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Two Species Interact in the Ecosystem with Michaelis Menten Harvesting Rate and Holling Type II Functional Response

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Abstract

In the study, a mathematical model was proposed two nonlinear ordinary differential equation to depict the relationship between prey and predator population. This approach provides insights into the behaviors of prey-predator interaction. In the context of our model, the Holling type II functional response is accompanied, by a Michaelis Menten harvesting rate on prey. In the investigation of the presence, boundedness of positive outcomes, and local stability at each feasible equilibrium point of this model are explored. The model includes four equilibrium points, with two being unstable, whereas the others are contingent upon specific conditions. Utilizing the Bendixson-Dulac criterion, the global stability of the equilibrium point representing predator extinction is analyzed. A theorem presents the global stability criteria. Lastly, two instances of numerical analysis demonstrate the validity of the theoretical approach.

Keywords: Stability, functional response, harvesting, Holling type II.

1. Introduction

A mathematical model comprising predators and prey, exhibiting mutual influence is analyzed through dynamics methods. As for the growth rate, its a conceptual structure that considers population limits in the context of growth dynamics[1]. The Lotka-volterra model, a core component of prey-predator model, employs a set of equations that describe the dynamics for both species. The population limits for prey and predators are addressed in [2] and [3]. In an ecosystems, the relationship between prey and predators induces changes in population dynamics. The impact on the species involved can be positive, negative or even nonexistent, depending on the dynamics between prey and predators [4]. According to the model, there is a recommendation for transformation to

$$\frac{dk_1}{dt} = k_1\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \hat{\mu}(k_1)k_2 \qquad (1)$$

$$\frac{dk_2}{dt} = -\tilde{c}k_2 + \tilde{n}\hat{\mu}(k_1)k_2$$

Here, k_1 and k_2 represent the densities among the prey and predator populations respectively, $\tilde{v}$ denotes the population limit, while $\tilde{c}$ signifies the predators death rate. $\tilde{w}$ denotes the internal growth rate. The function $\hat{\mu}(k_1)$ characterizes the functional response, illustrating the rate at which each predator kills prey per unit time. Through a logical reasoning involving the allocation of a predators time into two periods, pursuit and consumption of prey, the Holling type II functional response is evident. In a mathematical context, arguments regarding temporal distinctions in species behavior were employed and elaborated upon in [5]. Harvesting various organisms serves a range of purposes. Either as a result of population growth or in the pursuit of economic progress, the ever increasing use of natural resources persists these actions often result in the eradication of various organisms globally. Reducing the negative impact can be achieved by harvesting these specific organisms. For business proposes, the exploitation of natural resources also take place. As harvesting increases, the like hood of predator populations facing extinction also rises[6]. To analyze local stability, one typically examines the Jacobian matrix of the system at the equilibrium points. However, the Jacobian matrix of a nonlinear system can have complex eigenvalues, leading to oscillatory behavior near equilibrium points. The analysis can be intricate and might not straight forwarding extent to global stability.

In prey-predator dynamics, various methods of harvesting are proposed and studied. The commonly employed harvesting functions include constants proportionate, and nonlinear Michaelis-Menten type[7]. Harvesting isn't necessarily carried out with a constant yield or intensity[8].

$$\frac{dk_1}{dt} = k_1\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \hat{\mu}(k_1)k_2 - \hat{h}(k_1) \qquad (2)$$

$$\frac{dk_2}{dt} = -\tilde{c}k_2 + \tilde{n}\hat{\mu}(k_1)k_2$$

The remainder of this paper is structured in the following manner: Section 2 presents the formulation of the mathematical model. Section 3 delves into the analysis of positivity and boundedness concerning the prey predator equation. Section 4 explores the determination of various equilibrium points, which is conducted in section 5. Section 6 deals with global stability of the predator existence equilibrium point. Section 7 exhibits the outcomes obtained through numerical suggestion conclusively, the last section discusses the outcomes of this research.

