Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 46 No. 7
July 2025

Common Fixed Point Theorems in New Generalized Fuzzy Metric Space with Graph

Swathi M[1*], R. Sumitra[2], R. Hemavathy[3]

[1,2,3]Department of Mathematics, Queen Mary's College, Chennai - 600 004, Tamil Nadu, India (Affiliated to University of Madras)

Abstract

The purpose of this paper is to prove common fixed point theorem in generalized fuzzy metric space. Also, we define new generalized fuzzy metric space endowed with graph and prove common fixed point theorem in this new setting. Our theorems generalize existing results. In addition, we bolster our findings with illustrative examples that validate our theorems.

Keywords: Fixed point; common fixed point; generalized mapping; weakly compatible; generalized fuzzy metric space.
Classification: 47H10,55M20,54H25

1. Introduction

Fixed point theory in functional analysis is an essential framework with broader applications across mathematical and other disciplines. Banach[1] formulated the Banach contraction principle, which is celebrated for several aspects. Kramosil and Michalek[2] developed the concept of fuzzy metric spaces by applying t −norms to the fuzzy set theory introduced by L. A. Zadeh[3]. Later, George and Veeramani[4] modified the definition of fuzzy metric space. Hussain et al.[5] and Nadaban[6] proposed the concept of fuzzy b −metric space as a generalization of fuzzy metric space.

Jungck[7] proposed the concept of commutative mapping and established common fixed point theorem which fascinated many researchers leading to development in this area.

Generalized metric spaces are widely studied in fixed point theory because they encompass a broad range of previously investigated spaces. Motivated by Jleli and Samet[8] generalized metric space, M.S. Ashraf et al.[9] introduced generalized fuzzy metric space which generalizes fuzzy metric space, fuzzy b −metric space and dislocated fuzzy metric space. Recently L.N. Mishra et al.[10], Thangthong and Charoensawan[11] proved some common fixed point theorems in Jleli and Samet[8] generalized metric space which motivated us to prove common fixed point theorem for commuting, weakly compatible and admissible mappings in generalized metric space.

The notion of a metric space endowed with a graph was introduced by Jachymski[12] which excited among researchers and the fixed point theorems in different spaces endowed with graph was established. In 2014, S. Shukla[13] gave the concept of fuzzy metric space endowed with graph. Charoensawan and Atiponrat[14] proposed the concept of new generalized metric space endowed with graph and proved the common fixed point theorem for generalized contraction. Motivated by Charoensawan and Atiponrat[14] we introduce the concept of new generalized fuzzy metric space endowed with graph and proved the common fixed point theorem in this new setting for generalized Ciric contraction.

We structure our paper as following. In section 2, we give the preliminary definitions and results which will be used in consequent sections. The main results in section 3 are subdivided into three subsections. In subsection 3.1, we proved common fixed point theorem for commuting and weakly compatible mapping in generalized fuzzy metric space. In the next subsection 3.2, we defined the admissible contraction mapping in generalized fuzzy metric space and proved common fixed point theorem in the foresaid space. Finally, in section 4 we introduced the new generalized fuzzy metric space endowed with graph and proved common fixed point theorem in this new setting.

2. Preliminaries

Jleli and Samet[8] defined the following generalized metric space which generalized many existing metric spaces.

Definition 2.1[8] Consider a non-void set Ξ and a mapping $G_{ms}: \Xi \times \Xi \to [0,1]$. For each $\nu \in \Xi$, we set $\mathcal{S}(G_{ms}, \Xi, \nu) = \left\{\nu_n \subseteq \Xi: \lim_{n\to\infty} G_{ms}(\nu_n, \nu) = 0\right\}$. For all $\nu, \mu \in \Xi$, with the following conditions:
$(\mathrm{G_{ms}}1)$ $G_{ms}(\nu, \mu) = 0 \Rightarrow \nu = \mu$;
$(\mathrm{G_{ms}}2)$ $G_{ms}(\nu, \mu) = G_{ms}(\mu, \nu)$;

$(\mathrm{G_{ms}}3)$ there exists $s' \geq 1$ such that if $\nu_n \in S(G_{ms}, \Xi, \nu)$ then

$$G_{ms}(\nu, \mu) \leq s' \limsup_{n\to\infty} G_{ms}(\nu_n, \mu).$$

Then G_{ms} is a generalized metric and (G_{ms}, Ξ) is termed as generalized metric space (briefly GMS).

Definition 2.2[15] A binary operation $\star: [0,1] \times [0,1] \to [0,1]$ is a continuous t −norm if it fulfills the subsequent requirements: (i) $\star$ is associate and commutative; (ii) $\star$ is continuous; (iii) $b_1 \star 1 = b_1$ for all $b_1 \in [0,1]$ and (iv) $b_1 \star b_2 \leq b_3 \star b_4$ whenever $b_1 \leq b_3$ and $b_2 \leq b_4$ for all $b_1, b_2, b_3, b_4 \in [0,1]$.

Definition 2.3[4] Consider a non-void set Ξ, $\star$ is a continuous t −norm and a mapping $F_{ms}: \Xi \times \Xi \times (0, \infty) \to [0,1]$. For all $\nu, \mu \in \Xi$ and $t, s > 0$, with the following conditions:

$(\mathrm{F_{ms}}1)$ $F_{ms}(\nu, \mu, t) > 0$;
$(\mathrm{F_{ms}}2)$ $F_{ms}(\nu, \mu, t) = 1 \Leftrightarrow \nu = \mu$;
$(\mathrm{F_{ms}}3)$ $F_{ms}(\nu, \mu, t) = F_{ms}(\mu, \nu, t)$;
$(\mathrm{F_{ms}}4)$ $F_{ms}(\nu, \mu, t + s) \geq F_{ms}(\nu, \mu, t) \star F_{ms}(\nu, \mu, s)$;
$(\mathrm{F_{ms}}5)$ $F_{ms}(\nu, \mu, \cdot): (0, \infty) \to [0,1]$ is continuous.

Then $\mathrm{F_{ms}}$ is a fuzzy metric and $(\mathrm{F_{ms}}, \Xi, \star)$ is termed as fuzzy metric space (briefly FMS).

Definition 2.4[5] Consider a non-void set Ξ with $b \geq 1$, $\star$ is a continuous t −norm. A mapping $F_b: \Xi \times \Xi \times (0, \infty) \to [0,1]$. For all $\nu, \mu \in \Xi$ and $t, s > 0$, with the following conditions:

$(\mathrm{F_b}1)$ $F_b(\nu, \mu, t) > 0$;
$(\mathrm{F_b}2)$ $F_b(\nu, \mu, t) = 1 \Leftrightarrow \nu = \mu$;
$(\mathrm{F_b}3)$ $F_b(\nu, \mu, t) = F_b(\mu, \nu, t)$;
$(\mathrm{F_b}4)$ $F_b(\nu, \mu, b(t + s)) \geq F_b(\nu, \mu, t) \star F_b(\nu, \mu, s)$ and
$(\mathrm{F_b}5)$ $F_b(\nu, \mu, \cdot): (0, \infty) \to [0,1]$ is continuous.

Then F_b is a fuzzy b −metric and $(F_b, \Xi, \star)$ is termed as fuzzy b −metric space (briefly $\mathrm{F_b}MS$).

M. Dinarvand[16] gave the following definition of triangular β −admissible in FMS.

Definition 2.5[16] *Let* $(F_{ms}, \Xi, \star)$ *be* FMS *and* $\beta: \Xi \times \Xi \times (0, \infty) \to (0, \infty)$ *then* $U: \Xi \to \Xi$ *is labelled as triangular* β *−admissible*

(1) $\beta(q_1, q_2, t) \leq 1$ imply $\beta(Uq_1, Uq_2, t) \leq 1$
(2) $\beta(q_1, q_3, t) \leq 1$ *and* $\beta(q_3, q_2, t) \leq 1$ *imply* $\beta(q_1, q_2, t) \leq 1$
for every $q_1, q_2, q_3 \in \Xi$ *and* $t > 0$.

Presented below are the definition and results of a generalized fuzzy metric space as introduced by M.S. Ashraf et al. [9].

