# Synergistic Approaches to Consolidated Values in Cooperative Game Theory

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**Abstract**— Marginalism shares the value of a game based on the contributions of players to the game while Egalitarianism divides the total worth of a game equally among all players in it. But both have their limitations in allocating gain to non-productive players. Consolidation of both values helps in reducing some limitations. So, in TU games, developing solution concepts by consolidating egalitarianism and marginalism has been widely studied. In this chapter, we reviewed different consolidated values, their axiomatization and characterization found in game-theoretic literature.
**Keywords—** Shapley value, egalitarian value, null player, solidarity value.

## I. Introduction

Game theory is a theoretical framework for conceiving social situations among independent participants and competitors. There are certain situations where more than one individual or party with antithetical ideas endeavour to make real-life decisions. The party that makes decisions try to influence others' welfare. The set of circumstances has a result that depends on the action of the decision-makers. They may sometimes cooperate or may compete with the other members of the group. The decision to choose an action by a player should be at least as good as every other action available to him according to his preferences. The formation of organizations like the United Nations by various countries for peace and security, to boost their economy and health sector are examples of cooperation on the international level. On the other hand, land and river disputes between countries, military conflicts, voting in a two-party system, and traders' reaction to a price change in stock markets are instances of competition. Such models are formulated using game theory.

Players in cooperative game theory make binding agreements. As a result of this, they form coalitions and enforce cooperative behaviour. Although maximizing their utility is the player's individual goal, they achieve that by cooperation. For instance, consider the situation of the peasant-landlord problem [5] where a landowner owns a large piece of land on which some peasants work. The landlord does not work himself in the field and needs at least one peasant to produce an output. On the other hand, the peasants cannot produce anything on their own as they require land to cultivate. So, cooperation between the landlord and the peasants is necessary to produce some output. A situation of this type is modeled by cooperative game theory.

In cooperative game theory, the main challenge is to distribute the total value generated by all the players among them in a fair way. For this purpose, different solution concepts have been proposed over years with different objectives. The Shapley value [13], Egalitarian value [8], and Solidarity value [11] are some important solutions in the cooperative game framework. For this, we use the standard cooperative game-theoretic solution concepts, called the Shapley value and the Equal division rule. The Shapley value is based on the marginalist viewpoint, i.e., it rewards players on the basis of their capabilities of producing worth. On the other hand, the Equal Division rule builds on egalitarianism; there is no distinction between a more productive and less productive player. Each of these solution concepts follows some standard properties such as efficiency, symmetry, additivity and so on. In addition to the other standard properties, the Shapley value has the null player property that assigns zero payoffs to every unproductive player. This idea is interesting in the sense that the replacement of the null player property with some alternative player properties leads to another variant of the Shapley value that has been proposed in [16]. It is also exciting to see that many new solution concepts have also been developed by replacing these standard properties.

Shapley value is an extreme case of marginalism which does not care about the unproductive players in the group, and thus, it gives nothing to them. But for the survival of a group, it is necessary to share worth with those who cannot contribute because of some acceptable reasons. For instance, people on maternity leave or a person with a disability might not be able to contribute to a game. It is seen in our society to provide helping hands t such people. On the other hand, equal division value is an extreme case of egalitarianism that does not bother about the contributions of players in the game and distribute the total worth among the players involved. This is not applicable to cooperative business scenarios where participants who invest more demand for larger sharing of the profit. Since both Shapley value and equal division value have their limitations in certain domains, researchers have consolidated both values to reduce their restrictions and obtain a better solution to distribute the value of the game.

