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Trajectory-Based Methods for Indefinite Quadratic Programming Via Dynamical Systems

Mintu Kumar Sah [a*], Neha Varma [b]

∗Corresponding Author: a, b University Department of Mathematics, Lalit Narayan Mithila University,
Darbhanga, Bihar-846008, India.

Abstract: This paper presents a comprehensive study on solving indefinite quadratic programming problems through the framework of dynamical systems. Specifically, we construct and analyze dynamical systems associated with two core DC (Difference of Convex functions) programming algorithms tailored for indefinite quadratic programs. Leveraging tools from non-smooth analysis and the theory of ordinary differential equations, we rigorously establish the existence and uniqueness of global solutions to these systems. Furthermore, we prove the convergence of system trajectories to Karush-Kuhn-Tucker (KKT) points, offering a novel trajectory-based verification method for KKT optimality conditions. The proposed approach not only provides theoretical insights into the behavior of such systems but also contributes to the development of robust algorithms for non-convex optimization problems.

Keywords: Infinite Quadratic programming, Dynamical system, KKT point set, Convergence.

1. Introduction

Quadratic Programming (QP) serves as a fundamental tool in the field of mathematical optimization and plays a pivotal role in a wide array of practical and theoretical applications across disciplines such as engineering design, economic modeling, portfolio optimization, machine learning, and operations research. At its core, QP involves the minimization (or maximization) of a quadratic objective function subject to linear constraints [1],[2]. When the objective function is convex—that is, when its associated Hessian matrix is positive semi-definite—the problem is well-posed and efficiently solvable using well-established techniques such as interior-point methods, active-set strategies, or gradient-based algorithms. The landscape changes drastically when the quadratic objective function is indefinite, meaning the Hessian matrix is not positive semi-definite. In this case, the problem becomes non-convex, leading to multiple local minima, saddle points, and possibly disconnected feasible regions. Such characteristics render the optimization problem significantly more difficult, as classical convex optimization methods may fail to locate the global minimum or even converge at all. These challenges underscore the necessity for more advanced methods capable of navigating the complex geometry of the solution space in non-convex settings [3].

In recent years, continuous-time dynamical systems have emerged as promising and powerful alternatives for addressing the inherent difficulties of indefinite quadratic programming. These systems model the optimization process as a trajectory governed by differential equations, where the state of the system evolves over time under the influence of carefully designed vector fields. By exploiting the rich theory of differential equations and dynamical systems, researchers have developed methods that not only provide insights into the qualitative behavior of the optimization process but also ensure global convergence under certain conditions. This trajectory-based approach has opened new avenues for solving non-convex QPs with theoretical rigor and practical applicability, making it a compelling area of ongoing research and development ([4], [5], [6], [7]).

In this study, we explore a novel trajectory-based approach for solving indefinite quadratic programs using dynamical systems inspired by DC (Difference of Convex functions) programming algorithms. These algorithms decompose non-convex problems into differences of convex functions, facilitating iterative refinement toward optimal solutions.

By constructing corresponding dynamical systems, we investigate their global behavior, particularly focusing on the existence, uniqueness, and convergence properties of their solutions ([8], [9], [10]).

We rigorously establish that the trajectories of the proposed dynamical systems not only exist globally and uniquely but also converge to the Karush-Kuhn-Tucker (KKT) points of the original optimization problem. This convergence provides a natural and efficient mechanism for KKT point verification, bridging the gap between theoretical optimality conditions and practical solution methods [11], [12].

Our results offer a robust theoretical foundation for employing dynamical systems in solving indefinite QPs, providing valuable insights into the stability and convergence characteristics of such systems. This work opens new avenues for the development of continuous-time algorithms with strong theoretical guarantees, suitable for a wide range of non-convex optimization applications.

The paper is organized as follows: Section 1 introduces the topic, setting the context for the study. Section 2 explores the fundamental concepts of dynamical systems and solution of Infinite quadratic programs. Section 3 presents the evolution of dynamical system trajectories. Section 4 assesses the convergence of dynamical system trajectories to KKT Points. Section 5 addresses the numerical problem: KKT point verification via trajectory convergence. Finally, Section 5 concludes the paper, summarizing key findings and implications.