2. Mathematical Modeling

In the absence of natural predators, the prey population increases rapidly, and predators hunt the prey with a Holling type II functional response. The predator's rate of prey consumption is determined by $\hat{\mu}(k_1) = \frac{\tilde{I}k_1}{k_1 + \tilde{j}}$. Modeled as follows, the Micaelis-Menten harvesting rate is described $\hat{h}(k_1) = \frac{x\hat{R}k_1}{\widetilde{o_1}\hat{R} + \widetilde{o_2}\, k_1}$
These elements combined, the format formulation of the model is as follows:

$$\frac{dk_1}{dt} = k_1\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{\tilde{I}k_1k_2}{k_1 + \tilde{j}} - \frac{x\hat{R}k_1}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1} \tag{3}$$

$$\frac{dk_2}{dt} = -\tilde{c}k_2 + \frac{\tilde{n}\tilde{I}k_1k_2}{k_1 + \tilde{j}}$$

Where,
$\tilde{I}$- rate of predation.
$\tilde{j}$- half saturation constant.
x - catchability coefficient.
$\hat{R}$- extrinsic effort dedicated to harvesting.
$\widetilde{o_1}, \widetilde{o_2}$- appropriate constant.
$\tilde{n}$- natality rate of predators by each individual relative to the highest intake for each individual.

3. Positivity and Boundedness

Theorem: 3.1 If the equation is positive at all time, $k_1 \geq 0, k_2 \geq 0$ then every solution of equation(3) is positive.

Proof:

Let the first equation from the model

$$\frac{dk_1}{dt} = k_1\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{\tilde{I}k_1k_2}{k_1 + \tilde{j}} - \frac{x\hat{R}k_1}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}$$

$$\frac{dk_1}{dt} = k_1\left[\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{\tilde{I}k_1}{k_1 + \tilde{j}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}\right]$$

$$\frac{dk_1}{k_1} = \left[\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{\tilde{I}k_1}{k_1 + \tilde{j}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}\right] dt$$

$$\frac{dk_1}{k_1} = \tilde{\kappa}(k_1, k_2)\, dt$$

Where $\tilde{\kappa}(k_1, k_2) = \left[\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{\tilde{I}k_1}{k_1 + \tilde{j}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}\right]$

Integrating over the interval [0, t] we obtain,

$k_1(t) = k_1(0)e^{\int \tilde{\kappa}(k_1,k_2)\, dt} > 0\ \forall$ t, since $k_1(0) \geq 0$

Next consider the second equation

$$\frac{dk_2}{dt} = -\tilde{c}k_2 + \frac{\tilde{n}\tilde{I}k_1k_2}{k_1 + \tilde{j}}$$

$$\frac{dk_2}{dt} = k_2\left[-\tilde{c} + \frac{\tilde{n}\tilde{I}k_1}{k_1 + \tilde{j}}\right]$$

$$\frac{dk_2}{k_2} = \left[-\tilde{c} + \frac{\tilde{n}\tilde{I}k_1}{k_1 + \tilde{j}}\right] dt$$

$$\frac{dk_2}{k_2} = \tilde{\varsigma}(k_1, k_2)dt$$

Where $\tilde{\varsigma}(k_1, k_2) = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_1}{k_1 + \tilde{j}}$

Integrating over the interval [0, t] we obtain,

$k_2(t) = k_2(0)e^{\int \tilde{\varsigma}(k_1,k_2)dt} > 0\ \forall$ t, since $k_2(0) \geq 0$

Thus, the outcomes of equation(3) may remain uniformly non-negative.

Theorem: 3.2 The domain $R_+^2 = \{(k_1, k_2) \in R^2 : k_1 \geq 0, k_2 \geq 0\}$ every solution of the equation(3) are bounded.

