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Efficient Optimization of Integer Quadratic Programming Through State Variables Reduction Using Separable Algorithms

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Abstract

Introduction: This paper tackles significant challenges in nonlinear programming and integer quadratic programming problems (IQPPs), presenting innovative solution methodologies. It introduces advanced decision-variable reduction techniques for IQPPs, optimizing solutions by minimizing state variables. The study establishes necessary and sufficient conditions for IQPPs and develops strategies to identify dominated terms in problem formulations. Variable reduction is further refined by analyzing problem data and upper bounds, enabling certain variables to be fixed at zero. A detailed computational analysis highlights the efficiency of these methods across various IQPP scenarios. Additionally, the paper provides insights into separable IQPPs, offering a streamlined framework to enhance comprehension and facilitate intuitive problem-solving. MATLAB-based simulations and graphical representations underscore the practical applicability and robustness of the proposed techniques.

Keywords: Nonlinear programming problems, Quadratic programming, Simplex methods, Separable programming algorithms.

1. Introduction

Integer quadratic programming problems (IQPPs) are optimization challenges characterized by a quadratic objective function and decision variables restricted to integer values. These problems are integral to various domains, including engineering, economics, and operations research, where solutions often involve discrete decision-making. Recent advancements in IQPP research have focused on overcoming computational complexities through innovative solution methods, including exact algorithms, heuristics, and metaheuristics. Attention has also been given to mixed-integer quadratic programming (MIQP) problems, which combine integer and continuous variables. Approaches like decom position techniques, branch-and-bound algorithms, and reformulation strategies have proven effective for MIQPs. The study of IQPPs emphasizes the development of efficient, scalable methods that maintain a balance between solution quality and computational complexity, addressing real-world challenges in optimization and computational intelligence. ([1], [2]). Optimization problems are mathematical frameworks designed to identify the optimal solution from a set of feasible alternatives, often involving the minimization or maximization of an objective function under given constraints. These problems are widely applicable across disciplines such as engineering, economics, operations research, and ma chine learning. Many real-world scenarios, including capacity planning, material cutting, and logistics network design, can be formulated as integer quadratic programming problems (IQPPs), which are often high-dimensional and computationally intensive. Addressing these challenges requires innovative methods to efficiently solve large-scale IQPPs ([3], [4]). Variable reduction techniques have gained considerable attention for addressing large-scale, complex optimization problems. A notable contribution in this area was made by Babayev and Mardanov in 1994 [5], who proposed an innovative approach that compares pairs of columns in the constraint matrix of integer programming problems. This technique focuses on reducing the number of integer variables in various instances, including Knapsack Problems (KP), Multidimensional KPs, and general integer programming problems. The authors established specific conditions under which a variable can be fixed at zero in the optimal solution and validated their method through empirical studies across diverse datasets. Additionally, the reduction of Integer Polynomial Programming Problems to Zero-One Linear Programming Problems has been explored by Lawrence J. Watters, further advancing the field of variable reduction in optimization [6]. Applying variable reduction techniques prior to implementing the Hashian algorithm has been shown to significantly reduce problem complexity and mitigate overflow risks associated with direct application. However, many existing methods are predominantly designed for linear or binary optimization problems ([7], [8], [9]). In 2007, Hua introduced a novel variable reduction approach specifically for convex integer quadratic programming problems (IQPPs), leveraging the optimal values of continuous relaxations and feasible solutions to IQPPs. Zhu and Broughan further contributed by establishing necessary and sufficient conditions for identifying reducible variables in general integer linear programming matrices, enhancing understanding of reduction strategies ([10], [11]). In addition, Sun and Gu addressed nonlinear integer programming in the context of postal design problems [12], while A. Billionnet and E. Soutif proposed an exact and efficient Lagrangian decomposition technique for the 0–1 quadratic knapsack problem. These advancements underscore ongoing efforts to tackle the computational complexities of IQPPs, fostering scalable and efficient optimization solutions [13]. Building upon the above discussion, this article proposes a novel variable reduction technique for general integer quadratic programming problems (GIQPPs) that allows certain variables to be fixed at zero without compromising optimality. Specific conditions for identifying removable decision variables in quadratic integer programming problems are presented. By analyzing problem data and the upper bounds of variables, we establish criteria under which some variables can be fixed at zero. The effectiveness of these conditions is validated through computational experiments using random data in MATLAB, accompanied by graphical illustrations of quadratic programming problems. The structure of this paper is as follows: Section 1 introduces the topic; Section 2 discusses the basic concepts of nonlinear and quadratic programming problems; Section 3 derives necessary and sufficient conditions for identifying dominated terms; Section 4 explains the separable technique for integer quadratic programming problems in a clear manner; Section 5 evaluates the proposed technique's efficacy using computational results from randomly generated GIQPPs and separable IQPPs; finally, Section 6 concludes the paper.

2. General Quadratic Programming Problems

2.1 Nonlinear Programming: General Overview The general nonlinear programming problem is written in the following form:

Optimize (max or min) $Z = f(x) = f(x_1, x_2, ..., x_n);$

subject to the constraints:

$$g(x) = g_i(x_1,\ x_2, ..., x_n) \le b_i,\quad i = 1, 2, \dots, m;$$

$$= b_i \quad i = 1, 2, \dots, m; \quad (1)$$

$$\ge b_i, \quad i = 1, 2, \dots, m;$$

and $x = (x_1, x_2, \dots, x_n)$; $\forall x_j \ge 0$, $j = 1, 2, \dots, n$;

where $f(x) = f(x_1, x_2, \dots, x_n)$ is real-valued objective function of n decision variables and $g(x) = g_i(x_1, x_2, \dots, x_n)$ is real-valued functions of n decision variables in which at least one of these functions is nonlinear.