Definition 2.6[9] Consider a non-void set Ξ, $\star$ is a continuous t −norm and a mapping $G_f: \Xi \times \Xi \times (0, \infty) \to [0,1]$. For each $\nu \in \Xi$, we set

$$\mathcal{S}(G_f, \Xi, \nu) = \left\{\nu_n \subset \Xi: \lim_{n\to\infty} G_f(\nu_n, \nu, t) = 1 \forall t > 0\right\}.$$

For all $\nu, \mu \in \Xi$ and $t > 0$, with the following conditions:

$(\mathrm{G_f}MS1)$ $G_f(\nu, \mu, t) > 0$;
$(\mathrm{G_f}MS2)$ $G_f(\nu, \mu, t) = 1 \Rightarrow \nu = \mu$;
$(\mathrm{G_f}MS3)$ $G_f(\nu, \mu, t) = G_f(\mu, \nu, t)$;
$(\mathrm{G_f}MS4)$ there exists $s' \geq 1$ such that if $\nu_n \in S(G_f, \Xi, \nu)$ then

$$G_f(\nu, \mu, t) \geq \limsup_{n\to\infty} G_f(\nu_n, \mu, t/s');$$

$(\mathrm{G_f}MS5)$ $G_f(\nu, \mu, \cdot): (0, \infty) \to [0,1]$ is continuous and $\lim_{n\to\infty} G_f(\nu, \mu, t) = 1$.

Then $\mathrm{G_f}$ is a generalized fuzzy metric and $(G_f, \Xi, \star)$ is termed as generalized fuzzy metric space (briefly $GFMS$).

Example 2.7[9] With a generalized metric space (Ξ, G_{ms}), create a mapping $G_f: \Xi \times \Xi \times (0, \infty) \to [0,1]$ such that

$G_f(\nu, \mu, t) = e^{-G_{ms}(\nu, \mu)/t}$ and

$$\mathcal{S}(G_f, \Xi, \nu) = \left\{\nu_n \subset \Xi: \lim_{n\to\infty} G_f(\nu_n, \nu, t) = 1\right\} \forall \nu \in \Xi \ \ and \ \ t > 0$$

then $(G_f, \Xi, \star)$ is $GFMS$ with t −norm $\star$ is given by $a_1 \star a_2 = a_1 a_2$.

Example 2.8 With a generalized metric space (Ξ, G_{ms}), create a mapping $G_f: \Xi \times \Xi \times (0, \infty) \to [0,1]$ such that $G_f(\nu, \mu, t) = t/t + G_{ms}(\nu, \mu)$ and

$\mathcal{S}(G_f, \Xi, \nu) = \left\{\nu_n \subset \Xi: \lim_{n\to\infty} G_f(\nu_n, \nu, t) = 1\right\} \forall \nu \in \Xi$ and $t > 0$. Let $\nu, \mu \in \Xi$ and $\nu_n \in \mathcal{S}(G_{ms}, \Xi, \nu)$. In view of G_{ms} being a generalized metric then from definition,

$$\lim_{n\to\infty} G_f(\nu_n, \nu, t) = t/t + 0 = 1$$

this gives $\nu_n \in \mathcal{S}(G_f, \Xi, \nu)$.

It can be inferred that

$$\begin{aligned} G_f(\nu, \mu, t) &= \frac{t}{t + G_{ms}(\nu, \mu)} \\ &\geq \frac{t}{t + s' \limsup_{n\to\infty} G_{ms}(\nu_n, \mu)} \\ &= \frac{t/s'}{t/s' + \limsup_{n\to\infty} G_{ms}(\nu_n, \mu)} \\ &= \limsup_{n\to\infty} \frac{t/s'}{t/s' + G_{ms}(\nu_n, \mu)} \\ &= \limsup_{n\to\infty} G_f(\nu_n, \mu, t/s') \end{aligned}$$

we obtain,

$G_f(\nu, \mu, t) \geq \limsup_{n\to\infty} G_f(\nu_n, \mu, t/s')$. then $(G_f, \Xi, \star)$ is $GFMS$ with t −norm $\star$ is given by $a_1 \star a_2 = a_1 a_2$.

Definition 2.9[9] Let $(G_f, \Xi, \star)$ be $GFMS$

(i) A sequence $\{\nu_p\} \in \Xi$ is labelled as G_f −convergent to ν if $\nu_n \in S(G_f, \Xi, \nu)$.

(ii) A sequence $\{\nu_p\} \in \Xi$ is labelled to be G_f −Cauchy sequence if $\lim_{p,q\to\infty} G_f(\nu_p, \nu_{p+q}, t) = 1$ for every $t > 0$.

(iii) Every G_f −Cauchy sequence is G_f −convergent is labelled as G_f −complete $GFMS$.

Proposition 2.10[9] (i) Every FMS is a $GFMS$.
(ii) Every F_bMS is a $GFMS$.

Proposition 2.11[9] Let $(G_f, \Xi, \star)$ be a $GFMS$ and a sequence $\{\nu_p\} \in \Xi$ such that $\{\nu_p\}$ is G_f −convergent to ν and μ then $\nu = \mu$.

Remark: For any sequence $\{\nu_n\} \in \Xi$, we denote

$$\delta(G_f, U, \nu_n, t) = \sup\{G_f(U^{n+i}(\nu_0), U^{n+j}(\nu_0), t) : \ i,j \in \mathbb{N} \cup 0 \ and \ t > 0\}.$$

3. Main Results

3.1 Common fixed point theorem in GFMS

Definition 3.1 Take a set of self functions U and K of a $GFMS$ $(G_f, \Xi, \star)$.
(i) An element $r \in \Xi$ is labelled as **coincidence point** of U and K equivalently $Ur = Kr$. We denote the set of coincidence points of U and K as $C_p(U,K) = r \in \Xi: Ur = Kr$.
(ii) An element $r \in \Xi$ is labelled as **common fixed point** of U and K equivalently $Ur = Kr = r$

Definition 3.2 A set of self functions U and K of a $GFMS$ $(G_f, \Xi, \star)$ is labelled as
(i) **commuting** if

$$G_f(UKd_1, KUd_1, t) = 1 \text{ for all } d_1 \in \Xi.$$

(ii) **weakly-commuting** if $G_f(UKd_1, KUd_1, t) \geq G_f(Ud_1, Kd_1, t)$ for all $d_1 \in \Xi$ and $t > 0$.

(iii) **compatible** if $\lim_{p\to\infty} G_f(UKd_p, KUd_p, t) = 1$ for every $t > 0$ and a sequence $\{d_p\} \in \Xi$ with a condition that $\lim_{p\to\infty} Ud_p = \lim_{p\to\infty} Kd_p = r$ for certain $r \in \Xi$.

(iv) $\mathbf{R}$ −**weakly commuting** if for certain $R > 0$,

$G_f(UKd_1, KUd_1, t) \geq G_f(Ud_1, Kd_1, t/R)$ for all $d_1 \in \Xi$ and $t > 0$.

(v) **weakly compatible** if they operate commutatively at the point of coincidence which means, if $Ur = Kr$ for certain $r \in \Xi$ results in $UKr = KUr$.

Definition 3.3 Let $(G_f, \Xi, \star)$ be a $GFMS$ and a function $U: \Xi \to \Xi$ is labelled continuous at a point $\nu_0 \in \Xi$, if $\{\nu_n\} \in S(G_f, \Xi, \nu_0)$ implies $\{U\nu_n\} \in S(G_f, \Xi, U\nu_0)$. Further, we say U is continuous on Ξ if it continuous at every $\nu \in \Xi$.

Theorem 3.4 Let $(G_f, \Xi, \star)$ be a complete $GFMS$. Take $U: \Xi \to \Xi$ a continuous transformation and $K: \Xi \to \Xi$ satisfying the following: (i) U commutes with K; (ii) $U(\Xi) \subseteq K(\Xi)$;(iii) $G_f(U(q_1), U(q_2), \rho t) \geq G_f(K(q_1), K(q_2), t)$ for $\rho \in (0,1)$; (iv) there exists $k_0 \in \Xi$ with $\delta(G_f, K, k_0, t) > 0$. Then U and K holds a unique common fixed point.

