beta) \tag{2.2}$$

Lemma 2.3: ([2, Lemma 3]

Assume that x(t) is an eventually positive solution of (1.1), h(t) is defined by (1.4) and β by (1.6) with $0 < \beta \leq 1/e$. Then

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq \lambda_0 , \tag{2.3}$$

where λ_0 is the smaller root of the transcendental equation $e^{\beta\lambda} = \lambda$. (2.4)

Lemma 2.4: (Grönwall inequality)

Assume that x(t) is a positive of $\dot{x}(t) - \sum_{i=1}^{n} q_i g_i\left(x(\tau_i(t))\right) \geq 0$, $t \geq t_0$, where the coefficients $q_i \in R$ and $\tau_i \in R^+$ for i = 1,2,…n are functions of positive real numbers and $x(t) \geq 0$, we have

$$x(s) \geq x(t) \exp \int_t^s \sum_{i=1}^{n} \frac{q_i(u)}{M_i}\, du, \quad s \geq t \geq t_0.$$

Theorem 2.4: ([1,Theorem 2.2])

Assume (H1) – (H3) , $0 < M_i < \infty$ and the following condition hold, where M_i is defined by (1.5). If

$$\liminf_{t\to\infty} \int_t^{\tau(t)} \sum_{i=1}^{n} q_i(s)\, ds > \frac{M^*}{e}, \tag{2.5}$$

then all solutions of (1.1) oscillate, where $M^* = \max\{M_i\}$ and

$\tau(t) = \min_{1 \leq i \leq n} \{\tau_i(t)\}$.

From (H3) and from (1.5) we can choose M_i for $1 \leq i \leq n$ such that

$g_i(x(\tau_i(t)) \geq \frac{1}{M_i} x(\tau_i(t))$ (See eq.(8) in [1]).

Thus (1.1) can be rewritten as

$$\dot{x}(t) - \sum_{i=1}^{n} \frac{q_i(t)}{M_i} x(\tau_i(t)) \geq 0 \qquad (2.6)$$

3. Main Result

Theorem 3.1:

Assume that h(t) is defined by (1.4) and for some j ϵ N

$$\limsup_{t\to\infty} \int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(t)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds > 1, \qquad (3.1)$$

where

$$R_j(t, \varepsilon) = \left(\sum_{i=1}^{n} \frac{q_i(t)}{M_i}\right)\left[1 + \int_t^{\tau(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_t^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_{j-1}(\psi, \varepsilon)\, d\psi\right) du\right) ds\right]$$
(3.2)

with $R_0 = \left(\sum_{i=1}^{n} \frac{q_i(t)}{M_i}\right)(\lambda_0)$ and λ_0 is the smaller root of the transcendental equation

$\lambda = e^{\beta\lambda}$.

Then all solutions of (2.6) are oscillatory.

Proof:

Assume for the sake of contradiction that, there exists a non-oscillatory solution x(t) of (2.6). Since -x(t) is also a solution of (2.6) ,we can confine our discussion only to the case where the solution x(t) is eventually positive. Then there exists a $t_1 > t_0$

such that x(t) > 0 and x($\tau_i(t)$) > 0 , 1 ≤ i ≤ n ,for all t ≥ t_1.

Thus from (2.6) , we have

$$\dot{x}(t) = \sum_{i=1}^{n} \frac{q_i(t)}{M_i} x(\tau_i(t)) \geq 0 \quad \text{for all } t \geq t_1,$$

which means that x(t) is an eventually nondecreasing function of positive integers. Considering $\tau_i(t) \geq h(t)$, (2.6) implies that

$$\dot{x}(t) - \sum_{i=1}^{n} \frac{q_i(t)}{M_i} x(h(t)) \geq 0 \quad \text{for all } t \geq t_1, \qquad (3.3)$$

Observe that (2.3) implies that ,for each ε > 0,there exists a t_ε such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \varepsilon \quad \text{for all } t \geq t_\varepsilon \geq t_1.$$
(3.4)