2.2 Necessary Kuhn-Tucker conditions for nonlinear programming problems (NLPs)

Maximize $Z = f(x)$,

subject to the constraints:

$g_i(x) = 0, \quad i = 1, 2, \dots, m; \qquad (2)$

and $x = x_i, \ge 0$ for all i.

In a nutshell, this is written as:

1. $\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} g_i = 0, j = 1,2,\dots,n$
2. $\sum_{i=1}^{m} \lambda_i\, g_i = 0, \quad i = 1,2,\dots,m;$
3. $\sum_{i=1}^{m} \lambda_i\, g_i \le 0, \quad i = 1,2,\dots,m;$
4. $\lambda_i \ge 0$

Note: If λ ≤ 0, these conditions also apply to minimization nonlinear programming (NLP) problems. The non negativity conditions $x = (x_1, x_2 \ldots\ldots x_n) \geq 0$ is taken for all these conditions 1-4. This represents the feasibilties condi tions.

The Kuhn-Tucker conditions for maximization NLP problem is rewritten as:

Maximize Z = f(x),

subject to the constraints

$g_i \leq 0,$ (3)

and -x ≤ 0, $i = 1,2,\ldots,m;$

where $x = (x_1, x_2 \ldots\ldots x_n)$

In this NLP problem, taking the m+n inequalities into equations. Take the m+n slack variables $g_i^2 \geq 0$ (i = 1,2,...,m,m+1,...,m+n) as:

$$g_i(x) + s_i^2 = 0, \quad i = 1,2,\ldots,m;$$

$$x_j + s_{m+j,}^2 \quad j = 1,2,\ldots,n$$

The Kuhn-Tucker necessary conditions for the maximum of f(x) is obtained as:

$$\frac{\partial}{\partial x_j} f(x) = \lambda \frac{\partial}{\partial x_j} g_i(x),$$

$$\sum_{i=1}^{m} \lambda_i g_i = \lambda_{m+j}, j = 1,2,\ldots,m$$

$$\lambda_i g_i(x) = 0, \quad i = 1,2,\ldots\ldots,m$$

$$\lambda_{m+j} x_j = 0$$

$$g_i(x) \leq 0, \quad \lambda_i, \lambda_{m+j}, x_j \geq 0, \quad for\ all\ i \text{ and } j.$$

The Kuhn-Tucker necessary conditions is taken into sufficient conditions when f(x) is concave and $g_i(x)$ is convex with respect to x. For minimization NLPPs, f(x) is taken as convex, while gi(x) is taken as concave in relation to x.

Lagrangian function is rewritten as:

$L(x,s,\lambda) = f(g_i) - \sum_{i=1}^{m} \lambda_i s_i^2$

Where $\lambda = (\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_m)T$ is the vector of Lagrange multiplier. The necessary conditions for an extreme point to be local optimum (max or min) can be obtained by solving the following equations:

$$\frac{\partial}{\partial x_j} L = \lambda \frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \frac{\partial}{\partial x_j} g_i(x)\,\lambda = 0, \quad j = 1,2,\ldots,n$$

$$\frac{\partial}{\partial x_j} L = -[g_i(x) + s_i^2] \quad i = 1,2,\ldots\ldots,n$$

$$\frac{\partial}{\partial x_j} L = -2x_i\lambda_i, \quad i = 1,2,\ldots\ldots,m$$

Thus, the Kuhn-Tucker necessary conditions to be satisfied at a local optimum (max or min) point is stated as follows:

$$\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \frac{\partial}{\partial x_j} g_i(x)\,\lambda = 0, \quad j = 1,2,\ldots,n$$

$$\lambda_i g_i(x) = 0$$

$$g_i(x) \leq 0,$$

$$\lambda_i \geq 0, \quad i = 1,2,\ldots\ldots,m$$

Remark If the provided nonlinear programming (NLP) problem is a minimization problem or if the constraints are of the form $g_i(x) \geq 0$, then $\lambda_i \leq 0$. Conversely, if the NLP problem is a maximization problem with constraints of the form $g_i(x) \leq 0$, then $\lambda_i \geq 0$.

Kuhn-Tucker Sufficient Conditions:

Theorem 2.1 The Kuhn-Tucker necessary conditions for the problem, Maximize Z = f(x),

subject to the constraints: $g_i(x) \leq 0$, i = 1,2,...,m; x ≥ 0,

are also sufficient conditions if *f*(x) is concave and all $g_i(x)$ are convex functions of *x*.

2.3 Quadratic Programming (QP) Problems

Mathematical modeling of quadratic programming problems is written as:

Optimize (Max or Min)

Z = $\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} d_{jk} x_j x_k,$

subject to the constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i; \quad (4)$$

And $x_j \geq 0$ for all *i* and *j.*

In matrix notation, the QP problem is rewritten as:

Optimize (Max or Min) $Z = c^T x + \frac{1}{2} x^T D x$,

subject to the constraints:

$Ax \le b$, (5)

and $x \ge 0$,

where $x = (x_1, x_2 \ldots\ldots x_n)^T$; $c = (c_1, c_2, \ldots, c_n)^T$; $b_1 = (b_1, b_2, \ldots, b_m)^T$, $D = [d_{jk}]$ is an n×n symmetric matrix, i.e., $d_{jk} = d_{kj}$; $A = [a_{ij}]$ is an m×n matrix.

2.4 Separable Programming Problems

Separable programming problem: A nonlinear programming (NLP) problem is termed a separable programming problem when its objective function can be expressed as a sum of multiple distinct one-variable functions, some of which may be nonlinear.

Separable convex programming: In separable convex programming, each individual function is convex. Specifically, the nonlinear function f(x) is convex when the problem involves minimization and concave when the problem involves maximization.

Separable Functions: A function $f(x_1, x_2 \ldots\ldots x_n)$ that is expressed as the sum of n single-variable functions, $f_1(x_1)$, $f_2(x_2)$,...., $f_n(x_n)$ such that:

$f(x_1, x_2 \ldots\ldots x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$

is said to be a separable function.