Proof:

Let $\omega(t) = k_1 + k_2$ then differentiate with respect to t and we get,

$$\frac{d\omega}{dt} = \frac{dk_1}{dt} + \frac{dk_2}{dt}$$

$$\frac{d\omega}{dt} = k_1\tilde{w}\left(1 - \frac{k_1}{\tilde{v}}\right) - \frac{x\hat{R}k_1}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}$$

$$\frac{d\omega}{dt} = k_1\left[\tilde{w} - \frac{k_1\tilde{w}}{\tilde{v}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}\right] - \tilde{c}k_2$$

For any $\xi > 0$.Given any $\epsilon > 0$, there exists t_0 such that t on $t \geq t_0$, we get

$$\frac{d\omega}{dt} + \xi\omega = k_1\left[\tilde{w} + \xi - \frac{k_1\tilde{w}}{\tilde{v}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1}\right] + (\xi - \tilde{c})k_2$$

In this case where ξ is a positive constant for $(\tilde{w} + \xi) \geq 0, (\xi - \tilde{c}) \geq 0$.

$$\frac{d\mu(t)}{dt} + \xi\mu \leq \frac{-\hat{B} \pm \sqrt{\hat{B}^2 - 4\hat{A}\hat{C}}}{2\hat{A}} + (\xi - \tilde{c})$$

$$= \eta + \epsilon$$

Where $\hat{A} = \widetilde{o_2}\tilde{w}$

$$\hat{B} = \tilde{w}\widetilde{o_1}\hat{R} - \widetilde{o_2}\tilde{w}\tilde{v} - \xi\widetilde{o_2}\tilde{v}$$

$$\hat{C} = x\hat{R}\tilde{v} - \tilde{w}\widetilde{o_1}\hat{R}\tilde{v} - \xi\widetilde{o_1}\hat{R}\tilde{v}$$

Applying Aziz-Alaoui and Okiye, lemma

$$0 \leq \mu(t) \leq \frac{\eta + \epsilon}{\xi}\left(1 - e^{-\xi(t - t_0)}\right) + \omega(t_0)e^{-\xi(t - t_0)}$$

limit $t \to \infty$ and letting $\epsilon \to 0$.

Hence, $0 \leq \omega(t) \leq \frac{\eta}{\xi}$

Therefore, the system of equations are bounded.

4. Presence of Equilibrium Point

Analyzing the local stability in the prey-predator model necessitates the equilibrium points in the dynamical systems. Model(3) equilibrium points are determined through the steady state equation $\frac{dk_2}{dt} = 0$ and $\frac{dk_1}{dt} = 0$. Algebraic computations are conducted to determine the trivial and nontrivial equilibrium points.

i. The initial equilibrium point $\widehat{E_1}(0,0)$ represents the absence of both prey and predators.

ii. The predator existence equilibrium point $\widehat{E_2}(k_{1(ii)}, 0)$ exists, as the prey population grows without predation.

Where $k_1 = \frac{-S_2 \pm \sqrt{{S_2}^2 - 4S_1S_3}}{2S_1}$

Therefore, $k_{1(i)} = \frac{-S_2 - \sqrt{{S_2}^2 - 4S_1S_3}}{2S_1}$ and

$$k_{1(ii)} = \frac{-S_2 + \sqrt{{S_2}^2 - 4S_1S_3}}{2S_1}$$

Here, $S_1 = \widetilde{w}\widetilde{o_2}$

$S_2 = \widetilde{w}\widetilde{o_1}\widehat{R} - \widetilde{w}\widetilde{o_2}\tilde{v}$

$S_3 = \tilde{v}\widehat{R}(x - \widehat{w}\widetilde{o_1})$

When $S_2 < 0$, $S_3 > 0$, if $x\tilde{v}\widehat{R} < \widehat{w}\widetilde{o_1}\tilde{v}\widehat{R}$, the equilibrium point $k_{1(ii)}$ is exist.

iii. The non-negative equilibrium point $\widehat{E_3}(k_1^*, k_2^*)$ exists when both coordinates k_1 and k_2 are positive.

5. Stability Analysis of Equilibrium Points

Regarding the equilibrium points, the jacobian matrix of model (3) is as follows:

$$J(k_1, k_2) = \begin{bmatrix} \hat{\theta}_{11} & \hat{\theta}_{12} \\ \hat{\theta}_{21} & \hat{\theta}_{22} \end{bmatrix}$$