2 Basic Concepts of Dynamical Systems and Solution of Infinite Quadratic Programs

Dynamical systems of solution of indefinite quadratic programs under a geometric constraint is written as:

$$min\left\{f\{x\} = \frac{1}{2}x^TQx + q^Tx \,\middle|\, x \in C\right. \qquad (1)$$

where $Q \in R^{n\times n}$ is a symmetric matrix, $q \in R^n$, $C \subset R^n$ is a nonempty closed convex set, and q^T represents the matrix transposition. This problem is well known in optimization theory. Many qualitative properties of (1) were established for the case where C is a polyhedral convex set. If C is a closed ball with center a and radius r > 0, i.e.,

$$C \;=\; \bar{B}(a,r) := \{\; x \in R^n \mid \| \; x - a \; \| \leq \; r\},$$

then by introducing the new variable y := x−a one gets the problem on minimizing a linear quadratic function on the ball B¯(0,r). Thus, (1) includes the trust-region sub problem [17]. Numerical methods to solve the problem were considered by GM Lee.

If $\bar{x} \;\in\; \mathrm{C}$ is a local solution of (1), the

$$\langle \mathrm{Q\bar{x}} + \mathrm{q}, \mathrm{y} - \mathrm{\bar{x}} \rangle \geq \; 0, \text{for all } \mathrm{y} \;\in\; \mathrm{C}, \qquad (2)$$

Definition 1 A vector $\bar{x} \;\in\; \mathrm{C}$ is said to be a Karush-Kuhn-Tucker point (a KKT point) of (1) if condition (2) is satisfied. The KKT point set of (1) is denoted by C∗.

Definition 2 Let $Q \in R^{n\times n}$ be a matrix, q ∈ R n , and C ⊂ R n a nonempty closed convex set. The affine variationl inequality with a geometric constraint set defined by the triple (Q,q,C) is the problem of finding ¯x ∈ C such that condition (2) is fulfilled.

The KKT point set of the indefinite quadratic programming problem under a geometric constraint (1) coincides with the solution set of the AVI (affine variational inequality) with a geometric constraint set defined by the triple (Q,q,C), where Q is a symmetric matrix. In general, the matrix Q of the AVI given by (Q,q,C) needs not be symmetric.

To solve (1) by DCA (Difference-of-Convex functions Algorithm), one uses the representation $f(x)=f_1(x)-f_2(x)$ for the objective function f(x) with

$$f_1(x) = \left[\frac{1}{2}\, x^T\, Q_1 x + q^T x\;\right] + \delta_C(x),$$

$$f_2(x) = \frac{1}{2}\, x^T\, Q_2 x,$$

where Q_1 and Q_2 are such symmetric positive definite n × n matrices that Q = Q1 − Q2, Here, $\delta_C(x)$ is the indicator function of C (i.e., $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = \infty$ for $x \notin C$). Denote by $\lambda_1(Q)$ and $\lambda_n(Q)$, respectively, the smallest eigenvalue and the largest eigenvalue of Q. Pham Dinh et al. [15] have defined :

(a) $Q_1 = \rho I$, $Q_2 = \rho I - Q$, where ρ > 0 is a real value such that $\rho > \lambda_n(Q)$;

(b) $Q_1 = Q+\rho I$, $Q_2 = \rho I$, where ρ > 0 is a real value satisfying the condition $\rho > -\lambda_1(Q)$.

Since an upper bound and a lower bound for the eigenvalues of a symmetric matrix can be easily computed. one can quickly find a real value ρ > 0 satisfying $\rho > \lambda_n(Q)$ (resp., $\rho > -\lambda_1(Q)$).

The choice made in (a) leads to the explicit iteration schem

$$x^{k+1} := P_C\left(x^k - \frac{1}{\rho}(Qx^k + q)\right) \quad (k = 0,1,2,\ldots) \qquad (3)$$

with x 0 ∈ C being an initial point and PC(u) denoting the metric projection of u ∈ R n on C.

Theorem 1 If C is a polyhedral convex set and (1) has a global solution, then for each x^0 ∈ C, the DCA sequence {x^k} constructed by the scheme (3) converges R-linearly to a KKT point of the problem, that is, there exists $\bar{x}$∈ C* such that

$$\limsup_{k\to\infty} ||x^k - \bar{x}||^{\frac{1}{k}} < 1$$

Theorem 2 (16) If C = B¯(0,r) with r > 0, then for each x^0 ∈ C, the DCA sequence {x^k} obtained by the scheme (3) converges to a KKT point of (1).