Combining inequalities (3.3) and (3.4) ,we obtain

$$\dot{x}(t) - \sum_{i=1}^{n} \frac{q_i(t)}{M_i} (\lambda_0 - \varepsilon)\, x(t) \geq 0 \text{ for all } t \geq t_\varepsilon \qquad (3.5)$$

or

$$\dot{x}(t) - R_0\, x(t) \geq 0 \text{ for all } t \geq t_\varepsilon$$
(3.6)

where $R_0 = \sum_{i=1}^{n} \frac{q_i(t)}{M_i} (\lambda_0 - \varepsilon)$

Applying Grönwall inequality in (3.6) ,we conclude that

$x(t) \geq x(s) \exp\left(\int_t^s R_0(\vartheta, \varepsilon)\, d\vartheta\right)$, $t \leq s \leq t_\varepsilon$ (3.7)

Now we divide (2.6) by x(t) > 0 and integrating on [t , s],so

$$\int_t^s \frac{\dot{x}(u)}{x(u)} du - \int_t^s \sum_{i=1}^n \frac{q_i(u)}{M_i} \frac{x(\tau_i(u))}{x(u)} du \geq 0$$

or

$$\int_t^s \frac{\dot{x}(u)}{x(u)} du \geq \int_t^s \sum_{i=1}^n \frac{q_i(u)}{M_i} \frac{x(\tau(u))}{x(u)} du,$$

$$\ln \frac{x(s)}{x(t)} \geq \left(\int_t^s \sum_{i=1}^n \frac{q_i(u)}{M_i} \frac{x(\tau(u))}{x(u)} du\right), \ t \leq s \leq t_\varepsilon.$$

$$x(s) \geq x(t) \exp\left(\int_t^s \sum_{i=1}^n \frac{q_i(u)}{M_i} \frac{x(\tau(u))}{x(u)}\right) du \tag{3.8}$$

since $\tau_i(u) > \tau(u) > u, u = t, s = \tau_i(u)$, then (3.7) becomes

$$x(\tau(u)) \geq x(u) \exp\left(\int_t^s R_0(\vartheta, \varepsilon)\, d\vartheta\right), \ t \geq s \geq t_\varepsilon \tag{3.9}$$

Combining (3.8) and (3.9) , we obtain for sufficiently large t,

$$x(s) \geq x(t) \exp\left(\int_t^s \left(\sum_{i=1}^n \frac{q_i(u)}{M_i}\right) \exp\left(\int_t^s R_0(\vartheta, \varepsilon)\, d\vartheta\right) du\right) \tag{3.10}$$

Hence,

$$x(\tau(s)) \geq x(t) \exp\left(\int_t^{\tau(s)} \left(\sum_{i=1}^n \frac{q_i(u)}{M_i}\right) \exp\left(\int_t^s R_0(\vartheta, \varepsilon)\, d\vartheta\right) du\right) \tag{3.11}$$

Integrating (2.6) from t to τ(t), we have

$$x(\tau(t)) - x(t) - \int_t^{\tau(t)} \left(\sum_{i=1}^n \frac{q_i(s)}{M_i}\right) x(\tau_i(s))\, ds \geq 0$$

$$x(\tau(t)) - x(t) - \int_t^{\tau(t)} \left(\sum_{i=1}^n \frac{q_i(s)}{M_i}\right) x(\tau(s))\, ds \geq 0 \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$x(\tau(t)) - x(t) - x(t) \int_t^{\tau(t)} \left(\sum_{i=1}^n \frac{q_i(s)}{M_i}\right) \exp\left(\int_t^{\tau(s)} \left(\sum_{i=1}^n \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_0(\vartheta, \varepsilon)\, d\vartheta\right) du\right) ds \geq 0.$$

Multiplying the last inequality by $\sum_{i=1}^n \frac{q_i(t)}{M_i}$, we get

$$\left(\sum_{i=1}^n \frac{q_i(t)}{M_i}\right) x(\tau(t)) - \left(\sum_{i=1}^n \frac{q_i(t)}{M_i}\right) x(t)$$

$$-\left(\sum_{i=1}^n \frac{q_i(t)}{M_i}\right) x(t) \int_t^{\tau(t)} \left(\sum_{i=1}^n \frac{q_i(s)}{M_i}\right) \exp\left(\int_t^{\tau(s)} \left(\sum_{i=1}^n \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_0(\vartheta, \varepsilon)\, d\vartheta\right) du\right) ds \geq 0 \quad .$$