Where

$$\hat{\theta}_{11} = \widetilde{w} - \frac{2\widetilde{w}k_1}{\tilde{v}} - \frac{\tilde{I}k_2}{(k_1 + \tilde{j})} + \frac{\tilde{I}k_1k_2}{(k_1 + \tilde{j})^2} - \frac{x\widehat{R}}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_1\right)} + \frac{x\widehat{R}\widetilde{o_2}k_1}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_1\right)^2}$$

$$\hat{\theta}_{12} = -\frac{\tilde{I}k_1}{(k_1 + \tilde{j})}$$

$$\hat{\theta}_{21} = \frac{\tilde{n}\tilde{I}k_2}{(k_1 + \tilde{j})} - \frac{\tilde{n}\tilde{I}k_1k_2}{(k_1 + \tilde{j})^2}$$

$$\hat{\theta}_{22} = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_1}{(k_1 + \tilde{j})}$$

Theorem: 5.1 System(3) shows a saddle at the equilibrium point $\{k_1 = 0, k_2 = 0\}$.

Proof:

The jacobian matrix is,

$$J(0,0) = \begin{bmatrix} \widetilde{w} - \frac{x}{\widetilde{o_1}} & 0 \\ 0 & -\tilde{c} \end{bmatrix}$$

The eigenvalues are $\lambda_1 = \widetilde{w} - \frac{x}{\widetilde{o_1}}$, $\lambda_2 = -\tilde{c}$

If $\frac{x}{\widetilde{o_1}} > \widetilde{w}$, then $\widehat{E_1}$ is stable.

Theorem: 5.2 The system(3) exhibits stability an equilibrium point $\{k_1 = k_{1(ii)}, k_2 = 0\}$ when ever

$\widetilde{w} + \frac{x\widehat{R}\widetilde{o_2}k_{1(ii)}}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_{1(ii)}\right)^2} < \frac{2\widetilde{w}k_{1(ii)}}{\tilde{v}} + \frac{x\widehat{R}}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_{1(ii)}\right)}$ and $\tilde{c} > \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}$.

Proof:

The jacobian matrix is,

$$J(k_{1(ii)}, 0) = \begin{bmatrix} \widetilde{\varphi_{11}} & \widetilde{\varphi_{12}} \\ 0 & \widetilde{\varphi_{22}} \end{bmatrix}$$

Where

$$\widetilde{\varphi_{11}} = \widetilde{w} - \frac{2\widetilde{w}k_{1(ii)}}{\tilde{v}} - \frac{x\widehat{R}}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_{1(ii)}\right)} + \frac{x\widehat{R}\widetilde{o_2}k_{1(ii)}}{\left(\widetilde{o_1}\widehat{R} + \widetilde{o_2}k_{1(ii)}\right)^2}$$

$$\widetilde{\varphi_{12}} = -\frac{\tilde{I}k_{1(ii)}}{\left(k_{1(ii)} + \tilde{j}\right)}$$

$$\widetilde{\varphi_{21}} = 0$$

$$\widetilde{\varphi_{22}} = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}$$

The eigenvalue of $\widehat{E_2}(k_{1(ii)}, 0)$ are

$$\lambda_1 = \tilde{w} - \frac{2\tilde{w}k_{1(ii)}}{\tilde{v}} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})} + \frac{x\hat{R}\widetilde{o_2}k_{1(ii)}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})^2},$$

$$\lambda_2 = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}.$$

In the case where $\tilde{w} + \frac{x\hat{R}\widetilde{o_2}k_{1(ii)}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})^2} < \frac{2\tilde{w}k_{1(ii)}}{\tilde{v}} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})}$ and $\tilde{c} > \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}$, the equilibrium point is stable in $k_1 - k_2$ direction. However, if $\tilde{w} + \frac{x\hat{R}\widetilde{o_2}k_{1(ii)}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})^2} > \frac{2\tilde{w}k_{1(ii)}}{\tilde{v}} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})}$ and

$\tilde{c} > \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}$ it is unstable because it is stable point.

Note: If $\lambda^2 - Tr(\widehat{E_3})\lambda + Det(\widehat{E_3}) = 0$, then the necessary and sufficient requirement crucial for determining the outcomes for the linear stability characterization are

$Tr(\widehat{E_3}) < 0$ and $Det(\widehat{E_3}) > 0$.