Proof. With given hypothesis, we have the mapping U commutes with K. Let $k_0 \in \Xi$ and $U(k_0) = K^2(k_0)$ subsequently by induction

$$U^{m-1}(p_0) = K^m(p_0) \tag{3.1}$$

For $m \in \mathbb{N}$ and $i,j \in \mathbb{N}$, we obtain,

$$\begin{aligned} G_f(K^{m+i}(k_0), K^{m+j}(k_0), t) &= G_f(U^{m-1+i}(k_0), U^{m-1+j}(k_0), t) \\ &\geq G_f(K^{m-1+i}(k_0), K^{m-1+j}(k_0), t/\rho) \end{aligned}$$

Taking supremum,

$$\begin{aligned} &sup G_f(K^{m+i}(k_0), K^{m+j}(k_0), t) \\ &\geq sup\ G_f(K^{m-1+i}(k_0), K^{m-1+j}(k_0), t/\rho) \end{aligned}$$

$\delta(G_f, K, K^m(k_0), t) \geq \delta(G_f, K, K^{m-1}(k_0), t/\rho)$.
Subsequently, for every $m \in \mathbb{N}$, $\delta(G_f, K, K^m(k_0), t) \geq \delta(G_f, K, (k_0), t/\rho^m)$.
By employing the inequality mentioned above, for every $m, m_1 \in \mathbb{N}$, we obtain,

$$\begin{aligned} \delta(G_f, K^m, K^{m+m_1}(k_0), t) &\geq \delta(G_f, K, K^m(k_0), t) \\ &\geq \delta(G_f, K, (k_0), t/\rho^m). \end{aligned}$$

With $\delta(G_f, K, (k_0), t/\rho^m) > 0$ and $\rho \in (0,1)$ we end at $\lim_{m,m_1\to\infty} G_f(K^m(k_0), K^{m+m_1}(k_0), t) = 1$ which indicates $K^m(k_0)$ is a G_f −Cauchy sequence. With provided that $(G_f, \Xi, \star)$ is G_f −complete, there exists $l_\alpha \in \Xi$ so that $K^m(k_0)$ is G_f −convergent to l_α. Following equation (3.1), $U^m(k_0)$ is G_f −convergent to l_α.
From hypothesis (iii) and K is continuous, we get U is continuous. Thus, $U(K^m(k_0)) \to U(l_\alpha)$. Since U and K is commutative, for every m, we have $U(K^m(k_0)) = K(U^m(k_0))$ which intern implies $U(l_\alpha) = K(l_\alpha)$. Moreover, by commutative property, $K(K(l_\alpha) = K(U(l_\alpha)) = U(U(l_\alpha))$,

$$\begin{aligned} G_f(U(l_\alpha), U(U(l_\alpha)), t) &\geq G_f(K(l_\alpha), K(U(l_\alpha)), t/\rho) \\ &\geq G_f(U(l_\alpha), U(U(l_\alpha)), t/\rho) \\ &\geq G_f(K(l_\alpha), K(U(l_\alpha)), t/\rho^2) \\ &\geq \cdots \geq G_f(U(l_\alpha), U(U(l_\alpha)), t/\rho^m). \end{aligned}$$

From (G_fMS2) and (G_fMS5) we have,
$G_f(U(l_\alpha), U(U(l_\alpha)), t) = 1$, that is, $U(l_\alpha) = U(U(l_\alpha))$.
By Proposition 2.11, we conclude, $U(l_\alpha) = U(U(l_\alpha)) = K(U(l_\alpha))$ that is, $U(l_\alpha)$ is a common

fixed point of K and U.
Next, we prove the uniqueness, Assume $q_1 = K(q_1) = U(q_1)$ and $q_2 = K(q_2) = U(q_2)$

$$\begin{aligned} G_f(q_1, q_2, t) &= G_f(U(q_1), U(q_2), t) \\ &\geq G_f(K(q_1), K(q_2), t/\rho) \\ &= G_f(q_1, q_2, t/\rho) \\ &= G_f(U(q_1), U(q_2), t/\rho) \\ &\geq G_f(K(q_1), K(q_2), t/\rho^2) \\ &\geq \cdots \geq G_f(q_1, q_2, t/\rho^m) \to 1 \text{ as } m \to \infty \end{aligned}$$

$G_f(q_1, q_2, t) = 1$, that is, $q_1 = q_2$. □

Example 3.5 *Let $\varXi = [0,1]$ be a complete $GFMS$ with $(\varXi, G_{ms}) = |\nu - \mu|$ and $G_f(\nu, \mu, t) = t/t + |\nu - \mu|$ with $t > 0$ and $a_1 \star a_2 = a_1 a_2$. Define $U(\nu) = \nu/8$ and $K(\nu) = \nu/4$ on $\varXi$. Undoubtedly, $U(\varXi) \subseteq K(\varXi)$. Following this, we demonstrate U and K commute. $U(K(\nu)) = U(\nu/4) = \nu/4.8 = \nu/32$ and $K(U(\nu)) = K(\nu/8) = \nu/8.4 = \nu/32$ which implies $U(K(\nu)) = K(U(\nu))$.*
Now, we verify, $G_f(U(\nu), U(\mu), \rho t) \geq G_f(K(\nu), K(\mu), t)$. Taking $\rho = 1/2$

$$\begin{aligned} G_f(U(\nu), U(\mu), t/2) &= \frac{t/2}{t/2 + |U(\nu) - U(\mu)|} \\ &= \frac{t/2}{t/2 + |\nu/8 - \mu/8|} \\ &= \frac{t}{t + |\nu - \mu|/4} \\ &= \frac{t}{t + |\nu/4 - \nu/4|} \end{aligned}$$

$$G_f(U(\nu), U(\mu), t/2) \geq G_f(K(\nu), K(\mu), t)$$

Furthermore, *for $i, j \in \mathbb{N}$ and $t > 0$*

$$\delta(G_f, K, k_0, t) = \ sup\ \{G_f(K^i(k_0), K^j(k_0), t)\}$$

$$= sup\left(\frac{t}{t + \left|\frac{k_0}{2^i} - \frac{k_0}{2^j}\right|}\right)$$

$$= sup\left(\frac{t}{t + k_0\left|\frac{2^i - 2^j}{2^{i+j}}\right|}\right) = 1 > 0 \text{ for } k_0 = 0.$$

Thus, all the conditions of Theorem 3.4 holds, indicating that U and K share a unique common fixed point at 0.

Corollary 3.6 *Let $(F_{ms}, \varXi, \star)$ be a complete FMS. Take $U: \varXi \to \varXi$ a continuous transformation, $K: \varXi \to \varXi$ such that*

$$F_{ms}(U(q_1), U(q_2), \rho t) \geq F_{ms}(K(q_1), K(q_2), t) \text{ for } \rho \in (0,1)$$

Also, U commutes with K and $U(\varXi) \subseteq K(\varXi)$. If there exists $k_0 \in \varXi$ with

$$sup\{F_{ms}(U^{n+i}(k_0), U^{n+j}(k_0), t) : i, j \in \mathbb{N} \cup 0 \text{ and } t > 0\} > 0$$

then U and K holds a unique common fixed point.

Proof. Proof follows from Theorem 3.4 and proposition 2.10. □

Corollary 3.7 *Let $(F_b, \varXi, \star)$ be a complete $F_b MS$. Take $U: \varXi \to \varXi$ a continuous transformation, $K: \varXi \to \varXi$ such that*

$$F_b(U(q_1), U(q_2), \rho t) \geq F_b(K(q_1), K(q_2), t) \quad \text{for} \quad \rho \in (0,1).$$

Also, U commutes with K and $U(\varXi) \subseteq K(\varXi)$. If there exists $k_0 \in \varXi$ with

$$sup\{F_b(U^{n+i}(k_0), U^{n+j}(k_0), t) : i, j \in \mathbb{N} \cup 0 \text{ and } t > 0\} > 0$$

then U and K holds a unique common fixed point.

Proof. Proof follows from Theorem 3.4 and proposition 2.10. □

Theorem 3.8 *Let $(G_f, \varXi, \star)$ be a $GFMS$. A set of self functions U and K are weakly compatible on $\varXi$ satisfying the following: (i) $U(\varXi) \subseteq K(\varXi)$; (ii) $G_f(U(k_1), U(k_2), \rho t) \geq G_f(K(k_1), K(k_2), t)$ for $\rho \in (0,1)$; (iii) there exists $k_0 \in \varXi$ with $\delta(G_f, K, k_0, t) > 0$. If either U or K is complete then U and K holds a unique common fixed point.*

Proof. Following Theorem 3.4, we arrive, $\lim_{m\to\infty} G_f(K^m(k_0), K^{m+m_1}(k_0), t) = 1$ indicates $K^m(k_0)$ is a G_f −Cauchy sequence. Consider $K(\varXi)$ is complete then the subsequence of $K^m(k_0)$ has a limit in $K(\varXi)$, say l_α. Since $K^m(k_0)$ is a cauchy sequence with convergent subsequence which implies $K^m(k_0)$ is also convergent.
Consider

$$G_f(U^m k_0, U(h), \rho t) \geq G_f(K^m(k_0), K(h), t).$$

Taking limit $n \to \infty$

$$\begin{aligned} G_f(l_\alpha, U(h), \rho t) &\geq G_f(l_\alpha, K(h), t) \\ &\geq G_f(l_\alpha, l_\alpha, t) = 1. \end{aligned}$$

Then $U(h) = l_\alpha = K(h)$, that is, U and K have a coincidence point. Since U and K are weakly-compatible then $U(K(h)) = K(U(h))$ which implies $Ul_\alpha = Kl_\alpha$.
Now we prove l_α is a common fixed point of U and K.

$$G_f(U^m k_0, U(l_\alpha), \rho t) \geq G_f(K^m(k_0), K(l_\alpha), t).$$

Taking limit $n \to \infty$

$$\begin{aligned} G_f(l_\alpha, U(l_\alpha), \rho t) &\geq G_f(l_\alpha, K(l_\alpha), t) \\ &\geq G_f(l_\alpha, U(l_\alpha), t). \end{aligned}$$