Joosten [9] proposed the -egalitarian Shapley value which is a convex combination of the Shapley value and the equal division rule determined by the convexity parameter. It allocates a portion of the total worth equally among the players and the remaining portion according to the Shapley rule. Later, Choudhary et. al. [7] generalized the value by considering different values of convexity parameters for different sizes of the coalition. Another consolidated value is due to Choudhary et. al. [6], called k-SED value which allocates Shapley value to coalitions of size larger than k and equal division value to smaller coalitions. The value has been characterized using some well-known as well as freshly defined axioms of cooperative games. Borkotokey et. al [2] have studied the involvement of middlemen in cooperative set-up and how it affects the value distribution among players. Bora et. al. [1] have introduced a value function for a cooperative game involving middlemen. In this paper, we review these consolidated values highlighting their importance and applications.

## II. Preliminaries

This section is basically devoted to the discussion of the various definitions and axioms that are needed for the study of characterizations and axiomatizations of different consolidated values in the literature.

**Definition 1**. A TU cooperative game or simply a cooperative game is the pair $(P, u)$where $P$ is a finite set of players and $u: 2^P \to R$ is a characteristic function such that $u(\emptyset) = 0$.

We shall denote a TU game by its characteristic function $u$ only for simplicity and $G^P$ denote the class of all TU games with player set $P$. Also, let $V^P$ be the set of all value functions on the player set $P$. For a finite player set$P$, subsets of $P$ are called coalitions. For each coalition $T \epsilon 2^P, u(T)$ represents the worth of $T$. Note that we shall denote the number of players in a coalition with small letters. For instance, $|T| = t$.

**Definition 2. (Unanimity game)** The unanimity game $u^S$ for each non-empty coalition $S \epsilon 2^P$is defined by

$$u_S(T) = f(x) = \begin{cases} 1, & if \quad S \subseteq T \\ 0, & otherwise. \end{cases}$$

Here, $G^P$ is a linear space under the addition and scalar multiplication of functions given by $(\alpha u)(T) = \alpha . u(T)$ and $(u + v)(T) = u(T) + v(T)$, $\alpha \epsilon R, T \epsilon 2^P$. The class of unanimity games forms a basis for the space $G^P$.

**Definition 3. (Marginal Contribution)** The marginal contribution of coalition $T$ in the game $(P, u)$ is defined as $MC_j^v(T) = v(T) - v(T\backslash\{j\})$ for the player $j \in T$.

**Definition 4. (A-null player)** A player $j \epsilon P$ is said to be A-null player if $C$av $(T)$ = 0 for every coalition $T$ containing $j \epsilon P$.

Now, let us discuss the axioms that are used for the axiomatizations and characterizations of different consolidated values in game theoretic literatures. Let, $\varphi$ be a value on $V^P$.

**Axiom 1.Efficiency (EFF):** For all $j \in P$ and $v \in G^N$,$\sum_{j \in N} \Phi_j = v(P)$: i.e., the total utility is equal to the value of the grand coalition.

**Axiom 2.Additivity (ADD):** For two games $u; v \in V^P$, $\Phi_j(u + v) = \Phi_j(u) + \Phi_j(v)$ for all $j \in N$, i.e., the solution to the sum of two TU games must be the sum of the individual solutions of the two games.

**Axiom 3.Symmetry (SYM):** If players $i$ and $j$ are two substitutes in any game $v \in V^P$, then $\Phi_i(v) = \Phi_j(v)$, i.e., if two parties contribute the same amount to the coalition, they should be rewarded equally.

**Axiom 4.Null player condition (NPC):** For $j \in P$, if $v(S \cup j) = v(S)\ \forall\ S \subseteq P\backslash j$; then $\Phi_j(v) = 0$: i.e.; if a member of a coalition contributes nothing, then they should receive nothing in return.

In addition to these axioms, few other axioms are studied in the literature to characterize these values. We mention a few of them, which will be helpful for our characterization at a later stage.

**Axiom 5.Monotonicity (Mon):** For $i; j \in P$ if $v(S \cup i) \leq v(S \cup j)\ \forall\ S \subseteq P\{i, j\}$; then $\Phi_i(v) \leq \Phi_j(v)$; that is if a player contributes more, then he must receive more.

