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Review of Dominating Elementary Functions and their Representation in terms of Hypergeometric Functions

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Abstract

Hypergeometric functions plays an important role in special functions, which is a generalized form of a geometric series having the power series with the coefficients made of ratios of rational functions of constants. It isn't only a solution of the second order linear ordinary differential equations that is frequently encountered in mathematical, physical, and engineering problems but also behaves as a tool of expressing many elementary and nonelementary functions as well as nonelementary integrals in its form. In this paper we have studied the nature of dominating elementary functions in context of hypergeometric functions. The dominating elementary functions contain the extended trigonometric, hyperbolic, exponential and logarithmic functions having their monomial argument with general exponent r, whole divided by a monomial having general exponent m. Some corrections have been found in the dominating trigonometric and hyperbolic functions, which have been mentioned. The dominating elementary functions have been expressed directly or indirectly in terms of hypergeometric functions. The paper ends with a short note on limitations of the work and the future scope of research based on this paper.

Keywords: Elementary function, nonelementary function, hypergeometric function, dominating function, dominating elementary function.

Abstract

Hypergeometric functions plays an important role in special functions, which is a generalized form of a geometric series having the power series with the coefficients made of ratios of rational functions of constants. It isn't only a solution of the second order linear ordinary differential equations that is frequently encountered in mathematical, physical, and engineering problems but also behaves as a tool of expressing many elementary and nonelementary functions as well as nonelementary integrals in its form. In this paper we have studied the nature of dominating elementary functions in context of hypergeometric functions. The dominating elementary functions contain the extended trigonometric, hyperbolic, exponential and logarithmic functions having their monomial argument with general exponent r, whole divided by a monomial having general exponent m. Some corrections have been found in the dominating trigonometric and hyperbolic functions, which have been mentioned. The dominating elementary functions have been expressed directly or indirectly in terms of hypergeometric functions. The paper ends with a short note on limitations of the work and the future scope of research based on this paper.

Key-Words: Elementary function, nonelementary function, hypergeometric function, dominating function, dominating elementary function.

1. Introduction

The study of domination is one of the fastest areas in mathematical and allied sciences. The term domination was given formal definition by Berge in 1958 and Ore in 1962. Although the earliest idea of dominating sets has origin in the game of chess, in which one studies sets of chess pieces which cover various opposing pieces or various squares of the chess board. One such popular problem is 'queen's domination problem'. John Van Neumann also considered the term domination in diagraphs while solving problems in game theory (Dominating set - Wikipedia contributors, 2025; Yadav et al., 2009; Yadav & Sen, 2025).

Yadav et al. (2009, 2025) studied domination in the field of functions and propounded the dominating function represented by two sided infinite series as

$$f(x) = \sum_{n=1}^{\infty} \frac{B_n}{(ax+b)^n} + C_0 + \sum_{n=1}^{\infty} A_n (ax+b)^n$$

where A_n, B_n, C_0 are all real constants for all real n. They call a function f(x) a dominatable function or d-able function if it can be expressed in the above form.

They proved that all most all elementary functions are d-able functions.

Thereafter Yadav & Sen (2012, 2017) introduced dominating sequential functions of many elementary functions like of trigonometric, hyperbolic, exponential and logarithmic functions, which dominated all the previous classical elementary trigonometric, hyperbolic, exponential and logarithmic functions in the sense that for particular values of some arbitrary constants, the dominating sequential functions reduces to the classical elementary functions.

Recently Yadav & Yadav (2024) have reviewed the nonelementary integrals in context of hypergeometric functions. Chaudhary & Yadav (2024) studied the hypergeometric functions in context of nonelementary integrals. They discussed eleven propositions based on five questions on elementary functions, nonelementary functions, hypergeometric functions and nonelementary integrals. The present paper is an extension of these works for dominating elementary functions in context of hypergeometric functions.

2. Preliminary Ideas

To study the relations between dominating elementary and hypergeometric functions, we need to know the basic concepts of dominating elementary functions:

Elementary Functions: The functions like algebraic, trigonometric, hyperbolic, exponential and logarithmic functions have been treated as elementary functions, because all these can be expressed in a closed form expression (Closed form - Wikipedia contributors, 2025; Function (mathematics) - Wikipedia contributors, 2025; Yadav, 2024).