(3.13)

As $\dot{x}(t) \geq \sum_{i=1}^n \frac{q_i(t)}{M_i} x(\tau_i(t))$, we have

$$\dot{x}(t) \geq \sum_{i=1}^n \frac{q_i(t)}{M_i} x(\tau_i(t)) \geq \sum_{i=1}^n \frac{q_i(t)}{M_i} x(\tau(t), \tag{3.14}$$

Combining inequalities (3.13) and (3.14) , we have

$$\dot{x}(t) - \left(\sum_{i=1}^n \frac{q_i(t)}{M_i}\right) x(t)$$

$$-\left(\sum_{i=1}^n \frac{q_i(t)}{M_i}\right) x(t) \int_t^{\tau(t)} \left(\sum_{i=1}^n \frac{q_i(s)}{M_i}\right) \exp\left(\int_t^{\tau(s)} \left(\sum_{i=1}^n \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_0(\vartheta, \varepsilon)\, d\vartheta\right) du\right) ds \geq 0$$

(3.15)

Hence,

$\dot{x}$(t)−($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) [$1+\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})$ exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})$ exp ($\int_u^{\tau(u)} R_0(\vartheta,\varepsilon)\, d\vartheta$) du) ds)]x(t)≥0

$\dot{x}$(t) - $R_1(t,\varepsilon)$ x(t) ≥; 0 (3.16)

where

$R_1(t,\varepsilon)$=($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) [$1+\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})$ exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})$ exp ($\int_u^{\tau(u)} R_0(\vartheta,\varepsilon)\, d\vartheta$) du ds]

Integrating (3.16) on [t , s] leads to

x(s) ≥ x(t) exp($\int_t^{s} R_1(\psi,\varepsilon)$ dψ (3.17)

By considering the steps from (3.6) to (3.11) , we obtain

x(τ(u)) ≥ x(u) exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ (3.18)

Combining (3.8) and (3.18) ,we obtain

x(s) ≥ x(t) exp ($\int_t^{s}(\sum_{i=1}^{n}\frac{(q_i(u))}{M_i}$)exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du)

Since τ(s) > s,

x(τ(s)) ≥ x(t) exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$)exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du). (3.19)

By (3.12) and (3.19) , we have

x(τ(t)) – x(t) - x(t) $\int_t^{\tau(t)}\sum_{i=1}^{n}\frac{q_i(s)}{M_i}$ exp ($\int_t^{\tau(s)}\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$ exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du) ds ≥ 0,

Multiplying the last equation by $\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$,we find

x(τ(t))($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) − x(t)($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$ - x(t)($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) $\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}$)exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$)exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du) ds ≥ 0.

As $\tau_i(t) > \tau(t)$, we have from the above inequality,

x(τ_i (t))($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$)− x(t)($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) - x(t)($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) $\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}$) exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$) exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du) ds ≥ 0.

Again as $\dot{x}$(t) ≥ x(τ_i (t))($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$), we get

$\dot{x}$(t) – x(t))($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) - x(t)($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) $\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}$) exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$) exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du) ds ≥ 0,

$\dot{x}$(t) − ($\sum_{i=1}^{n}\frac{q_i(t)}{M_i}$) [1 + $\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}$) exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}$) exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du) ds] $x(t)$ ≥ 0

Therefore , for sufficiently large t,

$\dot{x}$(t) - $R_2(t,\varepsilon)x(t) \geq 0$, (3.20)

where ,

$R_2(t,\varepsilon) = (\sum_{i=1}^{n}\frac{q_i(t)}{M_i})$ [1 + $\int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})$ exp ($\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})$ exp($\int_u^{\tau(u)} R_1(\psi,\varepsilon)$ dψ) du)] ds .