Rewrite the formula in (3) equivalently as

$$\frac{x^{k+1} - x^k}{\eta} = \frac{1}{\eta}\left[P_C\left(x^k - \frac{1}{\rho}(Qx^k + q)\right) - x^k\right], \qquad (4)$$

where η > 0 is a constant. Suppose that the whole iteration sequence {x^k} lies on a continuously differentiable curve $\{z(t)\}_{t\geq 0}$ in R n and z(0) = x^0 , z(t_k) = x^k for all k, with {t_k} being an increasing sequence of positive real numbers satisfying the condition $\lim_{k\to\infty} t_k = \infty$. Then, by using the rough approximation

$$\dot{z}(t_k) \approx \frac{z(t_{k+1}) - z(t_k)}{\eta} = \frac{x^{k+1} - x^k}{\eta},$$

From (4) we get

$$\dot{z}(t_k) \approx \frac{1}{\eta}\left[P_C\left(z(t_k) - \frac{1}{\rho}(Qz(t_k) + q)\right) - z(t_k)\right]$$

Therefore, the curve $\{z(t)\}_{t\geq 0}$ can be approximated by the trajectory x(·) of the dynamical system

$$\begin{cases} \dot{x}(t_k) = \frac{1}{\eta}\left[P_C\left(x(t) - \frac{1}{\rho}(Qx(t) + q)\right) - z(t)\right], \\ x(0) = x^0. \end{cases} \qquad (5)$$

for all t>0

Where η and ρ are fixed positive constants and ρ is strictly larger than the largest eigenvalue of Q.

If the choice (b) is adopted, then one has an implicit iteration scheme, where x^{k+1} is computed via x^k by solving the strongly convex quadratic program

$$min\left\{\psi\{x\} = \frac{1}{2}x^T Qx + q^T x + \frac{\rho}{2}||x - u||^2 \;\middle|\; x \in C \right\} \qquad (6)$$

With u= x^k [5]. Denote by F_C(u) the unique solution (6). Then, the map F_C: R_n→ C is Lipschitz continuous, i.e., there exists a constant ℓ>0 such that

$$||F_C(u^1) - F_C(u^2)|| \leq \ell||u^1 - u^2|| \quad \forall u^1, u^2 \in R^n \qquad (7)$$

This fact can be proved by the arguments for obtaining Theorem 2.1 in [15]. Now, the implicit iteration scheme under consideration is given by the formula

$$x^{k+1} = F_C(x^k) \qquad (k = 0,1,2,\ldots\ldots..) \qquad (8)$$

With x^0∈C being an initial point. The following convergence theorem was established by Cuong et al.[5].

Theorem 3 If C is a polyhedral convex set and (1) has a global solution, then for each x_0∈C, the DCA sequence {x^k} generated by the scheme (8) converges R-linearly to a point $\bar{x} \in C_*$.

Numerical tests on two families of randomly generated indefinite quadratic programs on polyhedral convex sets showed that in terms of the number of computation steps and the execution time, algorithm (8) is much more efficient than algorithm (3) when the algorithms are applied to the same problem. If the whole sequence{x^k} generated by the iteration scheme(8) lies on a continuously differentiable curve $\{z(t)\}_{t\geq 0}$ in R^n and z(0)=x^0, z(t_k)=x^k for all k, where {t_k} is an increasing sequence of positive real numbers with $\lim_{k\to\infty} t_k = \infty$, then the reasoning given in a preceding paragraph shows that the curve $\{z(t)\}_{t\geq 0}$ can be approximated by the trajectory x(·) of the dynamical system

$$\begin{cases} \dot{x}(t_k) = \frac{1}{\eta}[F_C(x(t)) - x(t)], \\ x(0) = x^0. \end{cases} \quad for\ all\ t \geq 0 \qquad (9)$$

Where η and ρ are fixed positive constants, ρ is strictly larger than $-\lambda_1(Q)$, and with $F_C(u)$ denoting the unique solution(6).

The next questions, which are the key ingredients of solving (1) by dynamical systems, arise in a natural way.