**Axiom 6.Nullifying Player Property (NPP):** For $j \in P$, if $v(S \cup j) = 0$ for all $S \subseteq P\{j\}$; then $\Phi_j(v) = 0$.

**Axiom 7.Fairness (F):** If two players $i$ and $j$ are symmetric in $v \in V^P$, then $\Phi_i(v + w) - \Phi_i(w) = \Phi_j(v + w) - \Phi_j(w)$.

**Axiom 8.Differential marginality (DM):** If $MC_i^v(S) - MC_j^v(S) = MC_i^u(S) - MC_j^u(S)$for all $S \subseteq P\backslash\{ i; j\}$ then $\Phi_i(v) - \Phi_j(v) = \Phi_i(u) - \Phi_j(u)$.

The Shapley value $\Phi^{Sh}: G^P \rightarrow R^p$ and egalitarian solution $\Phi^{ED} : G^P \rightarrow R^n$ for the game $(P; v)$ is given by formally given by

$$\Phi_j^{Sh}(v) = \sum_{S \subseteq N \backslash i} \frac{(p - s - 1)!\, s!}{p!} MC_i^v(S)$$

$$\Phi_j^{ED}(v) = \frac{v(P)}{p}$$

van den Brink in [14] and Kamijo et. al [10] provided some axiomatic characterization distinguishing the Shapley value and the egalitarian value for a TU-game. Different axiomatizations and characterizations of Shapley value and Egalitarian values are found in [8], [12], [13], [15]. Some notable characterizations are mentioned below.

**Theorem 1.** [13] *A value $\Phi$ is equal to $\Phi^{Sh}$ if and only if it satisfies EFF, SYM, ADD and NPC.*

**Theorem 2.** [8] *A value is equal to the $\Phi^{ED}$ if and only if it satisfies EFF, SYM, ADD and NPP.*

**Theorem 3.** [3] *The Shapley value, $\Phi^{Sh}$ is characterized by EFF, NPC, and DM.*

**Theorem 4.** [15] *A value $\Phi$ is equal to $\Phi^{Sh}$ if and only if it satisfies EFF, NPC and F.*

## III. Egalitarian Shapley value

The α-Egalitarian Shapley value is due to Joosten [9] that combines the Shapley value and the equal division value. For a real number $\alpha \in [0,1]$, the $\alpha$-Egalitarian Shapley Value, is given by

$\Phi^{\alpha} = \alpha\Phi^{Sh} + (1 - \alpha)\Phi^{ED}$ (1)

The Egalitarian Shapley value satisfies efficiency, additivity and symmetry. For $\alpha \neq 1$, they do not obey the null-player property. For $\alpha \neq 1$, they do not obey the null-player property. But it satisfies some other properties as Null-player in a productive environment property and desirability.

This value is important in the sense that it provides a portion of the generated wealth to null players. This seems practical as our society encourages donations to such people who can't generate value by themselves. Even the government in most countries provides free assistance to disable and unemployed personals. This is required for the survival of all the players in the group forming the grand coalition. An example is included below to illustrate its value distribution with Shapley value.

**Example1.** Consider the game of three players, namely$A, B$ and$C$. Suppose $A$ and $B$ can make an output of four units each. In contrast, their $C$ can't contribute anything to the game. Furthermore, we assume that $A$ and $C$ together can generate an output of ten units. Suppose they distribute their earnings according to Shapley value. Under this sharing situation, $C$ will receive nothing at the end. If $A$ and $B$ take the responsibility of $C$, then a portion of their income will be shifted to $C$. Then the game $(P, v)$ for $P = \{A, B, C\}$ can be represented as $v(\{A\}) = 4 = v(\{B\}),\ v(\{C\}) = 0,\ v(\{A, B\}) = 10, v(\{A, C\}) = 4 = v(\{B, C\})$ and $v(\{A, B, C\}) = 1$.