Repeating the above procedure ,it follows by induction that for sufficiently large t

$\dot{x}$(t) - $R_j(t,\varepsilon)x(t) \geq 0$, j ϵ N where ,

$R_j(\,t,\varepsilon\,) = (\sum_{i=1}^{n}\frac{q_i(t)}{M_i})\,[\,1 + \int_t^{\tau(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})\exp\,(\int_t^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp\,(\int_u^{\tau(u)}R_{j-1}(\,\psi,\varepsilon)\,d\psi)\,du)\,ds].$

Moreover ,since τ(s) ≥ h(s) ≥ h(t), by Grönwall inequality we have,

$x(\tau(s)) \geq x(h(t))\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\,d\psi)\,du).$ (3.21)

Integrating (2.6) from t to h(t) and using (3.21), we obtain

$x(h(t)) - x(t) - \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})\,x(\,\tau_i(s))ds \geq 0$

$x(h(t)) - x(t) - \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})x(\,\tau(s))ds \geq 0$

$x(h(t)) - x(t) - x(h(t))\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\,d\psi)du)ds \geq 0$

$x(h(t)) - x(t) - x(h(t))\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds \geq 0$ (3.22)

$x(h(t))[\,1 - \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds] - x(t) \geq 0$

The strict inequality is valid if we omit x(t) >0 on the left-hand side. Therefore,

$x(h(t))[\,1 - \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds] > 0$

or

$1 - \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds > 0.$

$1 > \int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds,$

or

$$\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds < 1$$

Taking the limit as t → ∞, we have

$\limsup_{t\to\infty}\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\,d\psi)du)ds < 1,$

Since ε may be taken as arbitrarily small, this inequality contradicts (3.1) .Thus the proof of the theorem is completed.

Theorem 3.2:

Assume that β defined by (1.6) with 0 < β ≤ 1/e and h(t) by (1.4).If for some j ∊ N,

$\limsup_{t\to\infty}\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q_i(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds > 1 - D(\beta).$ (3.23)

where R_j is defined by (3.2),then all solutions of (2.6) are oscillatory.

Proof:

Let x be an eventually positive solution of (2.6) .Then , by Theorem 3.2, inequality (3.22) is satisfied.

$x(h(t)) - x(t) - x(h(t))\int_t^{h(t)}(\sum_{i=1}^{n}\frac{q(s)}{M_i})\exp\,(\int_{h(t)}^{\tau(s)}(\sum_{i=1}^{n}\frac{q_i(u)}{M_i})\exp(\int_u^{\tau(u)}R_j(\,\psi,\varepsilon)\;d\psi)du)ds \geq 0$

$x(h(t))\,[1 - \int_t^{h(t)} (\sum_{i=1}^n \frac{q(s)}{M_i}) \exp(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds] \geq x(t)$

$[1 - \int_t^{h(t)} (\sum_{i=1}^n \frac{q(s)}{M_i}) \exp(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds] \geq \frac{x(t)}{x(h(t))}$,

$\int_t^{h(t)} (\sum_{i=1}^n \frac{q(s)}{M_i}) \exp(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds] \leq 1 - \frac{x(t)}{x(h(t))}$

$$\limsup_{t\to\infty} \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds \leq 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \qquad (3.24)$$

By Lemma 2.3,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_{\tau(s)}^{h(t)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_{\tau(u)}^{u} R_j(\psi, \varepsilon)\, d\psi)du)ds \leq 1 - D(\beta).$$

Since ε may be taken as arbitrarily small, this inequality contradicts (3.23).

Thus the proof of the theorem is completed.

Theorem 3.3:

Assume that β is defined by (1.6) with 0 < β ≤ 1/e and h(t) by (1.4).If for some j ϵ N

$$\limsup_{t\to\infty} \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds > \frac{1}{D(\beta)} - 1 \qquad (3.25)$$

where R_j is defined by (3.2),then all solutions of (2.6) are oscillatory.

Proof:

Assume for the sake of contradiction that, there exists a non-oscillatory solution x of (2.6) and that x is eventually positive. Then, as in the proof of Theorem 3.1, equation (3.21) is satisfied, which yields

$x(\tau(s)) \geq x(t) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du)$.