Then $U(l_\alpha) = K(l_\alpha) = l_\alpha$, that is, U and K have a common fixed point. □

Theorem 3.9 *Theorem 3.8 continues to be valid if weakly compatible property in Theorem 3.8 is substituted by*
(i) *weakly commuting*
(ii) $\boldsymbol{R}$ *−weakly commuting.*

Proof. (i) As all the hypothesis of Theorem 3.8 hold then U and K have a coincidence point. Let q_1 be the coincidence point of U and K then $U(q_1) = K(q_1)$.
Now, using weakly commuting

$$G_f(UK(q_1), KU(q_1), t) \geq G_f(U(q_1), K(q_1), t) = G_f(U(q_1), U(q_1), t) = 1.$$

Then $U\big(K(q_1)\big) = K\big(U(q_1)\big)$ which implies U and K are weakly compatible. Now, following theorem 3.8, we arrive U and K have a common fixed point.
(ii) Now, using R −weakly commuting

$$G_f(UK(q_1), KU(q_1), t) \geq G_f(U(q_1), K(q_1), t/R) = G_f(U(q_1), U(q_1), t/R) = 1.$$

Then $U\big(K(q_1)\big) = K\big(U(q_1)\big)$ which implies U and K are weakly compatible. Similarly, following theorem 3.8, we arrive U and K have a common fixed point. □

3.2 Common fixed point theorem for admissible contraction mapping in GFMS

Motivated by M. Dinarvand[16] we introduce the following definition of triangular−(β, G_f) − admissible in $GFMS$.

Definition 3.10 *Let* $(G_f, \Xi, \star)$ *be a* $GFMS$ *and* $U, K: \Xi \to \Xi$ *and* $\beta: \Xi \times \Xi \times (0, \infty) \to (0, \infty)$ U *is triangular−*(β, G_f) *−admissible with respect to* K *whenever*
(1β) $\beta(Kq_1, Kq_2, t) \leq 1$ *signifies* $\beta(Uq_1, Uq_2, t) \leq 1$ *and* $G_f(Kq_1, Kq_2, t) > 0$
(2β) $\beta(q_1, q_3, t) \leq 1$ *and* $\beta(q_3, q_2, t) \leq 1$ *signifies* $\beta(q_1, q_2, t) \leq 1$
for every $q_1, q_2, q_3 \in \Xi$ *and* $t > 0$.

Definition 3.11 *Let* $(G_f, \Xi, \star)$ *be a* $GFMS$ *and* $U, K: \Xi \to \Xi$ *and* $\beta: \Xi \times \Xi \times (0, \infty) \to (0, \infty)$. *A set of functions U and K is called an admissible* C_f *−contraction whenever*
$(1C_f)$ U *is triangular−*(β, G_f) *−admissible with respect to* K *and*
$(2C_f)$ *For every* $q_1, q_2 \in \Xi$ *and for some* $\rho \in (0,1)$ *with* $\beta(Kq_1, Kq_2, t) \leq 1$ *signifies*

$$\beta(Kq_1, Kq_2, t)\, G_f(Uq_1, Uq_2, \rho t) \geq \min\{G_f(Kq_1, Kq_2, t), G_f(Kq_2, Uq_2, t), G_f(Kq_1, Uq_1, t), G_f(Kq_1, Uq_2, t), G_f(Kq_2, Uq_1, t)\}.$$

Lemma 3.12 *Let* $(G_f, \Xi, \star)$ *be a GFMS,* $\beta: \Xi \times \Xi \times (0, \infty) \to (0, \infty)$ *and set of self functions* U *and* K *is a admissible* C_f *−contraction then for any* $\nu, \mu \in C_p(U, K)$ *satisfies the following*
(i) $\beta(Kq_1, Kq_1, t) \leq 1$ *then* $G_f(Kq_1, Kq_1, t) = 1$
(ii) $\beta(Kq_1, Kq_2, t) \leq 1$ *then* $G_f(Kq_1, Kq_2, t) = 1$.

Proof. (i) Let $q_1 \in C_p(U, K)$ then $Uq_1 = Kq_1$. Using U is triangular admissible, for $\beta(Kq_1, Kq_1, t) \leq 1$, we have $G_f(Kq_1, Kq_1, t) > 0$.

$$\begin{aligned} G_f(Kq_1, Kq_1, t) &= G_f(Uq_1, Uq_1, t) \\ &\geq \beta(Kq_1, Kq_1, t) G_f(Uq_1, Uq_1, t) \\ &\geq \min\{G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right), \\ &\quad G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right)\} \\ &\geq \min\{G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right)\} \\ &\geq G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right) \\ &\geq \cdots \geq G_f(Kq_1, Kq_1, t/\rho^n) \to 1 \text{ as } n \to \infty \text{ and } \end{aligned}$$

$\rho \in (0,1)$.
(ii) Let $q_1, q_2 \in C_p(U, K)$ then $Uq_1 = Kq_1$ and $Uq_2 = Kq_2$. Using U is triangular admissible, for $\beta(Kq_1, Kq_2, t) \leq 1$, we have $G_f(Kq_1, Kq_2, t) > 0$.

$$\begin{aligned} G_f(Kq_1, Kq_2, t) &= G_f(Uq_1, Uq_2, t) \\ &\geq \beta(Kq_1, Kq_2, t) G_f(Uq_1, Uq_2, t) \\ &\geq \min\{G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Uq_2, \tfrac{t}{\rho}\right), \\ &\quad G_f\left(Kq_1, Uq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Uq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Uq_1, \tfrac{t}{\rho}\right)\} \\ &\geq \min\{G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Kq_2, \tfrac{t}{\rho}\right), \\ &\quad G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Kq_1, \tfrac{t}{\rho}\right)\} \\ &\geq \min\{G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), \\ &\quad G_f\left(Kq_2, Kq_2, \tfrac{t}{\rho}\right)\} \end{aligned}$$

Case(i) If

$$\min\{G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Kq_2, \tfrac{t}{\rho}\right)\} = G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right)$$

then

$$\begin{aligned} G_f(Kq_1, Kq_2, t) &\geq G_f(Kq_1, Kq_2, t/\rho) \\ &\geq G_f(Kq_1, Kq_2, t/\rho^2) \\ &\geq \cdots \geq G_f(Kq_1, Kq_2, t/\rho^n) = 1 \text{as } n \to \end{aligned}$$

∞ and $\rho \in (0,1)$.

Thus, $G_f(Kq_1, Kq_2, t) = 1$.
Case(ii) If

$$\min\{G_f\left(Kq_1, Kq_2, \tfrac{t}{\rho}\right), G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right), G_f\left(Kq_2, Kq_2, \tfrac{t}{\rho}\right)\} = G_f\left(Kq_1, Kq_1, \tfrac{t}{\rho}\right)$$

then

$$G_f(Kq_1, Kq_2, t) \quad \geq G_f\left(Kq_1, Kq_1, \frac{t}{\rho}\right) = 1.$$

Case(iii) If

$min\{G_f\left(Kq_1, Kq_2, \frac{t}{\rho}\right), G_f\left(Kq_1, Kq_1, \frac{t}{\rho}\right),$
$G_f\left(Kq_2, Kq_2, \frac{t}{\rho}\right)\} = G_f\left(Kq_2, Kq_2, \frac{t}{\rho}\right)$
then
$G_f(Kq_1, Kq_2, t) \geq G_f\left(Kq_2, Kq_2, \frac{t}{\rho}\right) = 1.$ □

Theorem 3.13 Let $(G_f, \Xi, \star)$ be a $GFMS$, $\beta: \Xi \times \Xi \times (0,\infty) \to (0,\infty)$ and set of self functions then for any $\nu, \mu \in C_p(U,K)$ satisfies the following: (i) $U(\Xi) \subseteq K(\Xi)$; (ii) U and K is a admissible C_f −contraction; (iii) there exists $k_0 \in \Xi$ with $\beta(Kk_0, Uk_0, t) \leq 1$ and $\delta(G_f, U, k_0, t) > 0$; (iv) U and K are continuous; (v) U and K are commuting. Then U and K have a coincidence point. Furthermore, if for any $\nu, \mu \in C_p(U,K)$ we obtain $\beta(K\nu, K\mu, t) \leq 1$ then U and K have a common fixed point.

Proof. Take $k_0 \in \Xi$ such that $\beta(Kk_0, Uk_0, t) \leq 1$ and $\delta(G_f, U, k_0, t) > 0$.

Since $U(\Xi) \subset K(\Xi)$, for $m \in \mathbb{N}$, we construct $K^m(k_0) = U^{m-1}(k_0)$.