Problem1: Let$\{t_k\}\subset(0,\infty)$ be a sequence such that $\lim_{k\to\infty} t_k = \infty$, and suppose that $\lim_{k\to\infty} x(t_k) = \bar{x}$, where x(t) denotes the trajectory of either the dynamical system(5)or(9). Can it be established that ¯ x is a Karush-Kuhn-Tucker (KKT) point of the quadratic programming problem (1)?

Problem2: If the quadratic programming problem (1) admits a solution, then there exists a Karush-Kuhn-Tucker (KKT) point $\bar{x}$ such that $\lim_{k\to\infty} x(t_k) = \bar{x}$, where x(t) denotes the trajectory of either the dynamical system (5) or (9).

Regarding the positive constant ρ which has a great role in guaranteeing the convergence of the DCA sequences addressed in Theorem [20], the following question is of interest.

Problem 3: Is the condition ρ > λn(Q) (respectively, ρ > −λ1(Q)) necessary to ensure the convergence of the trajectory x(·) of (5) (respectively, of (9)) to a Karush-Kuhn-Tucker (KKT) point of (1), assuming that the problem admits a solution? It is interesting also to investigate the role of the constant η > 0 for the convergence of the trajectories of the dynamical systems (5) and of (9).

Problem 4: Does the constant η > 0 influence the convergence of the trajectory x(·) of (5) (respectively, of (9)) to a Karush-Kuhn-Tucker (KKT) point of (1), assuming that the problem admits a solution? Note that using dynamical systems to solve optimization problems, variation inequalities, as well as equilibrium problems, is a powerful approach that has been studied intensively worldwide.

3 **Evolution of Dynamical System Trajectories**

The initial value problem defined by the dynamical system (5) admits a unique global solution.

Theorem4 For any $x^0 \in R_n$, ρ>0, and η>0, there exists a unique C^1 function *x:R→Rn* satisfying the differential equation and the initial condition in(5).

Proof:-

Let $F(x) = \frac{1}{\eta}\left[P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right) - x\right]$ for every x $\in R_n$. Since the operator P_C: $R^n \to C$ is non expansive?, [ChapterI, Corollary2.4], one has for any *x, y* $\in R^n$ the estimates

$$
\begin{aligned}
&F(x) - F(y)\\
&\qquad = \left\|\frac{1}{\eta}\left[P_c\left(y - \frac{1}{\rho}\,(Qy+q)\right) - y\right] - \frac{1}{\eta}\left[P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right) - x\right]\right\|\\
&= \frac{1}{\eta}\left\|\left[P_c\left(y - \frac{1}{\rho}\,(Qy+q)\right) - P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right)\right]\right\|\\
&\qquad \le \frac{1}{\eta}\left[\left\|P_c\left(y - \frac{1}{\rho}\,(Qy+q)\right) - P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right)\right\| + \|y-x\|\right]\\
&\le \frac{1}{\eta}\left[\left\|\left(y - \frac{1}{\rho}\,(Qy+q)\right) - \left(x - \frac{1}{\rho}\,(Qx+q)\right)\right\| + \|y-x\|\right]\\
&\le \frac{1}{\eta}\left(\frac{1}{\rho}\,\|Q\| + 2\right)\|y-x\|.
\end{aligned}
$$

Thus the mapping *F:Rn→Rn* is globally Lipschitz with the constant $\frac{1}{\eta}\left(\frac{1}{\rho}\,\|Q\| + 2\right)$ Rewrite (5) as

$$\dot{x} = F(x),\;\; x(0) = x^0 \qquad (10)$$

Applying the Picard-Lindelof Theorem, we can assert that the initial value problem (10) has a locally unique solution. Now, to show that all solutions of the IVP in(10) are defined for all *t* $\in R^n$. There are constants *M* and *L* such that

$$\left|\left|F(x)\right|\right| \le M + L\left|\left|x\right|\right| \quad \forall x \in R^n. \qquad (11)$$