Optimize max or min $Z = \sum_{j=1}^{n} fj(xj)$; subject to the constraints: $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i, j = 1,2,\ldots,m$, and $x_j \ge 0$ for all j, where $f_j(x_j)$ is the j^{th} separable function to be approximated over a defined interval.

3 General Integer Quadratic Programming Problems (GIQPPs)

In this section, let us take the general quadratic programming problem (GIQPP):

(GIQPP) min $f(x) = x^T Q x + c^T x$,

subject to constraints:

$A_1 x \le b_1$

$A_2 x = b_2$ (6)

$x \in Z^n$

$x \ge 0$

Where $Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$

Q is a symmetric matrix;

$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$a_{mjJ} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T > 0$;

$$A_1 = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{m1} & \hat{a}_{m2} & \cdots & \hat{a}_{mn} \end{pmatrix}$$

$\hat{a}_{mj} {}^{=} (\hat{a}_{1j}, \hat{a}_{2j}, \ldots\ldots, \hat{a}_{mj})^T > 0$

$a_{mj} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T > 0$;

$c = (c_1, c_2, \ldots, c_n)^T \in R^n$;

$b_1 = (b_1, b_2, \ldots, b_{m1})^T \in R^{m1}$;

$b_2 = (b_1, b_2, \ldots, b_{m2})^T \in R^{m2}$;

$N = \{1,2, \ldots, n\}$; $M_1 = \{1,2, \ldots, m_1\}$; $M_2 = \{1,2, \ldots, m_2\}$. Let S be the feasible range of (GIQPP). If we remove some variables from (GIQPP), generally, the optimal solution and optimal value will change. However, if the optimal value of the problem which has been removed a variable, is equal to that of the original problem (GQPP), then we should only consider the new problem with lower dimensions.

Let (GIQPPk) be the new problem after removing the term k of (GIQPP):

(GIQPPk) min $f_k(y) = y^T Q_k y + d^T y$,

subject to constraints:

$A^r{}_1 y \le b_1$,

$A^r{}_2 y = b_2$, (7)

$y \in Z^{n-1}$,

$y \ge 0$,

where

$Q_r = \{q^r{}_{ij}\}_{(n-1)\times(n-1)}$ and $d = (c_1, \ldots, c_{r-1}, c_{r+1}, \ldots, c_n)^T \in R^{n-1}$;

$A_1 k = (a_1 \cdots a_{k-1}\ a_{k+1} \cdots a_n)$;

$A_2 k = (a_1 \cdots a_{k-1}\ a_{k+1} \cdots a_n)$.

Let S_k be the feasible range of (GIQPPk).

Definition 3.1. Let $x*$ be the optimal solution of (GIQPP) and $f(x*)$ be the corresponding optimal value. $y*$ is the optimal solution of (GIQPPk) and $f_k(y*)$ is the corresponding optimal value. If $f_k(y*) = f(x*)$, then we say term k can be removed. The corresponding integer variable x_k is called a dominated decision variable.

Theorem 3.1. Let $x \in R^n$ be a feasible integer solution of (GIQPP). Suppose $k \in N$ and for all $j \in N\setminus\{k\}$, there exist nonnegative integers l_j satisfying

$$\sum_{j\in n\setminus\{k\}} a_{kj} l_j \leq a_{kr} \ for\ k \in M_1;\ \text{and} \sum_{j\in n\setminus\{k\}} \hat{a}_{kj} l_j \leq \hat{a}_{kr} \ \text{for}\ k \in M_2.$$

If for all $j \in N\setminus\{k\}$, we set $y_j = x_j + l_j x_r$, then $y \in R^{n-1}$ is a feasible integer solution of (GQPPk). Additionally, $x_0 = (y_1, \ldots, y_{r-1}, 0, y_{r+1}, \ldots, y_n)^T$ is also a feasible integer solution of (GIQPP).

Theorem 3.2. If there exists a nonnegative integer vector $l \in S_k$ such that

$$l^T Q^k l \leq q_{kk};\ 2aTQ^+_k l + d^T l \leq 2a^T q_{-k} \leq c_k;$$

then x_k is a dominated variable in (GIQPP). Here $a = (a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)^T$, $Q^+_k = \{q^+_{ij}\}$, $Q^{-k} = \{q^-_{ij}\}$.

Remark. If a_j is the other upper bound of x_j, then the result of Theorem 3.2 also holds.

Denote $u = max_{i,\ j\in N\setminus\{k\}}\{q^+_{ij}\}$, $v = min_{j\in N\setminus\{k\}}\{q^-_{rj}\}$, $b = max_{j\in N\setminus\{k\}}\{a_j\}$, and $e = (1,1, \ldots, 1)^T_{(n-1)\times 1}$. With these symbols, we obtain the following result.

Corollary 3.1. If there exists a nonnegative integer vector $l \in S_k$ such that

$$l^T Q_k l \leq q_{kk};\ 2b(n-1)(ue_T\, l - v) + d_T\, l \leq c_k;$$

then x_k is a dominated variable in GIQPP.

Corollary 3.2. Assume for all $i, j \in N\setminus\{k\}$, $q_{ij} < 0$ and $q_{rj} > 0$ in GIQPP. If there exists a nonnegative integer vector $l \in S_k$ such that

$$l^T Q_k l \leq q_{kk};\ d^T l \leq c_k;$$

then x_k is a dominated variable in GIQPP.

4 Optimizing Integer Quadratic Programming Under Inequality Constraints

In this section, we have establish the GIQPP with inequality constraints:

$$\text{GQPP}_1 \quad \min f(x) = x^T Q_{x+} c^T x$$

subject to constraint: $A_1 x \leq b_1$;
$x \in Z^n$;

$$x \geq 0.$$

$GIQPPk_1$ is the new problem after removing the term r from GIQPP1:

$GIQPPk_1$ $\min f_r(y) = y^T Q_r y + d^T y$

subject to constraints: $A^r{}_1 y \leq b_1$;

$y \in Z^{n-1}$;

$y \geq 0$;

For convenience, we will continue to use the same notations as in the previous section, but with $M_2 = 0$. We will establish a necessary condition for identifying dominated terms.