Dominating Elementary Functions: We have called dominating trigonometric functions, dominating hyperbolic functions, dominating exponential functions and dominating logarithmic functions as dominating elementary functions (Yadav, 2009, 2012).

Hypergeometric Functions: It has been a practice to express a function in two different ways: one in closed form, which is generally known as elementary function and another in series i.e. writing a function in power series having many terms written individually with plus or minus sign. Hypergeometric function is one such function whose power series expansion is known as hypergeometric series (Chaudhary & Yadav, 2024; Hypergeometric function - Wikipedia contributors, 2025).

The 'hypergeometric series' term was first propounded by John Wallis in 1655 for the infinite series

$$1 + a + a\,(a+1) + \,a\,(a+1)(a+2) + \cdots$$

Thereafter this term was again used by Ernst Eduard Kummer (1810-1898) in 1836 for the series

$$1 + \frac{\propto.\beta}{1.\gamma}\mathrm{x} + \frac{\propto.(\propto+1).\beta.(\beta+1)}{1.2.\gamma.(\gamma+1)}\mathrm{x}^2 + \frac{\propto.(\propto+1).(\propto+2).\beta.(\beta+1).(\beta+2)}{1.2.3.\gamma.(\gamma+1).(\gamma+2)}\mathrm{x}^3 + \cdots$$

which is the popularly used form of the hypergeometric series (Chaudhary & Yadav, 2024; Hannah, 2013; Hypergeometric function - Wikipedia contributors, 2025). We define it in three parts:

General Hypergeometric Function: It is denoted and defined by

$$\mathrm{mFn}(\propto_1, \propto_2, \ldots, \propto_\mathrm{m}; \beta_1, \beta_2, \ldots, \beta_\mathrm{n}; \mathrm{x}) = \sum_{\mathrm{r}=0}^{\infty} \frac{(\propto_1)_\mathrm{r}.(\propto_2)_\mathrm{r}, \ldots, (\propto_\mathrm{m})_\mathrm{r}}{(\beta_1)_\mathrm{r}, (\beta_2)_\mathrm{r}, \ldots, (\beta_\mathrm{n})_\mathrm{r}} \frac{\mathrm{x}^\mathrm{r}}{\mathrm{r}!}$$

$$= 1 + \frac{\propto_1 \propto_2 \ldots \propto_\mathrm{m}}{\beta_1\beta_2 \ldots \beta_\mathrm{n}} \frac{\mathrm{x}^1}{1!} + \frac{\propto_1 (\propto_1+1)\, \propto_2 (\propto_2+1) \ldots \propto_\mathrm{m} (\propto_\mathrm{m}+1)}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)\ldots\beta_\mathrm{n}(\beta_\mathrm{n}+1)} \frac{\mathrm{x}^2}{2!} + \cdots$$

where $(\alpha)_\mathrm{r} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+\mathrm{r}-1)$ and $(\alpha)_0 = 1$ known as shifted factorial. The notation $\mathrm{mFn}(\propto_1, \propto_2, \ldots, \propto_\mathrm{m}; \beta_1, \beta_2, \ldots, \beta_\mathrm{n}; \mathrm{x})$ is called hypergeometric function and the series on the right hand side is called the hypergeometric series (Bailey, 1964; Chaudhary & Yadav, 2024; Du et al., 2002; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025).

Confluent Hypergeometric Function: Putting m = n = 1 in above function, we get the confluent hypergeometric function

$$1\mathrm{F1}(\propto, \beta, \mathrm{x}) = \sum_{\mathrm{r}=0}^{\infty} \frac{(\propto)_\mathrm{r}}{(\beta)_\mathrm{r}} \frac{\mathrm{x}^\mathrm{r}}{\mathrm{r}!} = 1 + \frac{\propto}{\beta}\frac{\mathrm{x}}{1!} + \frac{\propto(\propto+1)}{\beta(\beta+1)}\frac{\mathrm{x}^2}{2!} + \cdots$$

(Chaudhary & Yadav, 2024; Confluent hypergeometric function - Wikipedia contributors, 2025; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025).