To do so, the growth condition (11) is fulfilled with

$$
\begin{aligned}
\|F(x)\| &= \left\|\frac{1}{\eta}\left[P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right) - x\right]\right\|\\
&= \frac{1}{\eta}\left\|\left[P_c\left(x - \frac{1}{\rho}\,(Qx+q)\right) - x\right]\right\|
\end{aligned}
$$

$$= \frac{1}{\eta} | \left\| \left[\left\{ P_c \left(x - \frac{1}{\rho}\,(Qx+q \right) - \left(x - \frac{1}{\rho}\,(Qx+q \right) \right\} - \frac{1}{\rho}\,(Qx+q \right] \right\| |$$

$$\leq \frac{1}{\eta} \left[||\bar{x} - \left(x - \frac{1}{\rho}\,(Qx+q \right) || + \frac{1}{\rho}\,(Qx+q)|| \right]$$

$$\leq \frac{1}{\eta} \left[\left(||\bar{x}|| + \frac{2}{\rho} ||q|| \right) + \left(\frac{2}{\rho} ||Q|| + 1 \right) ||x|| \right].$$

Therefore, the growth condition (11) is fulfilled with

We have proved that there is a unique C^1functionx (·) defined on the whole real line satisfying the conditions in (5). □

Theorem 5 For any $x^0 \in R_n$, ρ>0, and η>0, the dynamical system (9) has a unique C^1 trajectory x(·), which is defined on the whole real line.

Proof: - Let ℓ be a positive constant satisfying the condition(7). For every vector *x∈Rn*,

Put $G(x) = \frac{1}{\eta}(F_C(x) - x)$

By(7), we have for any *x, y ∈Rn* the following

$$\| G(y) - G(x) \| = || \frac{1}{\eta}\ \left[(F_C(y) - y) - (F_C(x) - x) \right] ||$$

$$\leq \frac{1}{\eta}\ \left[||F_C(y) - F_C(x)|| + \| y - x \| \right.$$

$$\leq \frac{1}{\eta}\ (\ell + 1)\ \| y - x \|.$$

So, the mapping G:Rn→Rn is globally Lipschitz with the constant 1/η (ℓ+1). Rewrite the system (9) as ˙

$$\dot{x} = G(x),\ \ x(0) = x^0 \qquad (12)$$

This IVP has a locally unique solution by the Picard-Lindelof Theorem. There are constants M_1 and L_1 such that

$$||G(x)|| \leq M_1 + L_1 ||x|| \quad \forall x \in R^n. \qquad (13)$$

Indeed, fixing a point $\bar{x} \epsilon C$, we have for every *x ∈Rn* the estimates

$$\| G(x) \| = \left| \left| \frac{1}{\eta}\ [F_C(\bar{x}) - x] \right| \right|$$

$$= \frac{1}{\eta} ||\left(F_C(x) - F_C(\bar{x})\right) - (F_C(\bar{x}) - x)||$$

$$\leq \frac{1}{\eta} [\ \ell \| x - \bar{x} \| + \| F_C(\bar{x}) - x \|]$$

$$\leq \frac{1}{\eta}\ [\ell \| \bar{x} \| + \| F_C(\bar{x}) \| + (1+\ell) \| x \|]$$

Therefore, the growth condition (13) is satisfied if we choose

$$M_1 = \frac{1}{\eta}\ (\ell \| \bar{x} \| + \| F_C(\bar{x}) \|), \qquad L_1 = \frac{1+\ell}{\eta}$$

All solutions of the IVP in (12) are defined for all $t \in R_n$. We have seen that the system (9) possesses a uniqueC^1 trajectory x(·), which is defined on the whole real line. □

4 Convergence of Dynamical System Trajectories to KKT Points

Let F : R→R be defined by F(x)=αx+β for x ∈R, where α andβ are real constants, and let C =[−1,1]. Consider the (possibly non-convex) quadratic optimization problem

$$min \left\{ f\{x\} = \frac{1}{2}\alpha x^2 + \beta x \,\middle|\, x \in C \right. \qquad (14)$$

According to the generalized Fermat's rule , if $\bar{x} \epsilon C$ is a local solution of (14), then

$$\nabla \mathrm{f}(\bar{x})(\mathrm{y} - \bar{x}\,) \geq \qquad 0, \text{for all y} \in \mathrm{C}$$

$$\alpha\bar{x} + \beta)(y - \bar{x}) \geq 0, \qquad for\ all\ y \in C \qquad (15)$$

In agreement with the definition recalled in Section 2, $\bar{x} \in C$ is said to be a Karush-Kuhn-Tucker point (a KKT point) of (14) if the condition (15) is satisfied.