Theorem 4.1. In $GIQPP_1$, if x_k is a dominated decision variable, then ∃ a nonnegative integer vector $l \in S_k$ such that

$$l^T Q_k l + d^T l \leq q_{kk} + c_k;$$

where $S_k = \{y : A_1 ky \leq b_1, y \in Z^{n-1}, y \geq 0\}$.

Theorem 4.2. In $GIQPP_1$, if for $k, s \in N$; $s \neq r$, there exists a nonnegative integer ls such that

$\forall k \in M_1$, $a_k l_s \leq a_{kr}$

$q_{ss} l^2_s \leq q_{kk}$

$$\sum_{j\in n\setminus\{k\}} a_i q^+_{is} \quad + c_s l_s \leq c_k + \sum_{j\in n\setminus\{k\}} a_j q^-_{rj},$$

then x_k is a dominated decision variable.

Corollary 4.1. If x_k is a dominated decision variable in GIQPP1, then there exists a feasible solution x0 of GIQPP1

such that $f(x_0) \leq q_{kk} + c_k$. Furthermore, the optimal value of GIQPP1, $f(x^*)$, satisfies the inequality $f(x^*) \leq q_{kk} + c_k$.

Corollary 4.2. In GIQPP1, for $k \in N$, if $q_{kk} \geq 0$ and $c_k + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-} \geq 0$, then x_k is a dominated decision variable.

Corollary 4.3. In GIQPP1, if there exist $k \in N$ and $s \in N\backslash\{k\}$ satisfying $q_k \geq 0$, $q_{ss} \leq 0$, and

$$2\sum_{j\in n\backslash\{k\}} a_i q_{is}^{+} + c_s - \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|, \left|\frac{q_{kr}}{q_{ks}}\right|\right) \geq c_k + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-},$$

then x_k is a dominated decision variable.

Corollary 4.4. In GIQPP1, if $\exists\, k \in N$ and $s \in N\backslash\{k\}$ satisfying $q_{kk} \geq 0$, $q_{ss} \geq 0$, and

$$(2\sum_{j\in n\backslash\{k\}} a_i q_{is}^{+} + c_s) \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|, \left|\frac{q_{kr}}{q_{ks}}\right|^{\frac{1}{2}}\right) \geq c_k + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-};\ \frac{q_{kk}}{q_{ss}} \geq \frac{1}{2},$$

then x_k is a dominated decision variable.

Corollary 4.5. In GIQPP1, suppose $\exists\, k \in N$ satisfying $q_{kk} \geq 0$ and for all $i, j \in N\backslash\{k\}$ satisfying $q_{ij} < 0$. Assume

$s \in N\backslash\{k\}$, if

$$c_s - \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|, \left|\frac{q_{kr}}{q_{ks}}\right|\right) \geq c_k + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-},$$

holds, then x_k is a dominated variable.

Corollary 4.6. In GIQPP1, suppose $\exists k \in N$ and $s \in N\backslash\{k\}$ satisfying $q_{kk} < 0$ and $q_{ss} < 0$. If there exists an integer l_s satisfying

$$\left(\frac{q_{kk}}{q_{ss}}\right)^{\frac{1}{2}} \leq l_s \leq \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|, \left|\frac{q_{kr}}{q_{ks}}\right|\right),$$

And

$$2\sum_{j\in n\backslash\{k\}} a_i q_{is}^{+} + c_s - l_s + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-},$$

then x_k is a dominated decision variable.

Corollary 4.7. In GIQPP1, suppose $\exists k \in N$ satisfying $q_{kk} < 0$ and for all $i,j \in N\backslash\{k\}$ satisfying $q_{ij} < 0$. Assume

$s \in N\backslash\{k\}$, if there exists an integer l_s satisfying

$$\left(\frac{q_{kk}}{q_{ss}}\right)^{\frac{1}{2}} \leq l_s \leq \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|, \left|\frac{q_{kr}}{q_{ks}}\right|\right),$$

And

$$\frac{c_k}{l_s} \leq c_k + 2\sum_{j\in n\backslash\{k\}} a_j q_{rj}^{-},$$

then x_k is a dominated decision variable.

Example 4.1. Consider the following problem:

min$f(x)=f(x_1,x_2,x_3,x_4)=(-5x_1^2+40x_2^2+4x_3^2+3x_4^2+4x_1x_2-15x_1x_3+x_1x_4+3x_2x_3+9x_2x_4+2x_3x_4-2x_1+6x_2+8x_3+x_4)$

Subject to constraints: $3x_1 + 4x_2 + 2x_3 + 8x_4 \leq 30$

$2x_1 + 7x_2 + x_3 + 6x_4 \leq 40$

$x = (x_1, x_2, x_3, x_4) \geq 0,\ x \in X^4$

In the matrix form, the above can be written as,

$$\min f(x) = x^T \begin{pmatrix} -10 & 4 & -15 & 1 \\ 4 & 80 & 3 & 9 \\ -15 & 3 & 8 & 2 \\ 1 & 9 & 2 & 6 \end{pmatrix} x + \begin{pmatrix} -2 \\ 6 \\ 8 \\ 1 \end{pmatrix}^T \mathrm{x}$$

Subject to constraints: $Ax \leq b$

where,

$$A = \begin{pmatrix} 3 & 4 & 2 & 8 \\ 2 & 7 & 1 & 6 \end{pmatrix};\ b = \begin{pmatrix} 30 \\ 40 \end{pmatrix};\ x = (x_1, x_2, x_3, x_4)^T$$

$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_4 \geq 0;\ x \in X^4$.