Integrating (2.6) from t to h(t) ,we have

$x(h(t)) - x(t) - \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i})\, x(\tau_i(s))\, ds \geq 0$

or

$x(h(t)) - x(t) - \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i})\, x(\tau(s))\, ds \geq 0$

Thus by (3.21), the last inequality gives

$x(h(t)) - x(t) - \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i})\, x(t) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du)\, ds \geq 0$

or

$x(h(t)) - x(t) - x(t) \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du)\, ds \geq 0.$

Thus, for all sufficiently large t , it holds.

$x(h(t)) \geq [\,1 + \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du)ds]\, x(t)$

$\int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du)ds) \leq \frac{x(h(t))}{x(t)} - 1$

Letting t → ∞ ,we have

$$\limsup_{t\to\infty} \int_t^{h(t)} (\sum_{i=1}^n \frac{q_i(s)}{M_i}) \exp(\int_t^{\tau(s)} (\sum_{i=1}^n \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)du)ds \leq \liminf_{t\to\infty} \frac{x(h(t))}{x(t)} - 1.$$

which ,in view of (2.4) gives

$$\limsup_{t\to\infty}\int_t^{h(t)}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)\exp\left(\int_t^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \le \frac{1}{D(\beta)}-1 .$$

Since ε is arbitrarily small, this inequality contradicts (3.25).The proof of the theorem is complete.

Theorem 3.4:

Assume that β is defined by (1.6) with $0 < \beta \le 1/e$ and h(t) by (1.4). If for some $j \in N$

$$\limsup_{t\to\infty}\int_t^{h(t)}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)\exp\left(\int_{h(s)}^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right)ds>\left(\frac{1+\ln\lambda_0}{\lambda_0}\right)-D(\beta), \quad (3.26)$$

where R_j is defined by (3.2) and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$,

then all solutions of (2.6) are oscillatory.

Proof:

Assume, for the sake of contradiction that, there exists a non-oscillatory solution *x*(t) of (2.6) and that *x*(t) is eventually positive. Then, as in Theorem 3.1,(3.21) is satisfied. By (2.3),for each ε > 0, there exists a t_ε such that

$$\lambda_0 - \varepsilon < \frac{x(h(t))}{x(t)} \text{ for all } t \ge t_\varepsilon , \quad (3.27)$$

Since $\frac{x(h(t))}{x(s)}$ is non increasing in s ,we have

$$1 = \frac{x(h(t))}{x(h(t))} \ge \frac{x(h(t))}{x(s)} \ge \frac{x(h(t))}{x(t)} ,\ t_\varepsilon \ge h(t) \ge s \ge t,$$

In particular for $\varepsilon \in (0, \lambda_0 - 1)$, by continuity there exists a $t^* \in (t, h(t)]$ such that

$$1 > \lambda_0 - \varepsilon = \frac{x(h(t))}{x(t^*)} \quad (3.28)$$

By (3.21), we have

$$x(\tau(s)) \ge x(h(s))\exp\left(\int_{h(s)}^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right). \quad (3.29)$$

Integrating (2.6) from t to t^* ,we have

$$x(t^*) - x(t) \ge \int_t^{t^*}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)x(\tau(s))\,ds$$

By using (3.29) along with h(s) ≥ h(t) in combination with the non decreasingness of x ,we have

$$x(t^*) - x(t) \ge x(h(t))\int_t^{t^*}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)\exp\left(\int_{h(s)}^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right)ds$$

or,

$$\int_t^{t^*}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)\exp\left(\int_{h(s)}^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \le \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$$

In view of (3.28) and Lemma 2.3 for the ε considered, there exists a $t_\varepsilon \ge t_\varepsilon'$, such that

$$\int_t^{t^*}\left(\sum_{i=1}^{n}\frac{q_i(s)}{M_i}\right)\exp\left(\int_{h(s)}^{\tau(s)}\left(\sum_{i=1}^{n}\frac{q_i(u)}{M_i}\right)\exp\left(\int_u^{\tau(u)}R_j(\psi,\varepsilon)\,d\psi\right)du\right)ds < \frac{1}{\lambda_0-\varepsilon} - \frac{x(t)}{x(h(t))} ,\quad t_\varepsilon' \ge t \quad (3.30)$$