For some $x^0 \in C$, let x(·) be the solution of the following dynamical system

$$\begin{cases} \dot{x}(t_k) = \frac{1}{\eta}\left[P_C\left(\ x(t)\ - \frac{1}{\rho}\,(Qx(t) + \beta \right) - x(t)\ \right], \\ \ \ x(0) = x^0. \end{cases} \qquad (16)$$

$$for\ all\ t \geq 0$$

where η and ρ are fixed positive constants.

Proposition 1 For any *(α,β) ∈ R×R* satisfying *α < 0* and for any $x^0 \in [-1,1]$, the unique solution *x(t)* of (16) converges to a KKT point of (14) as t → ∞.

Proposition 2 For any *(α,β) ∈ R×R* satisfying α ≥ 0 and for any $x^0 \in [-1,1]$, the unique solution x(t) of (16) converges to a KKT point of (14) as *t → ∞,* provided that ρ −α ≥ 0.

Theorem 6 For any *(α,β)∈R×R* and for any

$x^0 \in [-1,1]$, the unique solution $x(t)$ of (16) converges to a KKT point of (14) as t →∞ if either $\alpha \le 0$ or $\alpha > 0$ and $\rho \ge \alpha$.

5 Numerical Simulation: KKT Point Verification via Trajectory Convergence

Let us consider the optimization problem:

$$Minimize\ f(x) = x_1^2 + x_2^2 \qquad (1)$$

$$\text{subject to } x_1+x_2-1=0$$

Let $x(t) \in R^2$ be a trajectory (solution path) of a dynamical system governed either by:–

(i) a projected gradient flow (e.g., equation (5)), or,

(ii) a neural network-based dynamical model (e.g., equation (9)).

Assume the following discrete-time samples are taken from the trajectory x(t) at increasing time values $t_k \to \infty$:

$x(t_1) = (0.7,0.5)$,

$x(t_2) = (0.6,0.4)$,

$x(t_3)=(0.55,0.4)$,

$x(t_4)=(0.51,0.4)$,

$x(t_5)=(0.5001,0.4999)$.

Let us assume that: $\lim_{k\to\infty} t_k = \bar{x} = (0.5,0.5)$.

Question

Is $\bar{x} = (0.5,0.5)$. a KKT point of the optimization problem (1)?

Solution

To check if $\bar{x}$ is a KKT point, we verify the KKT conditions.

Step 1: Lagrangian Function

Define the Lagrangian:

$L(x,\lambda) = x^2 + x^2 + \lambda(x1 + x2 - 1)$

Step 2: KKT Conditions

The KKT conditions for equality-constrained optimization are:

(i) Stationarity:

$$\nabla_X L(x,\lambda) = 0 \Rightarrow \begin{cases} 2x_1 + \lambda = 0 \\ 2x_2 + \lambda = 0 \end{cases}$$

(ii) **Primal Feasibility**:

$x_1 + x_2 - 1 = 0$

Step 3: Substitute $\bar{x}$ = (0.5,0.5)

(i) Check primal feasibility:

0.5 + 0.5−1=0 ⇒Satisfied

(ii) Solve stationarity: From the stationarity conditions:

2(0.5) + λ=0⇒λ=−12(0.5) λ=0⇒λ =−1

Thus, the same λ satisfies both equations.

Conclusion

All KKT conditions are satisfied. Hence

$$\boxed{\bar{x} = (0.5,0.5) \text{ is a KKT point of problem (1)}}$$

6 Conclusion

In this work, we have investigated the solution of indefinite quadratic programming problems by formulating and ana-lyzing associated dynamical systems derived from two principal DC programming algorithms. Through the application of non-smooth analysis and the theory of ordinary differential equations, we established both the existence and uniqueness of global solutions to the proposed systems. Importantly, we demonstrated that the trajectories of these dynamical systems converge to the Karush-Kuhn-Tucker (KKT) points, thereby validating the optimality of the solutions obtained. This trajectory-based framework offers a promising and theoretically sound approach for verifying KKT conditions in non-convex optimization scenarios. The results lay a solid foundation for the development of efficient continuous-time algorithms for solving a broader class of indefinite quadratic programming problems.

7. Declarations and Statements

7.1 Conflict of Interest

This is not applicable.

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7.3 Author Contributions

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7.5 Data Availability

The authors affirm that all data supporting the findings of this study are provided within the article.

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