The value transfer from $A$ and $B$ to $C$ under $\Phi^{\alpha-ES}$value is shown in the table, taking $\alpha = \frac{1}{2}$.

| Players | Shapley Value | $\Phi^{\alpha-ES}$ Value | Value Shifted |
|---|---|---|---|
| Brother A | 5 | $\frac{25}{6}$ | $-\frac{5}{6}$ |
| Brother B | 5 | $\frac{25}{6}$ | $-\frac{5}{6}$ |
| Brother C | 0 | $\frac{10}{6}$ | $+\frac{10}{6}$ |

It is seen that player $C$ received a one-sixth portion of the game for his survival. This value can be manipulated by changing the value of $\alpha$ depending on the game's conditions and the players' wishes. Thus, it seems a convenient way of allocation about situations mentioned above.

***Definition 3.1.Null-player in a productive environment property (NPE):*** *For all $v \in G^P$ and $i \in P$ such that $i$ is a null-player in $v$ and $v(P) \geq 0$, we have $\Phi_i(v) \geq 0$.*

This property guarantees null-players to obtain a non-negative value whenever the worth of the grand coalition is non-negative. This is quite arguable. Since the whole society is productive for $v(P) \geq 0$, it is not necessary for any player to end up with a negative pay-off as they do not do any harm to the society.

***Definition 3.2.Desirability:*** *For $i, j \in P$, if $v(S \cup i) \geq v(S \cup j)\ \forall\ S \subseteq P\backslash\{i, j\}$, then $\Phi_i^\alpha \geq \Phi_j^\alpha$.*

Desirability compares two players in a game and ensures that their payoffs are not opposite to their productivities measured by marginal contributions. Replacing the symmetric property by desirability does not only rule out adverse incentives, but also allows to weaker linearity into additivity. Now we state some important lemmas to be used in the axiomatization of $\alpha$-Egalitarian Shapley value and generalized $\alpha$-Egalitarian Shapley.

**Lemma 3.1.** *If a solution $\Phi$ satisfies desirability, then it also satisfies symmetry.*

**Lemma 3.2.** Every value $\Phi$ that satisfies efficiency, additivity and desirability also satisfies linearity.

**Lemma 3.3.** A value $\Phi$ on $G^P$ is an ESL-value if and only if there exists a unique collection of real constants $B^\Phi = (\alpha_s : s \in \{0,1,2, \dots, p\})$ with $\alpha_n = 1$ and $\alpha_0 = 0$ such that for every game $\Phi_i(v) = \Phi_i^{Sh}(B^\Phi v)$ where $(B^\Phi v)(S) = \alpha_s v(S)$ for each coalition of size $s$.

Now we mention two important characterization of α-Egalitarian Shapley value.

**Theorem 3.1.** [4] *A TU-value $\Phi$ satisfies EFF, LIN, SYM, and the NPE if and only if there exist $\alpha \leq 1$ such that $\Phi = \Phi^\alpha$.*

**Theorem 3.2.** [4] *A TU-value $\Phi$ satisfies ADD, EFF, D, and the NPE if and only if there exists an $\alpha \in [0,1]$ such that $\Phi = \Phi^\alpha$.*

## IV. Generalized egalitarian Shapley value

In this section, a generalized egalitarian Shapley value is discussed for the class $G^P$ by decomposing it into its subspaces based on the coalition size. Interestingly enough, this class of values incorporates the subclass of all the α-Egalitarian Shapley values including the Shapley value and the Equal Division in particular. We proceed as follows.