Dividing (2.6) by x(t) and integrating from $t*$ to $h(t)$, we find

$$\int_{t*}^{h(t)} \frac{\dot{x}(s)}{x(s)} ds \geq \int_{t*}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \frac{x(\tau(s))}{x(s)} ds$$

And using (3.29) , we get

$$\int_{t*}^{h(t)} \frac{\dot{x}(s)}{x(s)} ds \geq \int_{t*}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \frac{x(h(s))}{x(s)} \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds$$

$$\int_{t*}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \frac{x(h(s))}{x(s)} \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \leq \int_{t*}^{h(t)} \frac{\dot{x}(s)}{x(s)} ds \qquad (3.31)$$

By (3.27), for $t_{\varepsilon}' \geq h(t) \geq s$,we have $\frac{x(h(s))}{x(s)} > \lambda_0 - \varepsilon$, so from (3.31) we get

$$(\lambda_0 - \varepsilon) \int_{t*}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \leq \int_{t*}^{h(t)} \frac{\dot{x}(s)}{x(s)} ds.$$

Hence for sufficiently large t,we have

$$\int_{t*}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \leq \frac{1}{(\lambda_0 - \varepsilon)} \int_{t*}^{h(t)} \frac{\dot{x}(s)}{x(s)} ds.$$

$$\leq \frac{1}{(\lambda_0 - \varepsilon)} \ln \frac{x(h(t))}{x(t*)}$$

$$\leq \frac{\ln((\lambda_0 - \varepsilon)}{(\lambda_0 - \varepsilon)} \qquad (3.32)$$

Adding (3.30) and (3.32), and then taking the limit as t → ∞, we have

$$\limsup_{t \to \infty} \int_{t}^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds < \frac{1}{\lambda_0 - \varepsilon} - \liminf_{t \to \infty} \frac{x(t)}{x(h(t))} + \frac{\ln((\lambda_0 - \varepsilon)}{(\lambda_0 - \varepsilon)}$$

$$< \frac{1 + \ln(\lambda_0 - \varepsilon)}{(\lambda_0 - \varepsilon)} - D(\beta).$$

Since ε may be taken as arbitrarily small, this inequality contradicts (3.26).

Thus the proof of the theorem is completed.

Theorem 3.5:

Assume that h(t) is defined by (1.3) and for j ϵ N

$$\liminf_{t \to \infty} \int_{t}^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_{u}^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds > \frac{1}{e}, \qquad (3.33)$$

where R_j is defined by (3.2) .Then all solutions of (2.6) are oscillatory.

Proof:

Assume for the sake of contradiction that, there exists a non-oscillatory solution x(t) of (2.6).

Since -x(t) is also a solution of (2.6),we can confine our discussion only to the case where the solution x(t) is eventually positive. Then there exists a $t_1 > t_0$ such that x(t) > 0 and $x(\tau_i(t)) > 0$,

$1 \leq i \leq m$ for all $t \geq t_1$. Thus, from (2.6) we have,

$\dot{x}(t) - \sum_{i=1}^{n} \frac{q_i(t)}{M_i} x(\tau_i(t)) \geq 0$ for all $t \geq t_1$,

which means that x(t) is an eventually non-decreasing function of positive numbers. Moreover, as in previous theorem, (3.29) is satisfied.

Dividing (2.6) by x(t) and integrating from t to h(t) , for some $t_2 \geq t_1$, we get

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \frac{x(\tau_i(s))}{x(s)}\, ds$$

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \frac{x(\tau(s))}{x(s)}\, ds \quad (3.34)$$

Combining inequalities (3.29) and (3.34),we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \frac{x(h(s))}{x(s)} \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \geq 0$$

Taking into account that *x* is non-increasing and h(s) > s, the last inequality becomes

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \quad (3.35)$$

From (3.33) ,it follows that there exists a constant c > 0 such that for sufficiently large t

$$\int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \geq c > \frac{1}{e}$$

Choose c' such that $c > c' > \frac{1}{e}$. For every ε > 0 such that $c - \varepsilon > c'$, we have

$$\int_t^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \geq c > c - \varepsilon > c' > \frac{1}{e} \quad (3.36)$$

Combining inequalities (3.35) and (3.36), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq c', \quad t \geq t_3.$$

Thus

$$\left(\frac{x(h(t))}{x(t)}\right) \geq e^{c'} \geq ec' > 1,$$

which gives ,for some $t \geq t_4 \geq t_3$,

$x(h(t)) \geq (ec')\, x(t)$.