We find that $q_{44} + c_4 = 7$ is larger. Thus we should let $x_4 = 0$ in the optimal solution from Theorem 4.1. The optimal solution is $x = (1,0,1,0)^T$.

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Using the above Corollaries, For k = 4, q_{44} = 6 > 0 and q_{4j} = 0 for j = 1,2,3; c_4 = 1 > 0. x_4 is a dominated decision variable. We remove x4 first. For k = 2, s = 1; $c_1 \le \frac{a_{22}}{a_{21}} -c_2.x_2$ is a dominated decision variable. Remove x_2. The new problem can be written as:

$$\min f(z) = \frac{1}{2} z^T \begin{pmatrix} -10 & -15 \\ -15 & 8 \end{pmatrix} z + \begin{pmatrix} -2 \\ 8 \end{pmatrix}^T z$$

Subject to constraints: $3z_1 + 2z_2 \le 30;$

$2z_1 + z_2 \le 40;$

$z = [z_1, z_2]; z_1 \ge 0; z_2 \ge 0; z \ge 0;$

$z \in Z^2$.

The optimal solution is

$$Z^* = \frac{1}{2}\begin{pmatrix} 10 \\ 0 \end{pmatrix} \text{ and } g(z^*) = -20.$$

So for the original problem, the optimal solution is

$$x^* = \frac{1}{2}\begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } f(x^*) = -20$$

Example 4.2. Consider the following problem:

min $f(x) = f(x_1,x_2,x_3,x_4, x_5,x_6) = (5x_1^2+7.5x_2^2+5x_3^2-3x_4^2-1.5x_5^2+2.5x_6^2+2x_1x_2 +18x_1x_3 +15x_1x_4 +14x_1x_5 +12x_1x_6 +24x_2x_3 +19x_2x_4 +3x_2x_5 +14x_2x_6 -6x_3x_4 +10x_3x_5 +4x_3x_6 +3x_4x_5 +x_4x_6 +7x_5x_6 -7x_1 +5x_2 +x_3 +4x_4 +8x_5 +6x_6)$

Subject to constraints: $9x_1 +6x_2 +6x_3 +9x_4 +3x_5 +5x_6 \le 200$

$5x_1 +6x_2 +6x_3 +10x_4 +2x_5 +7x_6 \le 200$

$x = (x_1,x_2,x_3,x_4) \ge 0,\ x \in X^4$

In the matrix form, the above can be written as,

$$\min f(x) = \frac{1}{2} x^T \begin{pmatrix} 10 & 2 & 18 & 15 & 14 & 12 \\ 2 & 15 & 24 & 19 & 3 & 14 \\ 18 & 24 & 10 & -6 & 10 & 4 \\ 15 & 19 & -6 & -6 & 3 & 1 \\ 14 & 3 & 10 & 3 & -3 & 7 \\ 12 & 14 & 4 & 1 & 7 & 5 \end{pmatrix} x + \begin{pmatrix} -7 \\ 5 \\ 1 \\ 4 \\ 8 \\ 6 \end{pmatrix}^T x$$

Subject to constraints: $Ax \le b$

where,

$$A = \begin{pmatrix} 9 & 6 & 6 & 9 & 3 & 5 \\ 5 & 6 & 6 & 10 & 2 & 7 \end{pmatrix}; b = \begin{pmatrix} 200 \\ 200 \end{pmatrix};\ x = (x_1, x_2, x_3, x_4, x_5, x_6)^T.$$

$x_1 \ge 0; x_2 \ge 0; x_3 \ge 0; x_4 \ge 0; x_5 \ge 0; x_6 \ge 0;\ x \in X^4.$

Applying the above corollaries, for $k = 2$, q_{22} = 15 > 0 and q_{4j} = 0 for j = 1,3,4,5,6; c_2 = 5 > 0. x_4 is a dominated decision variable. For the same reason, there exists k = 6 satisfying above corollaries, so x_6 is a dominated decision variable. Remove x_2 first and x_6.

Then the new problem can be written as:

$$\min g(x) = z^T \begin{pmatrix} 10 & 18 & 15 & 14 \\ 18 & 10 & -6 & 10 \\ 15 & -6 & -6 & 3 \\ 14 & 10 & 3 & -3 \end{pmatrix} z + \begin{pmatrix} -7 \\ 1 \\ 4 \\ 8 \end{pmatrix}^T z$$

Subject to constraints: $9z_1 +6z_2 +9z_3 +3z_4 \le 200;$

$5z_1 +6z_2 +10z_3 +2z_4 \le 200;$

$z \ge 0;$

$z = [z_1,z_2, z_3,z_4]; z_1 \ge 0; z_2 \ge 0; z_3 \ge 0; z_4 \ge 0; z \ge 0;$

$z \in Z^4$

$$\text{The optimal solution is } z^* = \frac{1}{2}\begin{pmatrix} 22 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } g(z^*) = -154$$

So for the original problem, the optimal solution is

$$x^* = \frac{1}{2}\begin{pmatrix} 22 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } f(x^*) = -154$$

We use MATLAB software to compute the two examples with continuous variables. The results yield the same optimal solution and optimal value for both cases. This confirms the applicability of this technique to general integer quadratic programming problems.

5 Separable Integer Quadratic Programming Problems: A General Framework

Separable integer quadratic programming (IQP) is a prevalent type of integer quadratic programming problem (IQPP) with extensive practical applications. Taking, $q_{ij} = 0$ for all $i,j \in N;\ i \neq j$.