With $\beta(Kk_0, Uk_0, t) = \beta(Kk_0, K^2k_0, t) \leq 1$ and using U is triangular−(β, G_f) −admissible with respect to K implies

$$\beta(Uk_0, U^2k_0, t) = \beta(K^2k_0, K^3k_0, t) \leq 1.$$

Proceeding this process, we arrive,
$\beta(K^m k_0, U^{m+1}k_0, t) \leq 1$ for $m \in \mathbb{N}$.

Since U is triangular−(β, G_f) −admissible with respect to K, we arrive
$\beta(K^{q_1}k_0, U^{q_2}k_0, t) \leq 1$ for $q_1, q_2 \in \mathbb{N}$.

For $m \in \mathbb{N}$ and $i, j \in \mathbb{N}$, we obtain

$$\begin{aligned}G_f(K^{m+i+1}(k_0), K^{m+j+1}(k_0), t) &= G_f(U^{m+i}(k_0), U^{m+j}(k_0), t)\\ &\geq \beta(K^{m+i}(k_0), K^{m+j}(k_0), t) G_f(U^{m+i}(k_0), U^{m+j}(k_0), t)\\ &\geq min\{G_f(K^{m+i}(k_0), K^{m+j}(k_0), t/\rho), G_f(K^{m+i}(k_0), U^{m+i}(k_0), t/\rho),\\ &G_f(K^{m+j}(k_0), U^{m+j}(k_0), t/\rho), G_f(K^{m+i}(k_0), U^{m+j}(k_0), t/\rho), G_f(K^{m+j}(k_0), U^{m+i}(k_0), t/\rho)\}.\end{aligned}$$

Taking supremum,

$$\begin{aligned}\delta(G_f, U, U^m k_0, t) &\geq min\left\{sup\ G_f(K^{m+i}(k_0), K^{m+j}(k_0), \tfrac{t}{\rho}), sup\ G_f\left(K^{m+i}(k_0), U^{m+i}(k_0), \tfrac{t}{\rho}\right),\right.\\ &sup\ G_f\left(K^{m+j}(k_0), U^{m+j}(k_0), \tfrac{t}{\rho}\right), sup\ G_f\left(K^{m+i}(k_0), U^{m+j}(k_0), \tfrac{t}{\rho}\right), sup\ G_f\left(K^{m+j}(k_0), U^{m+i}(k_0), \tfrac{t}{\rho}\right)\}\\ &\geq min\left\{sup\ G_f(U^{m-1+i}(k_0), U^{m-1+j}(k_0), \tfrac{t}{\rho}), sup\ G_f\left(U^{m-1+i}(k_0), U^{m+i}(k_0), \tfrac{t}{\rho}\right),\right.\\ &sup\ G_f\left(U^{m-1+j}(k_0), U^{m+j}(k_0), \tfrac{t}{\rho}\right), sup\ G_f\left(U^{m-1+i}(k_0), U^{m+j}(k_0), \tfrac{t}{\rho}\right), sup\ G_f\left(U^{m-1+j}(k_0), U^{m+i}(k_0), \tfrac{t}{\rho}\right)\}\\ &\geq min\{\delta\left(G_f, U, U^{m-1}k_0, \tfrac{t}{\rho}\right), \delta\left(G_f, U, U^m k_0, \tfrac{t}{\rho}\right)\}\end{aligned}$$

$$\delta(G_f, U, U^m k_0, t) \geq \delta\left(G_f, U, U^{m-1}k_0, \frac{t}{\rho}\right).$$

In the same way, it can be inferred that,

$$\delta(G_f, U, U^m k_0, t) \geq \delta(G_f, U, k_0, t/\rho^m).$$

By employing the inequality mentioned above, for every $m, m_1 \in \mathbb{N}$ and $m_1 > m$, we obtain,

$$\begin{aligned}G_f(K^m(k_0), K^{m_1}(k_0), t) &= G_f(U^{m-1}(k_0), U^{m_1-1}(k_0), t)\\ &\geq \delta(G_f, U, U^{m-1}(k_0), t)\\ &\geq \delta(G_f, U, k_0, t/\rho^{m-1}).\end{aligned}$$

With $\delta(G_f, U, (k_0), t/\rho^{m-1}) > 0$ and $\rho \in (0,1)$ we end at $\lim_{m,m_1\to\infty} G_f(K^m(k_0), K^{m_1}(k_0), t) = 1$ which indicates $K^m(k_0)$ is a G_f −Cauchy sequence. With provided that $(G_f, \Xi, \star)$ is G_f −complete, there exists $l_\alpha \in \Xi$ so that $K^m(k_0)$ is G_f −convergent to l_α, that is,

$\lim_{m\to\infty} G_f(K^n(k_0), l_\alpha, t) = \lim_{m\to\infty} G_f(U^n(k_0), l_\alpha, t) = 1.$
Thus, $K^n k_0$, $U^n k_0 \in S(G_f, \Xi, l_\alpha)$.
From given hypothesis, U and K is continuous, we obtain, $UK^n(k_0) \in S(G_f, \Xi, Ul_\alpha)$ and $KU^n(k_0) \in S(G_f, \Xi, Kl_\alpha)$. Since U and K are commuting, we get $KU^n(k_0) \in S(G_f, \Xi, Ul_\alpha)$.
From proposition 2.10, $Ul_\alpha = Kl_\alpha$. Then l_α is a coincidence point of U and K.
Let $Ul_\alpha = Kl_\alpha = l_1$, since U and K are commuting, $Kl_1 = K(Ul_\alpha) = U(Kl_\alpha) = Ul_1$, that is, $Kl_1 = Ul_1$ then l_1 is a coincidence point.
From hypothesis $\beta(Kl_\alpha, Kl_1, t) \leq 1$ and lemma (3.12), $Ul_1 = Kl_1 = Kl_\alpha = l_1$, that is, $Ul_1 = Kl_1 = l_1$ which implies l_1 is a common fixed point of U and K. □

Definition 3.14 *Consider $X = [0,1]$ and G_f defined as*

$$G_f(\nu,\mu,t) = \begin{cases} e^{-\nu+\mu/t}, & \text{if } \nu,\mu \neq 0; \\ e^{-\mu/2t}, & \text{if } \nu = 0; \\ e^{-\nu/2t}, & \text{if } \mu = 0. \end{cases}$$

Then $(G_f, \Xi, \star)$ is a generalized fuzzy metric space with t −norm $a \star b = ab$. Consider

$$\beta(\nu,\mu,t) = \begin{cases} 1 & \nu,\mu \in [0,1/n] \text{ with } \nu \neq 0 \text{ or } \mu = 0; \\ 2 & \text{otherwise} \end{cases}$$

Define the mappings U and K as $U(\nu) = \nu^4$ and $K(\nu) = \nu^2$.
Here $U(\Xi) \subset K(\Xi)$. Choose $k_0 = 1/4 \in \Xi$ then $\beta(K(k_0), U(k_0), t) = \beta(1/16, 1/256, t) = 1$ and $\delta(G_f, U, 1/4, t) > 0$.
Also, it is easy to see U and K are continuous and commuting. Now we show that U and K is an admissible C_f contraction.
Consider $\nu_1, \nu_2, \nu_3 \in \Xi$. If $\beta(K\nu_1, K\nu_2, t) \leq 1$ then ${\nu_1}^2, {\nu_2}^2 \in [0, 1/2]$ and $K\nu_1 = {\nu_1}^2 \neq 0$ or $K\nu_2 = {\nu_2}^2 = 0$. Consequently, ${\nu_1}^4, {\nu_2}^4 \in [0, 1/4]$ and $K\nu_1 = {\nu_1}^4 \neq 0$ or $K\nu_2 = {\nu_2}^4 = 0$ leads to $\beta(U\nu_1, U\nu_2, t) \leq 1$ and $G_f(K\nu_1, K\nu_2, t) > 0$.
If $\beta(\nu_1, \nu_3, t) \leq 1$ implies $\nu_1 \neq 0$ or $\nu_3 = 0$ and $\beta(\nu_3, \nu_2, t) \leq 1$ implies $\nu_3 \neq 0$ or $\nu_2 = 0$ then we have $\nu_1 \neq 0$ or $\nu_2 = 0$ and $\beta(\nu_1, \nu_2, t) \leq 1$. Therefore, we have $(1C_f)$, U is triangular−(β, G_f) −admissible with respect to K.
Next to prove $(2C_f)$, Consider $\nu_1, \nu_2 \in \Xi$ and $\rho = 1/2$ with $\beta(K\nu_1, K\nu_2, t) \leq 1$ then ${\nu_1}^2, {\nu_2}^2 \in [0, 1/2]$ and $K\nu_1 = {\nu_1}^2 \neq 0$ or $K\nu_2 = {\nu_2}^2 = 0$. Here we arrive two cases:
Case(i) If $K\nu_2 = 0$

$$\beta(K\nu_1, K\nu_2, t) G_f(U\nu_1, U\nu_2, \rho t) = \beta(K\nu_1, 0, t) G_f\left({\nu_1}^4, 0, \frac{t}{2}\right)$$
$$= e^{-\frac{{\nu_1}^4}{2(t/2)}} = e^{-\frac{{\nu_1}^4}{t}}$$
$$\geq e^{-\frac{{\nu_1}^4 + {\nu_1}^2}{t}} = G_f(K\nu_1, U\nu_1, t)$$
$$\geq min\{G_f(K\nu_1, K\nu_2, t), G_f(K\nu_2, U\nu_2, t), G_f(K\nu_1, U\nu_1, t), G_f(K\nu_1, U\nu_2, t), G_f(K\nu_2, U\nu_1, t)\}$$