Repeating the above procedure ,it follows by induction that for any positive integer k,

$\frac{x(h(t))}{x(t)} \geq (ec')^k$ for sufficiently large t.

Since $ec' > 1$, there is a k ϵ N satisfying $k > 2\left(\frac{\ln(2) - \ln(c')}{1 + \ln(c')}\right)$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (ec')^k > \left(\frac{2}{c'}\right)^2 \quad (3.37)$$

Next we split the integral in (3.36) into two integrals , each integral being no less than $\frac{c'}{2}$.

$$\int_t^{t_n} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \geq \frac{c'}{2}, \quad (3.38)$$

$$\int_{t_n}^{h(t)} \left(\sum_{i=1}^{n} \frac{q_i(s)}{M_i}\right) \exp\left(\int_{h(s)}^{\tau(s)} \left(\sum_{i=1}^{n} \frac{q_i(u)}{M_i}\right) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds \geq \frac{c'}{2}, \quad (3.39)$$

Integrating (2.6) from t to t_n , we deduce that

$x(t_n) - x(t)$) - $\int_t^{t_n} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} x(\tau_i(s))ds \geq 0$

or

$x(t_n) - x(t)$) - $\int_t^{t_n} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} x(\tau(s))\, ds \geq 0$

which in view of (3.29), gives

$x(t_n) - x(t)$) - $\int_t^{t_n} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} x(h(s)) \exp\left(\int_{h(s)}^{\tau(s)} (\sum_{i=1}^{n} \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du\right)ds \geq 0$

$x(t_n) - x(t)$) $-x(h(t)) \int_t^{t_n} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} (\sum_{i=1}^{n} \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du\right)ds \geq 0$

The strict inequality is valid if we omit $x(t) > 0$ on the left-hand side

$x(t_n) - x(h(t)) \int_t^{t_n} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} (\sum_{i=1}^{n} \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du\right)ds \geq 0$

Using the inequality (3.38), we conclude that

$x(t_n) > \frac{c'}{2} x(h(t)).$ (3.40)

Similarly, integrating (2.6) between t_n *to* $h(t)$ with the later application of (3.28) leads to

$x(h(t)) - x(t_n) - x(h(t_n)) \int_{t_n}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} (\sum_{i=1}^{n} \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du\right) ds \geq 0$

The strict inequality is valid if we omit $x(t_n)$ on the left hand side.

$x(h(t)) - x(h(t_n)) \int_{t_n}^{h(t)} \sum_{i=1}^{n} \frac{q_i(s)}{M_i} \exp\left(\int_{h(s)}^{\tau(s)} (\sum_{i=1}^{n} \frac{q_i(u)}{M_i}) \exp(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi)\, du\right)ds > 0$.

Using the inequality (3.39), we get

$x(h(t)) > (\frac{c'}{2})\, x(h(t_n))$ (3.41)

Combining (3.40) and (3.41),

$\frac{x(h(t_n))}{x(h(t))} < (\frac{2}{c'})^2$,

which contradicts (3.37) .

Thus the proof of the theorem is completed.

Example:

Consider the following advanced differential equation:

$\dot{x}(t) - \frac{1}{e} x(\tau_1(t)) \ln(e^{-|x(\tau_1(t))|} + 2) - \frac{2}{e} x(\tau_2(t)) \ln(e^{-|x(\tau_2(t))|} + 3) = 0, \ t \geq 0$ (3.42)

Where

$$\tau_1(t) = \begin{cases} t+1, & t \in [3.5k,\ 3.5k+1], \\ 3t-7k-1, & t \in [3.5k+1,\ 3.5k+1.5], \\ -t+7k+5, & t \in [3.5k+1.5,\ 3.5k+2], \\ t+1, & t \in [3.5k+2,\ 3.5k+2.5], \\ 3t-7k-4, & t \in [3.5k+2.5,\ 3.5k+3], \\ -t+7k+8, & t \in [3.5k+3,\ 3.5k+3.5], \end{cases}$$