### IV(A). The $\alpha - GES$ value

Consider an arbitrary $k$ that ranges over the sizes of the coalitions, namely $\{1,2,\dots,p\}$. Let us introduce two subspaces of $G^P$ as follows:

$$G_{<k}(P) = v \in G^P: v(S) = 0 \; \forall \; s \geq k$$

$$G_{>k}(P) = v \in G^P: v(S) = 0 \; \forall \; s \leq k$$

Therefore, every game $v \in G^P$ can be written as $v = v_{<k} + v_{\geq k}$ where $v_{<k} \in G_{<k}(P)$ and $v_{\geq k} \in G_{\geq k}(P)$ such that

$$v_{<k} = \begin{cases} v(S), & if \;\; s < k \\ 0, & otherwise. \end{cases}$$

$$v_{\geq k} = \begin{cases} v(S), & if \;\; s \geq k \\ 0, & otherwise. \end{cases}$$

Now start with the vector $\alpha = (\alpha_1, \alpha_2)$ such that $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, a fixed $k$ and apply $\Phi^{\alpha_1 - ES}$ for $G_{<k}$ and $\Phi^{\alpha_2 - ES}$ for $G_{\geq k}$. Consider the quantity $\Phi_k^\alpha$ given by

$$\Phi_k^\alpha = \Phi^{\alpha_1 - ES}(v < k) + \Phi^{\alpha_2 - ES}(v \geq k).$$

Now take $1 \leq 2 \leq p$ and $G_{k_1 \leq k_2} = \{v \in G^P : v(S) = 0 \; \forall \; s < k_1 \; and \; s \geq k_2\}$. Therefore

$$G^P = G_{<k_1} \oplus G_{k_1 \leq k_2} \oplus G_{\geq k_2}$$

Consequently, every $v \in G^P$ can be expressed as $v = v_{<k_1} + v_{k_1 \leq k_2} + v_{\geq k_2}$, such that

$$v_{k_1 \leq k_2} = \begin{cases} v(S), & if \;\; k_1 < s < k_2 \\ 0, & otherwise. \end{cases}$$

Again, taking $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in [0,1]$ with $k = (k_1, k_2)$, we have

$$\Phi_k^\alpha = \Phi_{<k_1}^{\alpha_1 - ES} + \Phi_{k_1 \leq k_2}^{\alpha_2 - ES} + \Phi_{\geq k_2}^{\alpha_1 - ES}$$

Proceeding in the same way, taking $k = (k_1, k_2, \dots, p)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in [0,1]^n$ such that $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_p \leq 1$, we obtain the generalized Egalitarian-Shapley value as

$$\begin{aligned} \Phi_k^{\alpha - GES}(v) &= \Phi_{<k}^{\alpha_1 - ES} + \Phi_{k_1 \leq k_2}^{\alpha_2 - ES} \\ &\quad + \Phi_{k_2 \leq k_3}^{\alpha_2 - ES} \\ &= \Phi^{\alpha_1 - ES}(v_{<2}) \\ &\quad + \sum_{k=2}^{p-1} \Phi^{\alpha_k - ES}(v_{k \leq k+1}) \\ &\quad + \Phi^{\alpha_p - ES}(v_{>p}) \end{aligned}$$

$$= \sum_{i \in S} \frac{(p-s)!(s-1)!}{p!} [v(S \cup i) - v(S)].$$

### IV(B). Characterization of $\alpha - GES$ value

Following [4] where the Egalitarian Shapley value is characterized using efficiency, null player in a productive environment, a weaker variation of the null player in a productive environment property is considered to characterize the generalized egalitarian Shapley value. This new axiom which we call the Null player in a non-negative environment property is defined as follows.

**Axiom 4.1. Null player in a non-negative environment property:** For all $v \in G^P$ and $i \in P$ such that i is a null-player in v and $v(S) \geq 0 \; \forall \; S \subseteq P$, we have $\Phi_i(v) \geq 0$.

Now, we show that the $\alpha$ −GES value is characterized by efficiency and the null player in a non-negative environment along with additivity and desirability.

**Theorem 4.1.** [7] *A value $\Phi$ satisfifies effificiency, additivity, desirability and the null player in a non-negative environment property if and only if there exists an* $\alpha$=$(\alpha_1, \alpha_2, \dots, \alpha_p) \in [0,1]^p$ *where* $0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p = 1$ *such that* $\Phi = \Phi^{a-GES}$.