Gauss Hypergeometric Function: On putting m = 2, n = 1 in general hypergeometric function, we get it,

$$\mathrm{F}(\propto, \beta; \gamma; \mathrm{x}) = 2\mathrm{F1}(\propto, \beta; \gamma; \mathrm{x}) = \sum_{\mathrm{r}=0}^{\infty} \frac{(\propto)_\mathrm{r}.(\beta)_\mathrm{r}}{(\gamma)_\mathrm{r}} \frac{\mathrm{x}^\mathrm{r}}{\mathrm{r}!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(\propto)_r . (\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$$

$$= 1 + \frac{\propto \beta}{\gamma} \frac{x}{1!} + \frac{\propto (\propto +1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \cdots$$

where x, ∝, β, γ may be real or complex, $|x| < 1$ and γ is a non-negative integer (Chaudhary & Yadav, 2024; Hannah, 2013; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2025). In our study we will use the notations $F(\propto, \beta; \gamma; x)$, ${}_2F_1(\propto, \beta; \gamma; x)$ or $2F1(\propto, \beta; \gamma; x)$ for it.

3. Discussion

Based on the above notations and concepts of different terms, let us review different dominating elementary functions as follows:

Dominating Trigonometric Functions: Yadav & Sen (2012, 2017) have defined it as: for real numbers m, r, six dominating trigonometric functions corresponding to six classical trigonometric functions are given by

$$dsinx_m^r = \frac{sinx^r}{x^m}$$

$$dcosx_m^r = \frac{cosx^r}{x^m}$$

$$dtanx_m^r = \frac{dsinx_m^r}{dcosx_m^r} = \frac{sinx^r}{cosx^r} = tanx^r$$

$$dcotx_m^r = \frac{dcosx_m^r}{dsinx_m^r} = \frac{cosx^r}{sinx^r} = cotx^r$$

$$dcosecx_m^r = \frac{1}{dsinx_m^r} = \frac{x^m}{sinx^r}$$

$$dsecx_m^r = \frac{1}{dcosx_m^r} = \frac{x^m}{cosx^r}$$

where 'd' denotes for dominating function. These functions have been named as dominating sine, dominating cosine, dominating tangent, dominating cotangent, dominating cosecant and dominating secant functions. Obviously these functions reduce to the classical trigonometric functions for m = 0 and r = 1.

When these new functions were applied on the classical formulae of trigonometric functions, Yadav & Sen (2017) got some more properties but different from the pattern of classical one. They found that

$$dtanx_m^r \neq \frac{dsinx_m^r}{dcosx_m^r}$$

$$dtanx_m^r \neq \frac{1}{dcotx_m^r}$$

$$dcotx_m^r \neq \frac{dcosx_m^r}{dsinx_m^r}$$

$$dcotx_m^r \neq \frac{1}{dtanx_m^r}$$

$$dcosecx_m^r \neq \frac{1}{dsinx_m^r}$$

$$dsinx_m^r \neq \frac{1}{dcosecx_m^r}$$

$$dcosx_m^r \neq \frac{1}{dsecx_m^r}$$

$$dsecx_m^r \neq \frac{1}{dcosx_m^r}$$

in which some were unproved.

To prove them, let us observe the changes in the classical pattern of the formulae by assuming the same pattern of basic trigonometric formulae as:

$$dtanx_m^r = \frac{dsinx_m^r}{dcosx_m^r} = tanx^r \qquad (i)$$

and

$$dtanx_m^r = \frac{tanx^r}{x^m} = \frac{tanx^r}{x^{m_1+m_2}} = \frac{\frac{sinx^r}{x^{m_1}}}{\frac{cosx^r}{x^{-m_2}}} = \frac{dsinx_{m_1}^r}{dcosx_{-m_2}^r} \qquad (ii)$$

where $m_1 + m_2 = m$, It depends on us that how we divide m into two parts m_1 and m_2 as per our requirement. Therefore we get

$$dtanx_m^r \neq \frac{dsinx_m^r}{dcosx_m^r}$$

Also we have

$$\frac{1}{dcotx_m^r} = \frac{1}{\frac{cotx^r}{x^m}} = \frac{x^m}{cotx^r} = x^m tanx^r$$

Thus we get that

$$dtanx_m^r \neq \frac{dsinx_m^r}{dcosx_m^r} \text{ and } dtanx_m^r \neq \frac{1}{dcotx_m^r}$$

If we take $m_1 = -m_2$ i.e. m = 0, then (i) and (ii) will coincide i.e. the traditional formula will be true.