$\tau_2(t) = \tau_1(t) + 1$,

and

$$h_1(t) = \inf_{s \geq t}\{\tau_1(s)\} = \begin{cases} t+1, & t \in [3.5k,\ 3.5k+1], \\ 3t-7k-1, & t \in [3.5k+1,\ 3.5k+1.3], \\ -t+7k+5, & t \in [3.5k+1.3,\ 3.5k+2], \\ t+1, & t \in [3.5k+2,\ 3.5k+2.5], \\ 3t-7k-4, & t \in [3.5k+2.5,\ 3.5k+2.8], \\ 3.5k+4.5, & t \in [3.5k+2.8,\ 3.5k+3.5], \end{cases}$$

and

$$h_2(t) = \inf_{s \geq t}\{\tau_2(s)\} = h_1(t) + 1,$$

$k \in N_0$, N_0 is the set of nonnegative integers. Then

$$\tau(t) = \min_{1 \leq i \leq 2}\{\tau_i(t)\} = \tau_1(t).$$

Let $t = 3.5k + 1.3$ and $\tau(t) = 3.5k + 3$

$$\beta = \lim_{t \to \infty} \inf \int_t^{\tau(t)} \sum_{i=1}^n q_i(s),$$

where

$$q_1(t) = \frac{1}{e}, \quad q_2(t) = \frac{2}{e},$$

hence,

$$\beta = \lim_{t \to \infty} \inf \int_{3.5k+1.3}^{3.5k+3} \left(\frac{1}{e} + \frac{2}{e}\right) = 1.87$$ and therefore the smaller root of

$e^{1.87\lambda} = \lambda$ is $\lambda_0 = -0.213$.

Also,

$$M_i = \lim_{|x| \to \infty} \sup \frac{x}{g_i(x)}, \quad 0 < M_i < \infty.$$

$$M_1 = \lim_{|x| \to \infty} \sup \frac{x}{g_1(x)} = 0.91$$

$$M_2 = \lim_{|x| \to \infty} \sup \frac{x}{g_2(x)} = 0.72.$$

Therefore,

$\sum_{i=1}^{2} \frac{q_i(t)}{M_i}$ = 1.418.

Let us prove that the solutions of (3.42) is oscillatory by showing (3.1) of Theorem 3.1 holds.

The function $G_j = \int_t^{h(t)} (\sum_{i=1}^{2} \frac{q_i(s)}{M_i}) \exp\left(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^{2} \frac{q_i(u)}{M_i}) \exp\left(\int_u^{\tau(u)} R_j(\psi, \varepsilon)\, d\psi\right) du\right) ds$

attains its maximum at t = 3.5k +1.3 , k ϵ N_0 , for every j ≥ 1.

Thus,

$$G_1 = \int_t^{h(t)} (\sum_{i=1}^{2} \frac{q_i(s)}{M_i}) \exp\left(\int_{h(t)}^{\tau(s)} (\sum_{i=1}^{2} \frac{q_i(u)}{M_i}) \exp\left(\int_u^{\tau(u)} R_1(\psi, \varepsilon)\, d\psi\right) du\right) ds$$

with

$$R_1 = (\sum_{i=1}^{2} \frac{q_i(t)}{M_i}) \left[1 + \int_t^{\tau(t)} (\sum_{i=1}^{2} \frac{q_i(s)}{M_i}) \exp\left(\int_t^{\tau(s)} (\sum_{i=1}^{2} \frac{q_i(u)}{M_i}) \exp\left(\int_u^{\tau(s)} R_0(\vartheta, \varepsilon)\, d\vartheta \right) du \right) ds\right]$$

and

$R_0 = (\sum_{i=1}^{2} \frac{q_i(t)}{M_i}) (\lambda_0) = -0.302$.

Thus,

R_1 = 7.152 and

G_1 = 2.36 .

$\limsup_{t \to \infty} G_1 = 2.36 > 1$ .Hence the inequality (3.1) is satisfied.

Therefore all solutions of (3.42) are oscillatory

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