## V. Egalitarian Shapley value based on coalition sizes

In the present world, marginalism is observed in coalitions of large sizes whereas egalitarianism is perceived in small coalitions. In the north-eastern part of India, most of the States experience large tribal populations; it is evident even today, that this argument is discernible. In the hills and plains, there exist different tribes to take care of the welfare of their communities. Both the state and the central government earmarked funds to those autonomous councils which are mainly based on their marginal. Marginalism governs this apportion to the larger coalition among tribes. However, most of the facilities in such an autonomous council are ideally based on community and people execute not only egalitarian distribution but also practice sharing of resources. It is evident that the players in a sufficiently small size coalition (e.g., within a tribe) are more affable, cognate and socially engaged. Hence, more often they allow egalitarian sharing of resources among themselves. However, when more people get into the coalition, it becomes sufficiently large and therefore disparate; the productive players refused to share their earnings with the less productive or the non-productive in equal proportion [7]. Motivated by such real-life problems, here we discuss the existence of a solution for a TU game that embeds both egalitarianism and marginalism based on the size of the coalitions.

### V(A). The k-SED value

Let maximum size of the coalition be *k* in which any player enjoys its payoff according to equal division rule. Any coalitions of size greater than *k* will divide its share by Shapley rule. Define

$$T^k(\pi) = \{i \in P: \pi(i) \leq k\}$$

where $\pi: N \to N$ is the permutation by which the grand coalition N is formed such that the players enter the coalition one by one. Let $\Pi(P)$ be the set of all such permutations. Then $T^k(\pi)$ represents the set of first '*k*' players that enters the coalition under the permutation $\pi$.

Let $T(\pi, i) = \{j \in P: \pi(i) > \pi(j)\}$. Then the coalitional contribution, $C_i(v)$ for the player $i \in v$under the permutation $\pi$ is

$$C_i(v) = \begin{cases} \dfrac{v(Pk(\pi))}{k}, & if \;\; \pi(i) \geq k \\ v(P(\pi, i) \cup i) - v(T(\pi, i)), & otherwise. \end{cases}$$

Then for $\pi \in \Pi(P)$, the k-SED value, $\varphi^{k-SED}: G^P \to R^p$ is given by

$$\varphi^{k-SED}(v) = \sum_{j \notin S |S|=k-1} \frac{(p-k)!\,(k-1)!}{p!}[v(S \cup j)] + \sum_{j \notin S |S| \geq k} \frac{(p-s-1)!\,s!}{p!}[v(S \cup j) - v(S)]$$

Now for *k* = 1, the above equation gives,

$$\Phi_j^{k-SED}(v) = \sum_{j\notin S|S|\neq 0} \frac{(p-s-1)!\,s!}{p!}[v(S \cup j) - v(S)] = Sh_j(v)$$

Also, for $k = p$, we have

$$\Phi_j^{k-SED}(v) = \sum_{j\notin S|S|\neq 0} \frac{(p-1)!}{p!}[v(S\cup j)] = \frac{v(P)}{P} = Eg_j(v)$$

Thus $\Phi^{k-SED}$value is equal to the equal division value for $k = p$ and it is equal to Shapley value for $k = 1$.

## V(B). Characterization of $k - SED$ value

For axiomatization and characterization of *k–SED* value, we put some new terms and axioms as follows and explore the relationship between them.

A player $i \in P = \{1,2,\ldots p\}$ is said to be k-NNP if $\forall S \subseteq P\backslash i$,

$$v(S\cup i) = \begin{cases} v(S), & s \geq k \\ 0, & otherwise. \end{cases}$$

***Axiom 5.1.*** *k-nullifying null player property (k-NNPP): A solution $\varphi: G^P \rightarrow R^p$ is said to satisfy k-NNPP if $\varphi_i(v) = 0 \,\forall$ k-NNP, $i \in P$.*

This property assures that any player which is non-productive in coalitions of size larger than k and annihilates the contributions of smaller coalitions of size less than k, must get zero payoffs.