Similarly we can find

$$dcotx_m^r = \frac{dcosx_m^r}{dsinx_m^r} = cotx^r \qquad (iii)$$

and

$$dcotx_m^r = \frac{cotx^r}{x^m} = \frac{cotx^r}{x^{m_1+m_2}} = \frac{\frac{cosx^r}{x^{m_1}}}{\frac{sinx^r}{x^{-m_2}}}$$
$$= \frac{dcosx_{m_1}^r}{dsinx_{-m_2}^r} \qquad (iv)$$

where $m_1 + m_2 = m$. If we take $m_1 = -m_2$ i.e. m = 0, then (iii) and (iv) will coincide i.e. the traditional formula will be true.

Also we have

$$dcotx_m^r \neq \frac{dcosx_m^r}{dsinx_m^r}$$

and

$$\frac{1}{dtanx_m^r} = \frac{1}{\frac{tanx^r}{x^m}} = \frac{x^m}{tanx^r} = x^m cotx^r$$

Thus we get

$$dcotx_m^r \neq \frac{dcosx_m^r}{dsinx_m^r} \text{ and } dcotx_m^r \neq \frac{1}{dtanx_m^r}$$

Similarly we can prove that

$$dcosecx_m^r = \frac{cosecx^r}{x^m} = \frac{1}{x^m sinx^r} = \frac{1}{x^m x^m \frac{sinx^r}{x^m}}$$
$$= \frac{1}{x^{2m} dsinx_m^r} \neq \frac{1}{dsinx_m^r}$$

$$dsinx_m^r = \frac{sinx^r}{x^m} = \frac{1}{x^m cosecx^r} = \frac{1}{x^m x^m \frac{cosecx^r}{x^m}}$$
$$= \frac{1}{x^{2m} dcosecx_m^r} \neq \frac{1}{dcosecx_m^r}$$

$$dcosx_m^r = \frac{cosx^r}{x^m} = \frac{1}{x^m secx^r} = \frac{1}{x^m x^m \frac{secx^r}{x^m}}$$
$$= \frac{1}{x^{2m} dsecx_m^r} \neq \frac{1}{dsecx_m^r}$$

$$dsecx_m^r = \frac{secx^r}{x^m} = \frac{1}{x^m cosx^r} = \frac{1}{x^m x^m \frac{cosx^r}{x^m}}$$
$$= \frac{1}{x^{2m} dcosx_m^r} \neq \frac{1}{dcosx_m^r}$$

Therefore the proper representations of dominating trigonometric functions should be given by

$$dsinx_m^r = \frac{sinx^r}{x^m}$$

$$dcosx_m^r = \frac{cosx^r}{x^m}$$

$$dtanx_m^r = \frac{tanx^r}{x^m}$$

$$dcotx_m^r = \frac{cotx^r}{x^m}$$

$$dcosecx_m^r = \frac{cosecx^r}{x^m}$$

$$dsecx_m^r = \frac{secx^r}{x^m}$$

Although Yadav & Sen (2012) didn't mentioned it clearly, but they meant the same expression in their study, which is clear from their calculations regarding these functions.