For example, if f(x) = $\sum_{j=1}^{n} f_j(x_j)$ is the nonlinear convex objective function. Separable convex programming problem is written as:

(GQPP2) $\min f(x) = \sum_{j\in N}(q_{jj}x_j^2 + x_j)$

subject to $A_1x \leq B_1$

$A_2x = B_2$

$x \in Z^n$, $x \geq 0$,

where $A_j = (a_1, \ldots, a_n)^T > 0;\ A_2 = (a_{0j}, \ldots, a_{0n}),\ a_{0j} > 0,\ j=1,\ldots.n.$

Theorem 5.1 In (GIQPP2), if there exists a nonnegative integer vector $l=(l_1,\ldots,l_{r-1},l_{r+1},\ldots,l_n)^T \in S_k$ such that

$$\sum_{j\in N\setminus\{r\}} q_{jj}l_j^2 \leq q_{kk}\ ;\ \sum_{j\in N\setminus\{k\}} (2a_jq_j^+l_j + c_jl_j) \leq c_k\ ;$$

then x_k is a dominated decision variable. Here a_j and S_k are the same as those in Section 2.

Theorem 5.2 In (GIQPP2), if ∃ a nonnegative integer vector $l=(l_1,\ldots,l_{r-1},l_{r+1},\ldots,l_n)^T \in S_k$ such that

$$\sum_{j\in N\setminus\{k\}} q_{jj}l_j^2 \leq q_{kk}\ ;\ 2\mathrm{bu} + \sum_{j\in N\setminus\{k\}} l_j + \sum_{j\in N\setminus\{k\}} c_jl_j \leq c_k\ ;$$

then x_k is a dominated decision variable in (GIQPP2). Here u = $\max_{j\in N\setminus\{k\}}\{f(q_j^+)\}$, b = $\max_{j\in N\setminus\{k\}}\{f(a_j)\}$.

Theorem 5.3 In (GIQPP2), assume for all $j\in N\setminus\{r\}$, $q_{jj} < 0$. If there exists a nonnegative integer vector $l = (l_1,\ldots,l_{r-1},l_{r+1},\ldots,l_n)^T \in S_k$ such that

$$\sum_{j\in N\setminus\{r\}} q_{jj}l_j^2 \leq q_{kk}\ ;\ \sum_{j\in N\setminus\{r\}} c_jl_j \leq c_j\ ;$$

then x_k is a dominated decision variable.

$$\min f(x) = \sum_{j\in N}(q_{jj}x_j^2 + c_jx_j)$$

subject to $A_1x \leq B_1$

$x \in Z^n$, $x \geq 0$,

Theorem 5.4 In (GIQPP3), if x_k is a dominated decision variable, then ∃ a nonnegative integer vector $l = (l1,\ldots,lr-1,lr+1,\ldots,ln)^T \in S_r$ such that

$$\sum_{j\in N\setminus\{k\}} (q_{jj}l_j^2 + c_jl_j) \leq q_{kk} + c_k$$

If x_k is a dominated decision variable, $q_{kk} + c_k$ is large enough. Thus we should let $x_k = 0$ to decrease the objective function value.

Theorem 5.5. In (GIQPP3), let $k\in N$ and $s\in N\setminus\{k\}$. If there exists a nonnegative integer l_s such that

$$\sum_{j\in M_1} a_{ks}l_s \leq a_{kr};\quad q_{ss}l_s^2 + q_{kk};\quad 2q_{ss}^+a_sl_s + c_sl_s \leq c_k$$

then x_k is a dominated decision variable.

Corollary 5.1. In (GIQPP3), for $k \in N$, if

$q_{kk} \geq 0$ and $c_k \geq 0;$

then x_k is a dominated decision variable.

Corollary 5.2. In (GIQPP3), for $k \in N$ and $s \in N\setminus\{k\}$, if $q_{kk} \geq 0$, $q_{ss} > 0$, and

$$(2q_{ss}a_s + c_s)\min_{k\in M_1}\left(\frac{a_{kr}}{a_{ks}}, \left(\frac{a_{kr}}{a_{ks}}\right)^{\frac{1}{2}}\right) \leq c_k;$$

then *xk* is a dominated decision variable.

Corollary 5.2. In (GIQPP3), for $k \in N$ and $s \in N\setminus\{r\}$, if $q_{kk} \geq 0$ and $q_{ss} \leq 0$ and

$$c_s \min_{k\in M_1}\left(\frac{a_{kr}}{a_{ks}}\right) \leq c_k;$$

then x_k is a dominated decision variable.

Corollary 5.3. In (GIQPP3), suppose there exist $k\in N$ and $s\in N\setminus\{r\}$ such that $q_{kk}< 0$ and $q_{ss} < 0$. If there exists an integer l_s satisfying

$$\left(\frac{q_{kk}}{q_{ss}}\right)^{\frac{1}{2}} \leq l_s \leq \min_{k\in M_1}\left(\frac{a_{kr}}{a_{ks}}, and\ \ c_s.l_s \leq c_k\right) \leq c_k$$

then x_k is a dominated decision variable.

Example 5.1. Consider the following problem:

min $f(x) = f(x_1,x_2,x_3,x_4,) = (-4x_1^2-2.5x_2^2+x_3^2+2x_4^2+x_1-4x_2+2x_3+5x_4)$

Subject to constraints: $5x_1+4x_2+4x_3+7x_4 \leq 30$

$9x_1 + 3x_2 + 6x_3 + 7x_4 \leq 100$

$x = (x_1, x_2, x_3, x_4) \geq 0,\ x \in X^4$

In the matrix form, the above can be written as,

$$\min f(x) = \frac{1}{2}x^T \begin{pmatrix} -8 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ -4 \\ 2 \\ 5 \end{pmatrix}^T x$$

Subject to constraints: $Ax \leq b$

where,

$$A = \begin{pmatrix} 5 & 4 & 4 & 7 \\ 9 & 3 & 6 & 7 \end{pmatrix};\ b = \begin{pmatrix} 30 \\ 100 \end{pmatrix};\ x = (x_1, x_2, x_3, x_4)^T$$

$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_4 \geq 0;\ x \in X^4$.