Theorem: 5.3 The internal equilibrium point $\{k_1 = k_1^*, k_2 = k_2^*\}$ is locally asymptotically stable if

$\tilde{w} < \tilde{\tilde{c}} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}}{(k_1^* + \tilde{j})}(k_2^* - \tilde{n}k_1^*) - \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} - \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}$ and

$$\hat{R} > \frac{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}{x[\widetilde{o_2}k_1^* - (\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)]}$$

$$\left[\frac{1}{\tilde{c}(k_1^* + \tilde{j}) + \tilde{n}\tilde{I}k_1^*}\left(\frac{\tilde{n}\tilde{I}^2k_1^{*2}k_2^*}{(k_1^* + \tilde{j})^2} - \frac{\tilde{n}\tilde{I}^2k_1^*k_2^*}{(k_1^* + \tilde{j})}\right) - \tilde{w} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} - \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2}\right]$$

Proof:

The jacobian matrix is

$$J(k_1^*, k_2^*) = \begin{bmatrix} \widetilde{\aleph_{11}} & \widetilde{\aleph_{12}} \\ \widetilde{\aleph_{21}} & \widetilde{\aleph_{22}} \end{bmatrix}$$

Here

$$\widetilde{\aleph_{11}} = \tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} + \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} + \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}$$

$$\widetilde{\aleph_{12}} = -\frac{\tilde{I}k_1^*}{(k_1^* + \tilde{j})}$$

$$\widetilde{\aleph_{21}} = \frac{\tilde{n}\tilde{I}k_2^*}{(k_1^* + \tilde{j})} - \frac{\tilde{n}\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2}$$

$$\widetilde{\aleph_{22}} = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{(k_1^* + \tilde{j})}$$

The characteristic equation,

$$\lambda^2 - Tr(\widehat{E_3})\lambda + Det(\widehat{E_3}) = 0$$

Where

$$Tr(\widehat{E_3}) = \tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} + \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} + \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2} - \tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{(k_1^* + \tilde{j})}$$

$$= \lambda_1 + \lambda_2$$

$$Det(\widehat{E_3}) = \left(\tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} + \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} + \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}\right)\left(-\tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{(k_1^* + \tilde{j})}\right) + \left(\frac{\tilde{n}\tilde{I}^2k_1^{*2}k_2^*}{(k_1^* + \tilde{j})^2} - \frac{\tilde{n}\tilde{I}^2k_1^*k_2^*}{(k_1^* + \tilde{j})^3}\right)$$

$$= \lambda_1\lambda_2 + \lambda_3$$

Here,

$$\lambda_1 = \tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} + \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} + \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}$$

$$\lambda_2 = -\tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{(k_1^* + \tilde{j})}$$

$$\lambda_3 = \frac{\tilde{n}\tilde{I}^2k_1^{*2}k_2^*}{(k_1^* + \tilde{j})^2} - \frac{\tilde{n}\tilde{I}^2k_1^*k_2^*}{(k_1^* + \tilde{j})^3}$$

By note, E_3 is stable if $Tr(\widehat{E_3}) = \lambda_1 + \lambda_2 < 0$ and $Det(\widehat{E_3}) = \lambda_1\lambda_2 + \lambda_3 > 0$ by solving we get

$\tilde{w} < \tilde{\tilde{c}} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}}{(k_1^* + \tilde{j})}(k_2^* - \tilde{n}k_1^*) - \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} - \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}$ and

$$\hat{R} > \frac{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}{x[\widetilde{o_2}k_1^* - (\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)]}$$

$$\left[\frac{1}{\tilde{c}(k_1^* + \tilde{j}) + \tilde{n}\tilde{I}k_1^*}\left(\frac{\tilde{n}\tilde{I}^2k_1^{*2}k_2^*}{(k_1^* + \tilde{j})^2} - \frac{\tilde{n}\tilde{I}^2k_1^*k_2^*}{(k_1^* + \tilde{j})^3}\right) - \tilde{w} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} - \frac{\tilde{I}k_1^*k_2^*}{(k_1^* + \tilde{j})^2}\right]$$

Therefore, the theorem is proved.

6. Global Stability Analysis

The section examines the global asymptotic stability of the predators extinction equilibrium point $\widehat{E_3}(k_1^*, k_2^*)$ in the equation(3) using the Bendixson-Dulac criterion.