Dominating Hyperbolic Functions: Yadav & Sen (2012, 2017) have defined it as: for real numbers m, r, six dominating hyperbolic functions corresponding to six classical hyperbolic functions similar to dominating trigonometric functions are as follows:

$$dsinhx_m^r = \frac{sinhx^r}{x^m}$$

$$dcoshx_m^r = \frac{coshx^r}{x^m}$$

$$dtanhx_m^r = \frac{dsinhx_m^r}{dcoshx_m^r} = \frac{sinhx^r}{coshx^r} = tanhx^r$$

$$dcothx_m^r = \frac{dcoshx_m^r}{dsinhx_m^r} = \frac{coshx^r}{sinhx^r} = cothx^r$$

$$dcosechx_m^r = \frac{1}{dsinhx_m^r} = \frac{x^m}{sinhx^r}$$

$$dsechx_m^r = \frac{1}{dcoshx_m^r} = \frac{x^m}{coshx^r}$$

where again 'd' denotes for the dominating function. These functions have been named as dominating sine hyperbolic, dominating cosine hyperbolic, dominating tangent hyperbolic, dominating cotangent hyperbolic, dominating cosecant hyperbolic and dominating secant hyperbolic functions. Obviously these functions reduce to the classical hyperbolic functions for m = 0 and r = 1.

When these new functions were applied on the previous formulae of hyperbolic functions, Yadav & Sen (2017) got some more properties but different from the pattern of classical one. They found that

$$dtanhx_m^r \neq \frac{dsinhx_m^r}{dcoshx_m^r}$$

$$dtanhx_m^r \neq \frac{1}{dcothx_m^r}$$

$$dcothx_m^r \neq \frac{dcoshx_m^r}{dsinhx_m^r}$$

$$dcothx_m^r \neq \frac{1}{dtanhx_m^r}$$

$$dcosechx_m^r \neq \frac{1}{dsinhx_m^r}$$

$$dsinhx_m^r \neq \frac{1}{dcosechx_m^r}$$

$$dcoshx_m^r \neq \frac{1}{dsechx_m^r}$$

$$dsechx_m^r \neq \frac{1}{dcoshx_m^r}$$

In which some were unproved. To prove them, let us observe the changes in the classical pattern of the formulae by assuming the same pattern of basic hyperbolic formulae as:

$$dtanhx_m^r = \frac{dsinhx_m^r}{dcoshx_m^r} = tanhx^r \qquad (v)$$

and

$$dtanhx_m^r = \frac{tanhx^r}{x^m} = \frac{tanhx^r}{x^{m_1+m_2}} = \frac{\frac{sinhx^r}{x^{m_1}}}{\frac{coshx^r}{x^{-m_2}}} = \frac{dsinhx_{m_1}^r}{dcoshx_{-m_2}^r} \qquad (vi)$$

where $m_1 + m_2 = m$. The same logic is applied on m, m_1, m_2 as have been discussed earlier. Therefore we get

$$dtanhx_m^r \neq \frac{dsinhx_m^r}{dcoshx_m^r}$$

Also we have

$$\frac{1}{dcothx_m^r} = \frac{1}{\frac{cothx^r}{x^m}} = \frac{x^m}{cothx^r} = x^m tanhx^r$$

Thus we get

$$dtanhx_m^r \neq \frac{dsinhx_m^r}{dcoshx_m^r} \text{ and } dtanhx_m^r \neq \frac{1}{dcothx_m^r}$$

If we take $m_1 = -m_2$ i.e. m = 0, then (v) and (vi) will coincide i.e. the traditional formula will work for this case.

Similarly we can find

$$dcothx_m^r = \frac{dcoshx_m^r}{dsinhx_m^r} = cothx^r \qquad (vii)$$

and

$$dcothx_m^r = \frac{cothx^r}{x^m} = \frac{cothx^r}{x^{m_1+m_2}} = \frac{\frac{coshx^r}{x^{m_1}}}{\frac{sinhx^r}{x^{-m_2}}} = \frac{dcoshx_{m_1}^r}{dsinhx_{-m_2}^r} \qquad (viii)$$

where $m_1 + m_2 = m$. If we take $m_1 = -m_2$ i.e. m = 0, then (vii) and (viii) will coincide i.e. the traditional formula will work for this case. Thus

$$dcothx_m^r \neq \frac{dcoshx_m^r}{dsinhx_m^r}$$

Also we have

$$\frac{1}{dtanhx_m^r} = \frac{1}{\frac{tanhx^r}{x^m}} = \frac{x^m}{tanhx^r} = x^m cothx^r$$