Case(ii) If $K\nu_2 \neq 0$

$$\beta(K\nu_1, K\nu_2, t) G_f(U\nu_1, U\nu_2, \rho t) = G_f\left({\nu_1}^4, {\nu_2}^4, \frac{t}{2}\right)$$
$$= e^{-\frac{{\nu_1}^4 + {\nu_2}^4}{t/2}} = e^{-\frac{2({\nu_1}^4 + {\nu_2}^4)}{t}}$$
$$\geq e^{-\frac{{\nu_1}^2 + {\nu_2}^2}{t}} = G_f(K\nu_1, K\nu_2, t)$$
$$\geq min\{G_f(K\nu_1, K\nu_2, t), G_f(K\nu_2, U\nu_2, t), G_f(K\nu_1, U\nu_1, t), G_f(K\nu_1, U\nu_2, t), G_f(K\nu_2, U\nu_1, t)\}.$$

Hence, U and K satisfies an admissible C_f contraction with $\rho = 1/2$. Thus, all conditions of Theorem 3.13 satisfied. Hence, U and K have a coincidence point, that is, 0. Also, 0 is the common fixed point for U and K in Ξ.

4. Common fixed point theorem in new generalized fuzzy metric space endowed with graph

Motivated by Charoensawan and Atiponrat[14] new generalized metric space endowed with graph, we introduce the concept of new generalized fuzzy metric space endowed with graph.

Definition 4.1 *Consider a non-void set Ξ, $\star$ is a continuous t −norm and a mapping $N_f: \Xi \times \Xi \times (0,\infty) \to [0,1]$. For all $\nu, \mu \in \Xi$ and $t > 0$, with the following conditions:*
$(N_f MS1)$ $N_f(\nu,\mu,t) > 0$;
$(N_f MS2)$ $N_f(\nu,\mu,t) = 1 \Rightarrow \nu = \mu$;
$(N_f MS3)$ $N_f(\nu,\mu,t) = N_f(\mu,\nu,t)$;
$(N_f MS4)$ *there exists $a \geq 1$ such that if $\nu_n \in S(N_f, \Xi, \nu)$ and $\mu_n \in S(N_f, \Xi, \mu)$ then*

$$N_f(\nu,\mu,t) \geq \limsup_{n\to\infty} N_f(\nu_n, \mu_n, t/a);$$

$(N_f MS5)$ $N_f(\nu,\mu,\cdot): (0,\infty) \to [0,1]$ is continuous and $\lim_{n\to\infty} N_f(\nu,\mu,t) = 1$.
Then N_f is a new generalized fuzzy metric and $(N_f, \Xi, \star)$ is termed as new generalized fuzzy metric space(briefly $NGFMS$).

The definitions of Cauchy, convergent, complete and continuous, compatible of $NGFMS$ are analogous from $GFMS$.

Consider $(N_f, \Xi, \star)$ a new generalized fuzzy metric space. Let Υ be a directed graph with set of vertices $K(\Upsilon) = \Xi$ and set of all edges $E(\Upsilon)$ with $\Delta \subseteq E(\Upsilon)$ and no parallel edges where Δ is the diagonal of $\Xi \times \Xi$. From now we consider $(N_f, \Xi, \star)$ a generalized metric space with graph Υ.

Definition 4.2 *Let $(N_f, \Xi, \star)$ be a $NGFMS$ endowed with directed graph Υ and $U: (N_f, \Xi, \star) \to (N_f, \Xi, \star)$ is labelled as Υ −continuous if for some $\nu \in \Xi$ with $\nu_m \in S(N_f, \Xi, \nu)$ so that $(\nu_m, \nu_{m+1}) \in E(\Upsilon)$ for every $m \in \mathbb{N}$, we arrive $U\nu_m \in S(N_f, \Xi, U\nu)$.*

Definition 4.3 *Let $(N_f, \Xi, \star)$ be a $NGFMS$ endowed with directed graph Υ and $U, K: \Xi \to \Xi$, we call U is K −edge preserving with respect to Υ if $(Kq_1, Kq_2) \in E(\Upsilon)$ signifies $(Uq_1, Uq_2) \in E(\Upsilon)$.*

Definition 4.4 *Let $(N_f, \Xi, \star)$ be a $NGFMS$ endowed with directed graph then $E(\Upsilon)$ fulfills the transitivity property if for every $q_1, q_2, q_3 \in \Xi$, we obtain $(q_1, q_3), (q_3, q_2) \in E(\Upsilon)$ signifies $(q_1, q_2) \in E(\Upsilon)$.*

Definition 4.5 *Let $(N_f, \Xi, \star)$ be a $NGFMS$ endowed with directed graph Υ and $U, K: \Xi \to \Xi$, the pair (U, K) is called N_f −contraction whenever*

$(1N_f)$ U is K −edge preserving with respect to Υ.
$(2N_f)$ For every $q_1, q_2 \in \Xi$ and for some $\rho \in (0,1)$ with $(Kq_1, Kq_2) \in E(\Upsilon)$ signifies

$$N_f(Uq_1, Uq_2, \rho t) \geq min\{N_f(Kq_1, Kq_2, t), N_f(Kq_2, Uq_2, t), N_f(Kq_1, Uq_1, t), N_f(Kq_1, Uq_2, t), N_f(Kq_2, Uq_1, t)\}.$$

Lemma 4.6 Let $(N_f, \Xi, \star)$ be a $NGFMS$ endowed with directed graph Υ and $U, K: \Xi \to \Xi$. The pair (U, K) is a N_f −contraction then for any $q_2, q_1 \in C_p(U, K)$ satisfies the following
(i) $N_f(Kq_2, Kq_2, t) > 0$ then $N_f(Kq_2, Kq_2, t) = 1$.
(ii) $N_f(Kq_2, Kq_1, t) > 0$ and $(Kq_2, Kq_1) \in E(\Upsilon)$ then $Kq_2 = Kq_1$.

Proof. (i) Let $q_2 \in C_p(U, K)$ then $Uq_2 = Kq_2$

$$\begin{aligned} N_f(Kq_2, Kq_2, t) &= N_f(Uq_2, Uq_2, t) \\ &\geq min\{N_f(Kq_2, Kq_2, t/\rho), N_f(Kq_2, Uq_2, t/\rho), N_f(Kq_2, Uq_2, t/\rho), N_f(Kq_2, Uq_2, t/\rho), N_f(Kq_2, Uq_2, t/\rho)\} \\ &\geq N_f(Kq_2, Uq_2, t/\rho) \geq N_f(Kq_2, Uq_2, t/\rho^2) \\ &\geq \cdots \geq N_f(Kq_2, Uq_2, t/\rho^n) = 1 \text{ as } n \to \infty \text{ and } \rho \in (0,1). \end{aligned}$$

(ii) Let $q_2, q_1 \in C_p(U, K)$ then $Uq_2 = Kq_2$ and $Uq_2 = Kq_2$.

$$\begin{aligned} N_f(Kq_2, Kq_1, t) &= N_f(Uq_2, Uq_1, t) \\ &\geq min\{N_f(Kq_2, Kq_1, t/\rho), N_f(Kq_2, Uq_2, t/\rho), N_f(Kq_1, Uq_1, t/\rho), N_f(Kq_2, Uq_1, t/\rho), N_f(Kq_1, Uq_2, t/\rho)\} \\ &\geq min\{N_f(Kq_2, Kq_1, t/\rho), N_f(Kq_2, Kq_2, t/\rho), N_f(Kq_1, Kq_1, t/\rho)\} \end{aligned}$$

we consider the following cases:
case(i) If

$$\min\{N_f(Kq_2, Kq_1, t/\rho), N_f(Kq_2, Kq_2, t/\rho), N_f(Kq_1, Kq_1, t/\rho)\} = N_f(Kq_2, Kq_1, t/\rho)$$

then

$$\begin{aligned} N_f(Kq_2, Kq_1, t) &\geq N_f(Kq_2, Kq_1, t/\rho) \\ &\geq N_f(Kq_2, Kq_1, t/\rho^2) \\ &\geq \cdots \geq N_f(Kq_2, Kq_1, t/\rho^n) = 1 \text{ as } n \to \infty \text{ and } \rho \in (0,1). \end{aligned}$$