Theorem: 6.1 The globally asymptotically stable for the positive equilibrium point $\widehat{E_3}(k_1^*, k_2^*)$ of equation system(3) is satisfied if $\frac{2k_1^*}{\tilde{v}} > 1$ and $\frac{k_2^*}{k_1^* + \tilde{j}} > \tilde{n}$.

Proof:

In the equilibrium point $\widehat{E_3}(k_1^*, k_2^*)$

$$I(k_1^*, k_2^*) = \frac{1}{k_1^* k_2^*}$$

Clearly, $I(k_1^*, k_2^*)$ is non-negative for both $k_1^* > 0$ and $k_2^* > 0$.

Let $M_1 = k_1^* \left[\tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{k_1^* + \tilde{j}} - \frac{x\hat{R}}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*}\right]$

$$M_2 = k_2^* \left[-\tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{k_1^* + \tilde{j}}\right]$$

$$M(k_1^*, k_2^*) = \frac{\partial M_1}{\partial t} + \frac{\partial M_2}{\partial t}$$

$$= \tilde{w} - \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} - \frac{\tilde{I}k_1^* k_2^*}{(k_1^* + \tilde{j})^2} - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)}\left(\frac{\widetilde{o_2}k_1^*}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*}\right) - \tilde{c} + \frac{\tilde{n}\tilde{I}k_1^*}{(k_1^* + \tilde{j})}$$

$$= \tilde{w}\left(1 - \frac{2k_1^*}{\tilde{v}}\right) - \frac{1}{k_1^* + \tilde{j}}\left(\frac{k_1^* k_2^*}{k_1^* + \tilde{j}} + k_2^* - \tilde{n}k_1^*\right) - \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)}\left(\frac{\widetilde{o_2}k_1^*}{\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*}\right) - \tilde{c}$$

$$< 0$$

If the condition is satisfied $\frac{2k_1^*}{\tilde{v}} > 1$ and $\frac{k_2^*}{k_1^* + \tilde{j}} > \tilde{n}$ then above equation is less than zero.

In the positive quadrant of the $k_1^* - k_2^*$ plane the value of $M(k_1^*, k_2^*)$ do not change the sign and are not identically zero. Thus, by the Bendixson-Dulac criterion, there is no closed curve in the $k_1^* - k_2^*$ plane of R_+^2.

$\widehat{E_3}$ is globally asymptotically stable, thus ensuring system stability.

7. Numerical Simulation

For this model(3), real world data are not accessible as it is not a case study. Consequently, consider using hypothetical data solely to demonstrate the result established in this section. The illustration were produced utilizing MATLAB software. To examine the essential characteristic relevant to the equation(3), one can vary parameters appropriately to confirmed through the assignment of two sets of well-defined parameters. The unique results obtained for each set of values ensure that the characteristics of the model are thoroughly analyzed. The parameter values are, $k_1 = 0.6, k_2 = 0, \tilde{w} = 1.5, \tilde{v} = 1.9, \tilde{j} = 0.72, x = 0.65, \hat{R} = 0.34, \widetilde{o_1} = 0.65, \widetilde{o_2} = 0.41, \tilde{c} = 0.6,$

$\tilde{n} = 0.56, \tilde{I} = 0.72$, in this case $\tilde{w} + \frac{x\hat{R}\widetilde{o_2}k_{1(ii)}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})^2} < \frac{2\tilde{w}k_{1(ii)}}{\tilde{v}} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_{1(ii)})}$ and $\tilde{c} > \frac{\tilde{n}\tilde{I}k_{1(ii)}}{(k_{1(ii)} + \tilde{j})}$. By theorem(5.2), the point $\widehat{E_2}(k_{1(ii)}, 0)$ is stable (see figure:1(a) and 1(b)). Let's consider the following parameter values:

$k_1 = 0.5, k_2 = 0.3, \tilde{w} = 1.9, \tilde{v} = 1.4, \tilde{j} = 0.34, x = 0.045, \hat{R} = 0.64, \widetilde{o_1} = 0.56, \widetilde{o_2} = 0.45, \tilde{c} = 0.04,$