We find that $q_{44} + c_4 = 9$ is large enough. Thus we should let $x_4 = 0$ in the optimal solution from Theorem 4.4. The optimal solution is x = $(0,20,0,0)^T$. Applying the above corollaries, ∃ $k = 3$, such that $q_{33} > 0;\ c_3 > 0$. x_3 is a dominated decision variable. For $k = 4$, since $q_{44} > 0;\ c_4 > 0$. x_4 is a dominated decision variable. For $k = 1;\ s = 2$, since $q_{11} < 0;\ q_{22} < 0$ and let $\tilde{l}_2 = 2$, then $-9-2\times2 \leq 2$. x_1 can be dominated. Remove x_1, x_3, andx_4

Then the new problem is

min $g(z) = -5z_2^2 - 4z_2$

Subject to constraints:$4z_2 \leq 30$

$3z_2 \leq 100$

$z_2 \geq 0$

$\forall z_2 \in Z$.

The optimal solution is $z^* = 7$ and $g(z^*) = -28$.

So the optimal solution of the original problem is $x^* = (0,7,0,0)^T$ and $f(x^*) = -28$.

Example 5.2.

Consider the following problem:

min $f(x) = f((x_1, x_2, x_3, x_4, x_5, x_6)) = (4x_1^2 - 3x_2^2 + 6x_3^2 + 5x_4^2 - x_5^2 - 6.5x_6^2 - 8x_1 + 4x_2 + 6x_3 + 8x_4 + 7x_5 - 5x_6)$

Subject to constraints:

$5x_1 + 4x_2 + 3x_3 + 3x_4 + 2x_5 + x_6 \leq 20$

$4x_1 + 8x_2 + 4x_3 + 7x_4 + 2x_5 + 3x_6 \leq 20$

$x = (x_1, x_2, x_3, x_4, x_5, x_6) \geq 0;\ x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_4 \geq 0;\ x_5 \geq 0;\ x_6 \geq 0;\ x \in X^4$

In the matrix form, the above can be written as,

$$\min f(x) = \frac{1}{2}x^T \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13 \end{pmatrix} x + \begin{pmatrix} -8 \\ 4 \\ 6 \\ 8 \\ 7 \\ -5 \end{pmatrix}^T x$$

Subject to constraints: $Ax \leq b$

where,

$$A = \begin{pmatrix} 5 & 4 & 3 & 3 & 2 & 1 \\ 4 & 8 & 4 & 7 & 2 & 3 \end{pmatrix};\ b = \begin{pmatrix} 20 \\ 20 \end{pmatrix};\ x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$$

$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_4 \geq 0;\ x_5 \geq 0;\ x_6 \geq 0;\ x \in X^4$

. Applying the above theorems and corollaries, ∃ $k = 2$ and $s = 6$ satisfying $q_{11} \geq q_{66} \geq \frac{1}{2}$ and $\min_{k \in M_1} \left(\frac{a_{k1}}{a_{k6}}\right)$, then $-7-1 \leq 1$.x_2 is a dominated decision variable. We remove x_2 first. For the same reason, there exist $k = 2$ and $s = 6$ satisfying the corollaries so x_3 is a dominated decision variable. For $k = 3$; $s = 6$, since $-5\times1 \leq -8$.x_5 is a dominated decision variable. For $k = 4$, since $q_{44} > 0;\ c_4 > 0$. x_4 can be dominated and removed. Remove x_2, x_3, x_4, and x_5.

Then the new problem is

min $g(z) = 8z_1^2 - 13z_2^2 - 8z_1 - 5z_2$

s.t. $5z_1 + z_2 \leq 20;\ 4z_1 + 3z_2 \leq 20;\ z_1 \geq 0;\ z_2 \geq 0;\ z \in Z^2$.

The optimal solution is $z^* = (2,4)^T$ and $g(z^*) = -36$.

6 Numerical Simulations:

In this section, we analyze experimental results using theories and corollaries of this paper to identify and eliminate or dominated terms of (GIQPP1) and (GIQPP3) respectively.

For (GIQPP1), two types of random test problems with integer data are investigated:

1. For uncorrelated data: p_{ij} and a_{kj} are distributed uniformly in [0,200]. c_j is distributed uniformly in [−100,100]. Let $q_{ij}=(p_{ij} + p_{ji})^2\times 100$.
2. For strongly correlated data: $p_{ij}=a_{kj}-100$, with a_{kj} distributed uniformly in [0,200]. Let $q_{ij}=(p_{ij}+p_{ji})^2$.

The quadratic coefficient matrix $(q^{ij})_{n\times n}$ in (GIQPP1) is symmetric. We apply $q_{ij} =(p_{ij}+p_{ji})^2\times 100$ in uncorrelated data. The results are presented in Tables 1 and 2. For each type of problem size and constraints, we generate 15 test problems. Tables 1 and 2 summarize the experimental results for uncorrelated data.

Table 1: The effectiveness of uncorrelated data in (GIQPP1).

Problem size	The number of constraints	Average remaining variables	Average dominated rate (%)
10	2	5	15.6
	3	9.41	48.4
	5	7.1	44.88
	10	7.02	49.19
50	2	47.1	78.18
	5	44	67.21
	10	39.3	67.18
	50	49.6	56.99
100	1	99	58.1
	2	50.5	49.5
	50	53.94	46.06
	100	55.82	44.18

Tables 1 and 2 show that all theories and corollary are very good for the problem GIQPP1 and GIQPP3.

Table 2: The effectiveness of uncorrelated data in (GIQPP3).