Thus

$$dcothx_m^r \neq \frac{1}{dtanhx_m^r}$$

Similarly we have

$$dcosechx_m^r = \frac{cosechx^r}{x^m} = \frac{1}{x^m sinhx^r} = \frac{1}{x^m x^m \frac{sinhx^r}{x^m}} = \frac{1}{x^{2m} dsinhx_m^r} \neq \frac{1}{dsinhx_m^r}$$

$$dsinhx_m^r = \frac{sinhx^r}{x^m} = \frac{1}{x^m cosechx^r} = \frac{1}{x^m x^m \frac{cosechx^r}{x^m}} = \frac{1}{x^{2m} dcosechx_m^r} \neq \frac{1}{dcosechx_m^r}$$

$$dcoshx_m^r = \frac{coshx^r}{x^m} = \frac{1}{x^m sechx^r} = \frac{1}{x^m x^m \frac{sechx^r}{x^m}} = \frac{1}{x^{2m} dsechx_m^r} \neq \frac{1}{dsechx_m^r}$$

$$dsechx_m^r = \frac{sechx^r}{x^m} = \frac{1}{x^m coshx^r} = \frac{1}{x^m x^m \frac{coshx^r}{x^m}} = \frac{1}{x^{2m} dcoshx_m^r} \neq \frac{1}{dcoshx_m^r}$$

Therefore the proper representations of dominating hyperbolic functions should be given as

$$dsinhx_m^r = \frac{sinhx^r}{x^m}$$

$$dcoshx_m^r = \frac{coshx^r}{x^m}$$

$$dtanhx_m^r = \frac{tanhx^r}{x^m}$$

$$dcothx_m^r = \frac{cothx^r}{x^m}$$

$$dcosechx_m^r = \frac{cosechx^r}{x^m}$$

$$dsechx_m^r = \frac{sechx^r}{x^m}$$

Although Yadav & Sen (2012) didn't mentioned it, but they meant the same expression in their study, which is clear from their calculations regarding these dominating hyperbolic functions.

Dominating Exponential Functions: As far as the dominating exponential function with base e is concerned, Yadav & Sen (2012, 2017) denoted and defined it by

$$de^{x_m^r} = \frac{e^{x^r}}{x^m}$$

For m = 0 and r = 1, we get the traditional exponential function

$$de^{x_0^1} = \frac{e^{x^1}}{x^0} = e^x$$

They have discussed two more properties on it as

$$de^{ix_m^r} = dcosx_m^r + i\, dsinx_m^r,$$
$$de^{-ix_m^r} = dcosx_m^r - i\, dsinx_m^r$$

For m = 0 and r = 1, these reduce to

$$de^{ix_0^1} = dcosx_0^1 + i\, dsinx_0^1 = cosx + i\, sinx = e^{ix}$$

$$de^{-ix_0^1} = dcosx_0^1 - i\, dsinx_0^1 = cosx - i\, sinx = e^{-ix}$$

The trigonometric functions and dominating functions can be expressed in terms of these dominating exponential functions.

Dominating Logarithmic Functions: Yadav & Sen (2012, 2017) denoted dominating logarithmic function as

$$dlnx_m^r = \frac{lnx^r}{x^m}$$

For m = 0 and r = 1, we have

$$dlnx_0^1 = \frac{lnx^1}{x^0} = lnx$$

which is the classical logarithmic function with base e. They discussed two more logarithmic functions as follows

$$dln(1 + x_m^r) = \frac{\ln(1 + x^r)}{x^m} \text{ and } dln(1 - x_m^r) = \frac{\ln(1 - x^r)}{x^m}$$

Relations of Dominating Elementary and Hypergeometric Functions: As far as the dominating elementary functions are concerned in context of hypergeometric functions, we here are concerned with the relations between them. Recently Chaudhary & Yadav (2024) have studied the relations between elementary, nonelementary and hypergeometric functions. Taking advantage of their work, we can directly link dominating elementary functions with hypergeometric functions as follows:

Chaudhary & Yadav (2024) have stated that all trigonometric and inverse trigonometric functions are hypergeometric functions but not vice versa and

$$\sin x = x.\,0F1\left(;\frac{3}{2};\frac{-x^2}{4}\right)$$

$$\Longrightarrow sinx^r = x^r.\,0F1\left(;\frac{3}{2};\frac{-x^{2r}}{4}\right)$$

$$\Longrightarrow \frac{sinx^r}{x^m} = \frac{x^r}{x^m}.\,0F1\left(;\frac{3}{2};\frac{-x^{2r}}{4}\right) = dsinx_m^r,$$

$$\cos x = 0F1\left(;\frac{1}{2};\frac{-x^2}{4}\right)$$

$$\Longrightarrow cosx^r = 0F1\left(;\frac{1}{2};\frac{-x^{2r}}{4}\right)$$

$$\Longrightarrow \frac{cosx^r}{x^m} = \frac{1}{x^m}.\,0F1\left(;\frac{1}{2};\frac{-x^{2r}}{4}\right) = dcosx_m^r$$

Similarly we can compute other dominating trigonometric functions in terms of hypergeometric functions using the identities like

$$d\tan x_m^r = \frac{\tan x^r}{x^m} = \frac{1}{x^m}\frac{x^r.\,0F1\left(;\frac{3}{2};\frac{-x^{2r}}{4}\right)}{0F1\left(;\frac{1}{2};\frac{-x^{2r}}{4}\right)}$$

Since there is not a single and simple hypergeometric expression that is directly equal to tanx, therefore all such dominating trigonometric functions can also not be expressed in terms of hypergeometric functions directly. Similarly

$$d\cot x_m^r = \frac{\cot x^r}{x^m} = \frac{1}{x^m}\frac{0F1\left(;\frac{1}{2};\frac{-x^{2r}}{4}\right)}{x^r.\,0F1\left(;\frac{3}{2};\frac{-x^{2r}}{4}\right)}$$

$$d\operatorname{cosec} x_m^r = \frac{\operatorname{cosec} x^r}{x^m} = \frac{1}{x^m}\frac{1}{0F1\left(;\frac{3}{2};\frac{-x^{2r}}{4}\right)}$$

$$d\sec x_m^r = \frac{\sec x^r}{x^m} = \frac{1}{x^m}\frac{1}{0F1\left(;\frac{1}{2};\frac{-x^{2r}}{4}\right)}$$

Chaudhary & Yadav (2024) have stated that all hyperbolic and inverse hyperbolic functions are hypergeometric functions but not vice versa. Thus we have

$$\sinh x = x.\,0F1\left(-;\frac{3}{2};\frac{x^2}{4}\right)$$

$$\Longrightarrow \sinh x^{r} = x^{r}.\,0F1\left(-;\frac{3}{2};\frac{x^{2r}}{4}\right)$$

$$\Longrightarrow \frac{\sinh x^{r}}{x^{m}} = \frac{x^{r}}{x^{m}}.\,0F1\left(-;\frac{3}{2};\frac{x^{2r}}{4}\right) = d\sinh x_m^r\,,$$

$$\cosh x = 0F1\left(-;\frac{1}{2};\frac{x^{2}}{4}\right)$$

$$\Longrightarrow \cosh x^{r} = 0F1\left(-;\frac{1}{2};\frac{x^{2r}}{4}\right)$$

$$\Longrightarrow \frac{\cosh x^{r}}{x^{m}} = \frac{1}{x^{m}}0F1\left(-;\frac{1}{2};\frac{x^{2r}}{4}\right) = d\cosh x_m^r$$

Similarly we can express other dominating hyperbolic functions in terms of hypergeometric functions using the basic identities, as there is no direct hypergeometric representation of tanhx, cothx, etc, as follows:

$$d\tanh x_m^r = \frac{\tanh x^{r}}{x^{m}} = \frac{1}{x^{m}}\frac{x^{r}.\,0F1\left(-;\frac{3}{2};\frac{x^{2r}}{4}\right)}{0F1\left(-;\frac{1}{2};\frac{x^{2r}}{4}\right)}$$