***Theorem 5.1.*** *[6]A solution $\varphi: G^P \rightarrow R^p$is equal to $\varphi^{\{k-SED\}}$if and only if it satisfies linearity, k-NNPP, symmetry and efficiency.*

The Shapley value is characterized by efficiency, coalitional strategic equivalence (CSE) and symmetry. Also, additivity and NPP implies CSE which states that the payoff of a player in a game does not change on addition of another game in which the player satisfies NPP. Here we replace CSE by k-CSE (k-coalitional strategic equivalence) property. This property states that the payoff of any player in a game does not change when a new game is added to it if the player is k-NNP in the added game

***Axiom 5.2*** *A solution $\varphi: G^P \rightarrow R^p$is said to hold k-CSE if $\varphi_i(u+v) = \varphi_i(v)$ for all $k - NNP$ players $i \in u$ and for all $u, v \in G^P$.*

***Proposition 5.1.*** *[6]A value $\varphi: G^P \rightarrow R^p$satisfies linearity and k-NNPP, then $\varphi$ satisfies k-CSE.*

***Theorem 5.2.*** *[6]A value $\varphi: G^P \rightarrow R^p$satisfies efficiency, k-CSE and symmetry if and only if $\varphi = \varphi^{k-SED}$.*

## VI. The game involving middlemen

A middleman is a player in a game that aids a coalition of producers to gain more worth. Here, we take into account a cooperative game in which two kinds of players co-exist, viz., the middlemen and the producers. In such a situation, both the middlemen and producers play complementary roles to fulfil each other's needs, i.e., gain worth in a cooperative situation. Then a value is assigned for the cooperative games containing these two types of players.

A middleman is an intermediary player in a game who cannot produce any resource or worth by himself independently but helps increase the worth of every coalition. For this, he gets involved in some intermediary activities and facilitates the coalitions of the producers to generate more worth through negotiation, bargaining, creating demand for the produced items in the market, etc.

Examples of these situations are abundantly seen in service-oriented markets. For instance, consider the activities of e-commerce giants like Amazon and Alibaba that connect the corporate and seller houses with the customers in need. Similar examples are found in stock markets where brokers play the role of middleman among the buyers and sellers of stocks. Therefore, middlemen activities in markets involving buyers and sellers are crucial domain to study.

For simplicity, let us call all those players who can generate worth individually or through joint action; the producers (this would include the seller, the buyer, the corporate, etc.), and those who facilitate the generation of more worth through their middlemen activities but cannot generate any worth of their own; middlemen. In this section, we study how the producers and middlemen influence each other in allocating their share of the generated worth.

## VI(A). The Middlemen

A TU-game $v \in G^P$ is said to be a TU game with middlemen [2] if there exists a non-empty set $M \subset N$ satisfying the following conditions.

$v(S) = 0 \; \forall \; S \subseteq M$.

$v(S \cup i) > \; v(S) \; \forall \; i \in M; \emptyset \neq S \subseteq P \backslash M$.

$v(S \cup i) > \; v(S \cup j) \forall \; i \in M; \; j \in P \backslash M; \emptyset \neq S \subseteq P \backslash \{M \cup j\}$.

The set M is called the set of Middlemen and the one with the remaining players i.e., $P \backslash M$ is the set of producers. Each player $i \in M$ is called a Middleman. To distinguish a TU game with Middlemen from an ordinary TU game, we denote the former by the triple $(P; M; v)$ where $P$ is the player set, M the set of Middlemen and v the characteristic function on $2^P$. For each middleman $i \in M$ the quantity $\frac{v(S \cup i)}{v(S)}$ represents her ability to intermediate among the players in $S$. Let us denote it by $\eta_S$.