Thus, $K(q_2) = K(q_1)$.

case(ii) If

$$\min\{N_f(Kq_2, Kq_1, t/\rho), N_f(Kq_2, Kq_2, t/\rho), N_f(Kq_1, Kq_1, t/\rho)\} = N_f(Kq_2, Kq_2, t/\rho)$$

then

$$N_f(Kq_2, Kq_1, t) \quad \geq N_f(Kq_2, Kq_2, t/\rho) = 1$$

Thus, $K(q_2) = K(q_1)$.

case(iii) If

$$\min\{N_f(Kq_2, Kq_1, t/\rho), N_f(Kq_2, Kq_2, t/\rho), N_f(Kq_1, Kq_1, t/\rho)\} = N_f(Kq_1, Kq_1, t/\rho)$$

then

$$N_f(Kq_2, Kq_1, t) \quad \geq N_f(Kq_1, Kq_1, t/\rho) = 1$$

Thus, $K(q_2) = K(q_1)$. □

Theorem 4.7 Let $(N_f, \Xi, \star)$ be a complete $NGFMS$ endowed with directed graph Υ and $U, K: \Xi \to \Xi$, the pair (U, K) is a N_f −contraction satisfies the following: (i) $U(\Xi) \subseteq K(\Xi)$; (ii) $E(\Upsilon)$ holds transitivity property; (iii) there exists $\nu_0 \in \Xi$ with $(K\nu_0, U\nu_0,) \in E(\Upsilon)$ and $\delta(N_f, U, \nu_0, t) > 0$; (iv) U is Υ −continuous and K is continuous and (v) U and K are compatible. Then U and K have a coincidence point. Furthermore, if $(K\nu, K\mu) \in E(\Upsilon)$ for any $\nu, \mu \in C_p(U, K)$ and U and K are commuting. Then U and K have a common fixed point.

Proof. Take $k_0 \in \Xi$ such that $(Uk_0, Kk_0) \in E(\Upsilon)$ and $\delta(N_f, U, k_0, t) > 0$.
Since $U(\Xi) \subset K(\Xi)$, for $m \in \mathbb{N}$, we construct $K^m(k_0) = U^{m-1}(k_0)$.
With $(Kk_0, Uk_0) = (Kk_0, K^2k_0) \in E(\Upsilon)$ and using $(1N_f)$ U is K −edge preserving with respect to Υ implies

$$(Uk_0, U^2k_0) = (K^2k_0, K^3k_0) \in E(\Upsilon),$$

proceeding this process, we arrive
$(K^mk_0, K^{m+1}k_0) \in E(\Upsilon)$ for $m \in \mathbb{N}$.

Since $E(\Upsilon)$ fulfills the transitivity property, we arrive, $(K^{k_1}k_0, K^{k_2}k_0) \in E(\Upsilon)$ for all $k_1, k_2 \in \mathbb{N}$ and $k_1 < k_2$.
For $m \in \mathbb{N}$ and $i, j \in \mathbb{N}$, we obtain,

$$\begin{aligned} N_f(K^{m+i+1}(k_0), K^{m+j+1}(k_0), t) &= N_f(U^{m+i}(k_0), U^{m+j}(k_0), t) \\ &\geq \min\left\{N_f\left(K^{m+i}(k_0), K^{m+j}(k_0), \frac{t}{\rho}\right), N_f\left(K^{m+i}(k_0), U^{m+i}(k_0), \frac{t}{\rho}\right), N_f\left(K^{m+j}(k_0), U^{m+j}(k_0), \frac{t}{\rho}\right),\right. \\ &\qquad \left. N_f\left(K^{m+i}(k_0), U^{m+j}(k_0), \frac{t}{\rho}\right), N_f\left(K^{m+j}(k_0), U^{m+i}(k_0), \frac{t}{\rho}\right)\right\} \end{aligned}$$

Taking supremum,

$$\begin{aligned} \delta(N_f, U, U^mk_0, t) \geq \min\Big\{& supN_f\left(K^{m+i}(k_0), K^{m+j}(k_0), \frac{t}{\rho}\right), supN_f\left(K^{m+i}(k_0), U^{m+i}(k_0), \frac{t}{\rho}\right), \\ & supN_f\left(K^{m+j}(k_0), U^{m+j}(k_0), \frac{t}{\rho}\right), supN_f\left(K^{m+i}(k_0), U^{m+j}(k_0), \frac{t}{\rho}\right), supN_f\left(K^{m+j}(k_0), U^{m+i}(k_0), \frac{t}{\rho}\right)\Big\} \end{aligned}$$

$$\delta(N_f, U, U^mk_0, t) \geq \ \delta(N_f, U, U^{m-1}k_0, t)$$

In the same way, it can be inferred that,

$$\delta\left(N_f, U, U^m k_0, t\right) \geq \delta\left(N_f, U, k_0, t/\rho^m\right).$$

By employing the inequality mentioned above, for every $m, m_1 \in \mathbb{N}$ and $m_1 > m$, we obtain,

$$\begin{aligned} N_f(K^m(k_0), K^{m_1}(k_0), t) &= N_f(U^{m-1}(k_0), U^{m_1-1}(k_0), t) \\ &\geq \delta\left(N_f, U, U^{m-1}(k_0), t\right) \\ &\geq \delta\left(N_f, U, k_0, t/\rho^{m-1}\right). \end{aligned}$$

With $\delta\left(N_f, U, (k_0), \frac{t}{\rho^m}\right) > 0$ and $\rho \in (0,1)$ we end at $\lim_{m,m_1\to\infty} N_f(K^m(k_0), K^{m_1}(k_0), t) = 1$ which indicates $K^m(k_0)$ is a N_f −Cauchy sequence. With provided that $\left(N_f, \Xi, \star\right)$ is N_f −complete, there exists $l_\alpha \in \Xi$ so that $K^m(k_0)$ is N_f −convergent to l_α, that is, $\lim_{m\to\infty} N_f(K^n(k_0), l_\alpha, t) = \lim_{m\to\infty} N_f(U^n(k_0), l_\alpha, t) = 1$. Thus, $K^n k_0, U^n k_0 \in S\left(N_f, \Xi, l_\alpha\right)$. From given hypothesis, U is Υ −continuous and K is continuous, we obtain, $UK^n(k_0) \in S\left(N_f, \Xi, Ul_\alpha\right)$ and $KU^n(k_0) \in S\left(N_f, \Xi, Kl_\alpha\right)$.

Utilizing U and K are compatible and $\left(N_f MS4\right)$,

$$\begin{aligned} N_f(Ul_\alpha, Kl_\alpha, t) &\geq a \lim_{n\to\infty} \sup N_f(UK^n(k_0), KU^n(k_0), t) \\ &= 1, \end{aligned}$$

we obtain $Ul_\alpha = Kl_\alpha$. Then l_α is a coincidence point of U and K. Let $Ul_\alpha = Kl_\alpha = l_1$, since U and K are commuting, $Kl_1 = K(Ul_\alpha) = U(Kl_\alpha) = Ul_1$, that is, $Kl_1 = Ul_1$ then l_1 is a coincidence point.

From given hypothesis and lemma 4.6, $(Kl_\alpha, Kl_1) \in E(\Upsilon)$, $Ul_1 = Kl_1 = Kl_\alpha = l_1$, that is, $Ul_1 = Kl_1 = l_1$ which implies l_1 is a common fixed point of U and K. □

Example 4.8 Let $\Xi = [0,1]$ and define N_f be a new generalized fuzzy metric space as

$$N_f(\nu, \mu, t) = \begin{cases} e^{-2(\nu+\mu)/t}, & \text{if } \nu, \mu \neq 0; \\ e^{-\mu/t}, & \text{if } \nu = 0; \\ e^{-\nu/t}, & \text{if } \mu = 0. \end{cases}$$

Then $\left(N_f, \Xi, \star\right)$ is a new generalized fuzzy metric space. Consider $E(\Xi) = (\nu, \mu): \nu \neq 0 \text{or} \mu = 0$. We will define $U, K: \Xi \to \Xi$ as

$$U(\nu) = \frac{\nu}{2\nu + 12}$$

$$K(\nu) = \frac{\nu}{6}$$

Here $U(\Xi) \subset K(\Xi)$. Choose $k_0 = 1$, then

$(Kk_0, Uk_0) = \left(K(1), U(1)\right) = (1/6, 1/14) \in E(\Xi)$.