Problem size	The number of constraints	Average remaining variables	Average dominated rate (%)
10	2	7.4	50.6
	3	6.5	47.4
	5	5.7	52.38
	10	6.8	49.4
50	2	45.3	65.01
	5	44.8	55.4
	10	40.3	97.0
	50	40.6	43.17
100	1	90	100
	2	82.5	17.5
	50	89.5	94.07
	100	97	48.97

The results also demonstrate that uncorrelated data is almost similar to the strong correlated data in the results of numerical analysis. This decision variable reduction technique has very low requirements in problem data and it can be widely used in general integer quadratic programming problems with linear inequality constraints. For GIQPP1 and GIQPP3, we just consider the one type of random generated test problems: uncorrelated data and correlated data. q_{jj} and c_j were distributed uniformly in [−100,100]. a_{kj} was distributed uniformly in [0,200].

From the conclusion of the experiment results for general integer quadratic programming problems, we consider the average remaining variables and average dominated rate in Table 1 and Table 2. Specially, for the separable integer quadratic programming problem with linear inequality constraints of integer variables are identified as being dominated and can be fixed at zero before a solution method is used. We also find that for the same problem size and the number of constraints will also affect the average dominated rate (in percentage) and average remaining variables of the integer variables in separable integer quadratic programming problems.

Throughout this research article, we have applied the following notations: q_{ij}^{+}= max{q_{ij},0}, q_{ij}^{-} = min{q_{ij},0}.

Figure 1 is plotted for QPPs 4.1.

Figure 2 is plotted for QPPs 4.2.

Figure 3 is plotted for QPPs 5.1.

Figure 4 is plotted for QPPs 5.2.

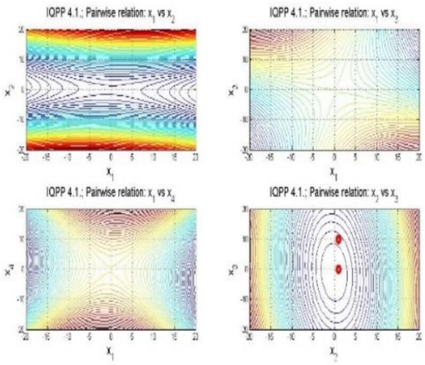

Fig.1: Phase portrait of Quadratic Programming

Problem 4.1 with different axes.

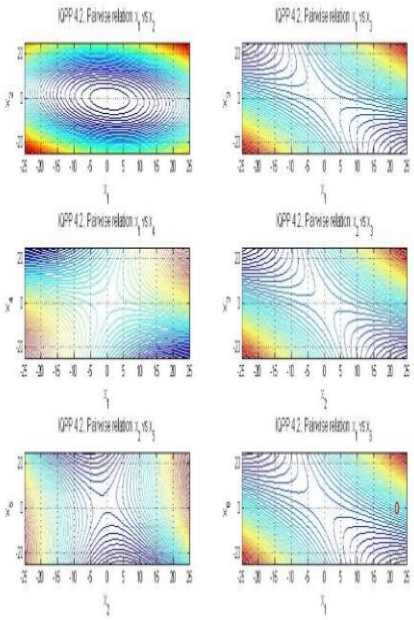

Fig.2: Phase portrait of Quadratic Programming Problem 4.2.

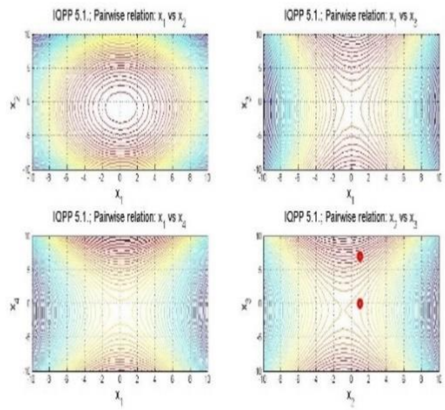

Fig.3: Phase portrait of Quadratic Programming

Problem 5.1 with different axes.

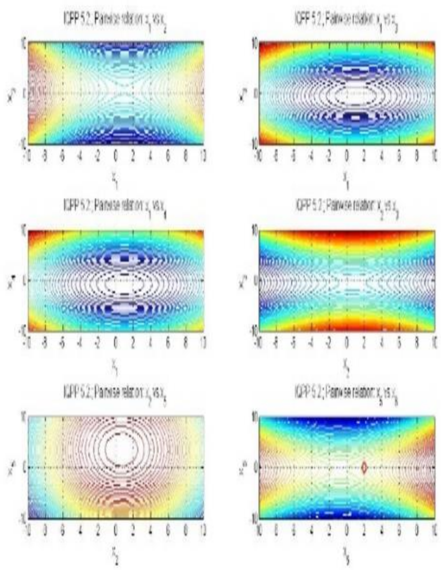

Fig.4: Phase portrait of Quadratic Programming Problem 5.2.

Matlab code for this:

```
num_variables = 10;
num_constraints = 2;
dominated_rate=randi([0,20],num_constraints,1);
average_dominated_rate=mean(dominated_rate)∗100;
min_remainingvariables=num_variables−max(dominated_rate);
max_remaining_variables=num_variables−min(dominated_rate);
average_remaining_variables=num_variables−mean(dominated_rate);
```

7. Conclusions

This paper investigates nonlinear programming and general integer quadratic programming problems (GIQPPs), focusing on advanced optimization techniques. We introduce an innovative variable reduction method for GIQPPs that reduces problem dimensionality by fixing certain decision variables at zero, maintaining optimality. We also establish necessary and sufficient conditions for identifying and eliminating dominated terms within GIQPPs. Extensive computational experiments using MATLAB validate the effectiveness of the proposed approach, demonstrating improved solution efficiency and accuracy. Our findings contribute valuable insights into optimizing GIQPPs, providing a robust framework for solving large-scale, real-world nonlinear programming problems.

8. Declarations and Statements

8.1 Conflict of Interest

This is not applicable.

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8.3 Author Contributions

All authors were actively involved in the conceptualization and development of this research article. Each author has reviewed and approved the final version of the manuscript.

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8.5 Data Availability

The authors affirm that all data supporting the findings of this study are provided within the article.

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