$\tilde{n} = 0.23, \tilde{I} = 0.6$, these values illustrates the effect of harvesting on prey density. The equilibrium points $\widehat{E_3}(k_1^*, k_2^*)$ is asymptotically stable(see figure:2(a) and 2(b)) in the case that the condition are

$\tilde{w} < \tilde{\tilde{c}} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}}{(k_1^* + \tilde{j})}(k_2^* - \tilde{n}k_1^*) - \frac{\tilde{I}k_1^* k_2^*}{(k_1^* + \tilde{j})^2} + \frac{x\hat{R}}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)} - \frac{x\hat{R}\widetilde{o_2}k_1^*}{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}$ and

$$\hat{R} > \frac{(\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)^2}{x[\widetilde{o_2}k_1^* - (\widetilde{o_1}\hat{R} + \widetilde{o_2}k_1^*)]}\left[\frac{1}{\tilde{c}(k_1^* + \tilde{j}) + \tilde{n}\tilde{I}k_1^*}\left(\frac{\tilde{n}\tilde{I}^2 k_1^{*2} k_2^*}{(k_1^* + \tilde{j})^2} - \frac{\tilde{n}\tilde{I}^2 k_1^* k_2^*}{(k_1^* + \tilde{j})^3}\right) - \tilde{w} + \frac{2\tilde{w}k_1^*}{\tilde{v}} - \frac{\tilde{I}k_2^*}{(k_1^* + \tilde{j})} - \frac{\tilde{I}k_1^* k_2^*}{(k_1^* + \tilde{j})^2}\right]$$

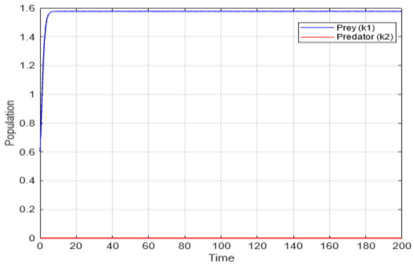

Figure: 1(a) – Population over time in prey-predator model

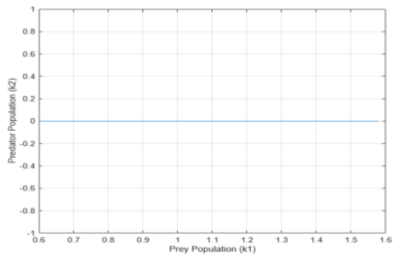

Figure:1(b) – Phase plot on $\widehat{E_2}$

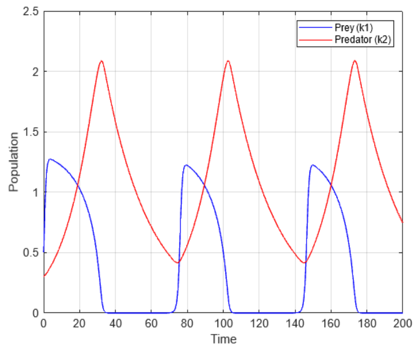

Figure: 2(a) – Population over time in prey-predator model

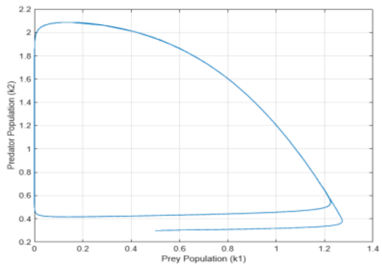

Figure: 2(b) – Phase plot on $\widehat{E_3}$

8. Conclusion

This paper focuses on studying the interactions of the prey-predator model. The Holling type II functional response for both prey and predator is evident in the ecological model. Local stability is examined in the behavior of the model, focusing on the presence of potential steady states. the system's global stability is established through the use of the Bendixson-Dulac criterion in $\widehat{E_2}$. The figure: 1(a) shows the prey absence and the predator population grows exponentially at time t. The figure: 1(b) shows the prey and predator populations in a phase plot. Figure: 2(a) illustrates the increase of the prey and predator populations over time t, while figure:2(b) presents the phase plot at equilibrium point $\widehat{E_3}$.

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