***Definition 6.1.*** *The set of middlemen M is said to satisfy anonymity for each $i \in M$ if $\eta_S = \frac{v(S \cup i)}{v(S)}$ is independent of $i$ and depends only on the size of $S$.*

If $M$ satisfy anonymity, then $\forall \; S \subseteq P, v(S \cup i) - v(S) = (\eta_S - 1) v(S)$. Thus, the marginal contribution of the $i \in M$ doesn't depend on $i$. It follows that each middleman has the same marginal contributions over the producers. This suggests that the middlemen and the producers in a group cannot be treated at par in terms of sharing their total worth and thus, a new solution concept that caters to the specific needs of both types of players. Throughout this paper, we assume that the set M of middlemen satisfy anonymity. Let us denote the class of TU games with Middlemen by $G_M$.

***Theorem 6.1.*** *[2] For two middlemen i and j, then $\Phi_i(P; M; v) = \Phi_j(P; M; v)$.*

## VI(B). A Value for the Class $G_M$

This section defines a new value for the class $G_M$ of TU games with Middlemen. Recall that such a game is denoted by $(P, M, v)$ where $M$ is the set of Middlemen. For each player $j \in P$, let us define two functions $\Phi_j^{Sh}(P; M; v)$ and $\Phi_j^{ED}(P; M; v)$ as

$$\Phi(P,M,v) = \begin{cases} \sum_{j\in N} \frac{(p-s-1)!\,s!}{p!}\, MC_j^v, & if\ j \in P\backslash M \\ \frac{1}{m}\sum_{j\in M} \frac{(p-s-1)!\,s!}{p!}(1-\eta_S)v(S), & if\ j \in M \end{cases}$$

Thus, $\Phi(P;M;v)$ mentioned above defines a value for the class $G_M$ of TU games with middlemen. We call this value as $M-SED$ value. In what follows next, we obtain a characterization $\Phi(P;M;v)$ in terms of some intuitive axioms.

**VI(C). .Axiomatic properties and characterizations of the $M-SED$ value**

$\Phi(P;M;v)$ defined in 6.2 satisfies EFF and ADD. In the following, we define a more axiom that will be required to characterize $\Phi(P;M;v)$.

***Definition 6.2. Null Producer (NP):*** *A player $i \in N$ is defined to be a $NP$ if $v(S \cup i) = v(S)$ for all $S \subseteq P\backslash M$.*

***Axiom 6.1. Null Producer Property (NPP):*** *A solution $\Phi: G^P \to R^n$ is said to satisfy NPP if $\Phi_j(v) = 0$ for all null producer $j \in P$.*

We assume that it is reasonable for all the players in $M$ to accept egalitarian sharing; they all receive the same value after playing the game. However, the producers receive their payoffs according to the Shapley value. So, their payoff will be equal if and only if their marginal contributions are equal over each coalition.

***Theorem 6.2.*** *[1] A value is equal to $\Phi(P;M;v)$ if and only if it satisfies EFF, NPP, SYM, and ADD.*

***Theorem 6.3.*** *[1] EFF, NPP, and DM characterizes the M-SED value uniquely.*

***Theorem 6.4.*** *[1] M-SED value is the unique value characterized by EFF, NPP, and F.*

## VII. Conclusion

The notions of marginalized and egalitarian solutions are valuable tools for distributing the value of a TU game. Marginalized solutions are more effective in cases when participants want payoffs commensurate to their contribution to the game. However, this value is unconcerned about the payoffs of null players in their group. Egalitarian solutions distribute the entire value uniformly, allowing null players to get profit from the game. However, this may result in a fall in the worth of individuals with high contributions, increasing the probability of that person leaving the grand coalition. Consolidated values were developed in cooperative game theory to address these restrictions. Initially, other consolidated values such as egalitarian Shapley value and generalized egalitarian Shapley value were examined. Then these values and allocated based on the sizes of the coalition in which players are operating and then the concept of middlemen is studied for egalitarian Shapley value. The characterization of all these values was investigated emphasizing their importance and applications.

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