$$d\coth x_m^r = \frac{\coth x^{r}}{x^{m}} = \frac{1}{x^{m}}\frac{0F1\left(-;\frac{1}{2};\frac{x^{2r}}{4}\right)}{x^{r}.\,0F1\left(-;\frac{3}{2};\frac{x^{2r}}{4}\right)}$$

$$d\operatorname{cosech} x_m^r = \frac{\operatorname{cosech} x^{r}}{x^{m}} = \frac{1}{x^{m}}\frac{1}{x^{r}.\,0F1\left(-;\frac{3}{2};\frac{x^{2r}}{4}\right)}$$

$$d\operatorname{sech} x_m^r = \frac{\operatorname{sech} x^{r}}{x^{m}} = \frac{1}{x^{m}}\frac{1}{0F1\left(-;\frac{1}{2};\frac{x^{2r}}{4}\right)}$$

Chaudhary & Yadav (2024) have stated that the exponential functions are hypergeometric functions but not vice versa and

$$e^{x} = 0F0(\,;\,;x) = 1F1(-;-;x) = F(a;a;x)$$

$$\Rightarrow e^{x^{r}} = 0F0(\,;\,;x^{r}) = 1F1(-;-;x^{r}) = F(a;a;x^{r})$$

$$\Rightarrow \frac{e^{x^{r}}}{x^{m}} = \frac{1}{x^{m}}0F0(\,;\,;x^{r}) = \frac{1}{x^{m}}1F1(-;-;x^{r}) = \frac{1}{x^{m}}F(a;a;x^{r}) = de^{x_m^r}$$

Chaudhary & Yadav (2024) have also stated that the logarithmic functions are hypergeometric functions but not vice versa. We have

$$\log(1+x) = x.\,2F1(1,1;2;-x)$$

$$\Rightarrow \log(1+x^{r}) = x^{r}.\,2F1(1,1;2;-x^{r})$$

$$\Rightarrow \frac{\log(1+x^{r})}{x^{m}} = \frac{x^{r}}{x^{m}}.\,2F1(1,1;2;-x^{r}) = d\log(1+x_m^r)\,,$$

$$\log(1-x) = -x.\,2F1(1,1;2;x)$$

$$\Rightarrow \log(1-x^{r}) = -x^{r}.\,2F1(1,1;2;x^{r})$$

$$\Rightarrow \frac{\log(1-x^{r})}{x^{m}} = -\frac{x^{r}}{x^{m}}.\,2F1(1,1;2;x^{r}) = d\log(1-x_m^r)\,,$$

$$\log x = -2\left(\frac{1-x}{1+x}\right).2F1\left(1,\frac{1}{2};\frac{3}{2};\left(\frac{1-x}{1+x}\right)^{2}\right)$$

$$\Rightarrow \log x^{r} = -2\left(\frac{1-x^{r}}{1+x^{r}}\right).2F1\left(1,\frac{1}{2};\frac{3}{2};\left(\frac{1-x^{r}}{1+x^{r}}\right)^{2}\right)$$

$$\Rightarrow \frac{\log x^{r}}{x^{m}} = -\frac{2}{x^{m}}\left(\frac{1-x^{r}}{1+x^{r}}\right).2F1\left(1,\frac{1}{2};\frac{3}{2};\left(\frac{1-x^{r}}{1+x^{r}}\right)^{2}\right) = d\log x_m^r$$

The above discussion and relations show that dominating trigonometric functions, dominating hyperbolic functions, dominating exponential functions, and dominating logarithmic functions are expressible in terms of different hypergeometric functions directly or indirectly because no direct expression in terms of hypergeometric functions has been found for many elementary and dominating functions.

4. Conclusion

From above discussion and relations, we can conclude that the dominating elementary functions doesn't have same behavior as the classical elementary functions have but every dominating elementary function can be expressed in terms of hypergeometric functions either directly or indirectly as the elementary functions are expressed. Therefore the elementary and dominating elementary functions have the similar expression in context of hypergeometric functions.

5. Limitations

The present paper is based on some simple relations only and it has not been applied on complex relations and on the relations between two different elementary functions. This gives the scope of future research in this field.

6. Future Scope of Research

The above discussion can be extended for all other formulae and properties of elementary functions for different dominating elementary functions.

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