$N_f(\nu, \mu, t) \geq e^{-2(\nu+\mu)/t} \geq e^{-4/t} \geq 0$ implies

$$\beta\left(G_f, U, 1, t\right) > 0.$$

Also, it is easy to see that, U is Υ −continuous and K is continuous. Consider $\nu_1, \nu_2 \in \Xi$ with $(K\nu_1, K\nu_2) \in E(\Xi)$. For $K\nu_1 \neq 0$ or $K\nu_2 \neq 0$ implies $\nu_1 \neq 0$ or $\nu_2 \neq 0$, we arrive $U\nu_1 \neq 0$ or $U\nu_2 \neq 0$ which implies $(U\nu_1, U\nu_2) \in E(\Xi)$ then U is K −edge preserving with respect to Υ. Next, we prove the transitivity property, Consider $(\nu_1, \nu_3), (\nu_3, \nu_2) \in E(\Upsilon)$ for $\nu_1, \nu_2 . \nu_3 \in \Xi$ then $\nu_3 \neq 0$ implies $\nu_1 \neq 0$ and $\nu_3 = 0$ implies $\nu_1 = 0$, that is, either $\nu_1 \neq 0$ or $\nu_2 = 0$ which implies $(\nu_1, \nu_2) \in E(\Upsilon)$.

Next, to prove that (U, K) satisfies N_f −contraction with $\rho = 1/2$. Let $\nu_1, \nu_2 \in \Xi$ and $(K\nu_1, K\nu_2) \in E(\Upsilon)$.

Case(i) If $K\nu_2 = 0$ then $U\nu_2 = 0$. For $\rho = 1/2$,

$$\begin{aligned} N_f(U\nu_1, U\nu_2, \rho t) &= N_f\left(\frac{\nu_1}{2\nu_1 + 12}, 0, \frac{t}{2}\right) \\ &= exp\left(-\frac{\frac{\nu_1}{2\nu_1+12}}{t/2}\right) \\ &\geq exp\left(-\frac{\nu_1/12}{t/2}\right) = exp\left(-\frac{\nu_1}{6t}\right) \\ &\geq N_f(K\nu_1, 0, t) = N_f(K\nu_1, K\nu_2, t) \\ &\geq min\{N_f(K\nu_1, K\nu_2, t), N_f(K\nu_2, U\nu_2, t), \\ &\quad N_f(K\nu_1, U\nu_1, t), N_f(K\nu_1, U\nu_2, t), N_f(K\nu_2, U\nu_1, t)\}. \end{aligned}$$

Case(ii) If $K\nu_2 \neq 0$ then $U\nu_2 \neq 0$. For $\rho = 1/2$

$$\begin{aligned} N_f(U\nu_1, U\nu_2, \rho t) &= N_f\left(\frac{\nu_1}{2\nu_1 + 12}, \frac{\nu_2}{2\nu_2 + 12}, \frac{t}{2}\right) \\ &= exp\left(-\frac{2\left(\frac{\nu_1}{2\nu_1+12} + \frac{\nu_2}{2\nu_2+12}\right)}{t/2}\right) \\ &= exp\left(-\frac{\frac{2(\nu_1+\nu_2)}{\nu_1+6}}{t}\right) \\ &\geq exp\left(-\frac{\frac{2(\nu_1+\nu_2)}{6}}{t}\right) \\ &= N_f(K\nu_1, K\nu_2, t) \\ &\geq min\{N_f(K\nu_1, K\nu_2, t), N_f(K\nu_2, U\nu_2, t), \\ &\quad N_f(K\nu_1, U\nu_1, t), N_f(K\nu_1, U\nu_2, t), N_f(K\nu_2, U\nu_1, t)\}. \end{aligned}$$

Consider a sequence $q_m \in \Xi$ satisfies

$$\lim_{m\to\infty} Uq_m = \lim_{m\to\infty} Kq_m = q$$

for some $q \in \Xi$.

$$\lim_{m\to\infty} \frac{q_m}{6} = \lim_{m\to\infty} \frac{q_m}{2q_m+12} = q.$$

Hence, we arrive $q = 0$ and

$$\begin{aligned} &\lim_{m\to\infty} N_f(KU(q_m), UK(q_m), t) \\ &= \lim_{m\to\infty}\left(\frac{q_m}{12q_m + 72}, \frac{q_m}{2q_m + 72}, t\right) \\ &= 1 \end{aligned}$$

which implies N_f −compatible. Then there is a coincidence point for (U, K), that is, 0. Then 0 is a common fixed point for U and K.

5. Conclusions

In this work, we have explored the common fixed point theorems within the framework of generalized fuzzy metric space. Furthermore, we introduced a new generalized fuzzy metric space endowed with a graph which effectively incorporates the connectivity of graph into the fuzzy metric framework. Within this extended setting, we established the common fixed point theorem and examples are included to support our theorems. Our results contribute to the ongoing development of fixed point theory in fuzzy and graph contexts, with potential implications for applied mathematics and computational models involving uncertainty.

6. References

[1] Stefan Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales." Fundamenta mathematicae 3(1), 133-181 (1922). https://eudml.org/doc/213289

[2] I. Kramosil and J. Michálek, ‘‘Fuzzy metrics and statistical metric spaces’’, Kybernetica, vol. 11, no. 5, 326–334(1975). https://eudml.org/doc/28711

[3] L. A. Zadeh, "Fuzzy sets" Information and control vol. 8, no. 3, 338-353 (1965). https://doi.org/10.1016/S0019-9958(65)90241-X

[4] A. George and P. Veeramani, ‘‘On some results in fuzzy metric spaces’’, Fuzzy Sets Systems, vol. 64, no. 3, 395–399(1994). https://doi.org/10.1016/0165-0114(94)90162-7

[5] N. Hussain, P. Salimi and V. Parvaneh, "Fixed point results for various contractions in parametric and fuzzy b-metric spaces", J. Nonlinear Sci. Appl., 8, no. 5, 719-739(2015). http://dx.doi.org/10.22436/jnsa.008.05.24

[6] S. Nădăban, "Fuzzy b-metric spaces", International Journal of Computers Communications and Control, 11, no. 2, 273-281(2016). http://dx.doi.org/10.15837/ijccc.2016.2.2443

[7] G. Jungck, "Commuting mappings and fixed points." The American Mathematical Monthly 83, no. 4, 261-263(1976).https://www.tandfonline.com/doi/pdf/10.1080/00029890.1976.11994093

[8] M. Jleli and B. Samet, ‘‘A generalized metric space and related fixed point theorems”, Fixed Point Theory Applications, vol. 2015, no. 1, 1–14(2015). http://dx.doi.org/10.1186/s13663-015-0312-7

[9] M. S. Ashraf, R. Ali, N. Hussain, "New fuzzy fixed point results in generalized fuzzy metric spaces with application to integral equations", IEEE Access. 8, 91653-91660(2020). http://dx.doi.org/10.1109/ACCESS.2020.2994130

[10] L. N. Mishra, A. Rani, M. Kumar, A. Rani, K. Jyoti, "Some common fixed point theorems in JS-metric spaces", Nonlinear Sci. Lett.A, vol. 9, No. 1, 73-85(2017).https://www.academia.edu/12854197 4/73_Some_common_fixed_point_theorems_in_JS_metric_spaces

[11] C. Thangthong and P. Charoensawan, "Common Fixed Point Theorems for Some Admissible Contraction Mapping in JS-Metric Spaces" Thai Journal of Mathematics (2019). https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/849

[12] J. Jachymski, "The contraction principle for mappings on a metric space with a graph", Proceedings of the American Mathematical Society 136 no. 4, 1359-1373 (2008).https://doi.org/10.1090/S0002-9939-07-09110-1

[13] S. Shukla, "Fixed point theorems of G-fuzzy contractions in fuzzy metric spaces endowed with a graph", Communications in Mathematics 22, no. 1, 1-12(2014).https://dml.cz/handle/10338.dmlcz/143900

[14] P. Charoensawan, and W. Atiponrat, "Coincidence Point Theorems for BKC-Contraction Mappings in Generalized Metric Spaces Endowed with a Directed Graph" Thai Journal of Mathematics, vol. 18, no.2, 539-549(2020).https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/1015

[15] B. Schweizer and A. Sklar, "Statistical metric spaces." Pacific Journal of Mathematics vol.: 10, no. 1, 313-334 (1960). http://dx.doi.org/10.2140/pjm.1960.10.313

[16] M. Dinarvand, "Some fixed point results for admissible Geraghty contraction type mappings in fuzzy metric spaces" Iranian journal of fuzzy systems 14, no. 3, 161-177(2017). https://doi.org/10.22111/ijfs.2017.3262