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Stein-type Inequalities for the fractional Fourier Transform

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Abstract: In this manuscript, we investigate the Stein-type inequalities for the fractional Fourier Transform in the classical Lorentz space.

Keywords: Classical Lorentz Space, Stein Inequality, Hardy-Littlewood-Stein Inequality, Fractional Fourier Transform.

introduction

The Fourier Transform of a complex-valued (Lebesgue) integrable function $\varphi(t)\epsilon L_1(\mathbb{R})$ on the real line is the complex valued function $\widehat{\phi}\,(\zeta)$ is defined by the integral \cite{1.} as follows,

$$\widehat{\phi}\,(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta t}\,\phi(t)\mathrm{d}t$$

so that its inverse is given by

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta t}\widehat{\phi}\,(\zeta)\mathrm{d}\,\zeta < \infty$$

The Fourier Transform and its name can be traced back to Jean-Baptiste Joseph Fourier's (1768-1830) publication on heat flow in 1822.

Firstly, Wiener developed the concept of the fractional Fourier transform (FrFT) in 1929 [1]. In 1980, Namias also explored the FrFT [2] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. This transformation is crucial for resolving a number of issues in signal processing, optics, and quantum physics [2,4,5,6,7,8,9,10,11,12]. A variety of mathematical analytic fields have examined the FrFT, which is a generalization of the Fourier transform. FrFT [5,17,18,19,20,21,22,23,24,25] of a function of a function φ $\in L_1(\mathbb{R})$ with parameter α denoted by $(\mathfrak{F}_\alpha\phi)(\zeta) = \widehat{\phi_\alpha}\,(\zeta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathfrak{F}_\alpha\phi)(\zeta) = \widehat{\phi_\alpha}\,(\zeta) = \int_{-\infty}^{\infty} K_\alpha\,(\eta,\zeta)\,\phi(\eta)d\eta$$

where the kernel $K_\alpha\,(\eta,\zeta)$ is given by

$$K_\alpha \;=\begin{cases} C_\alpha e^{\frac{i(\eta^2+\zeta^2)cot\alpha}{2}-\eta\zeta icsc\alpha,}, \alpha \neq n\pi, n \in \mathbb{Z} \\ \dfrac{1}{\sqrt{2\pi}}e^{-\eta\zeta i}, \alpha = \dfrac{1}{2\pi}, \end{cases}$$

and $C_\alpha = \sqrt{\frac{1-icot\alpha}{2\alpha}}$.

The inverse $of\ (\mathfrak{F}_\alpha\phi)(\zeta)$ is given by

$$\phi(\eta) = \int_{-\infty}^{\infty} \overline{K_\alpha\,(\eta,\zeta)}\,(\mathfrak{F}_\alpha\phi)(\zeta)d\zeta$$

and $\overline{K_\alpha\,(\eta,\zeta)} = K_{-\alpha}\,(\eta,\zeta)$.

The Stein inequality of Fourier Transform [6] is defined as follows

$$\|\hat{\phi}\|_{L_{i\prime,j}(\mathbb{R})} \leq \kappa\|\phi\|_{L_{i\prime,j}(\mathbb{R})}$$

where 1 < i < 2, $i'=\frac{i}{i-1}$ and 0 < j ≤ ∞; $L_{i',j}(\mathbb{R})$ is the classical Lorentz space.

In both pure and applied mathematics, the L(i, j)-spaces are quite interesting. The Stein inequality, called as the Hardy-Littlewood-Stein inequality in recognition of the contributions of Hardy and Littlewood [23] or the Fourier inequality, has been thoroughly examined in scholarly works. This inequality has been studied in Lebesque spaces with generic weights by several writers. Lars Erik Persson [24] published one of the first research on Stein's inequality in Lorentz spaces, as far as we know. The goal of this research work is to derive Stein-type inequality for the FrFT of inequality (3).

2 Primary Findings

Let μ be the one dimensional Lebesque measure given in $\mathbb{R}$. Let φ be a measurable mapping defined on $\mathbb{R}$.

A mapping

$$\phi^*(x) = inf\{\sigma: \mu\{y\epsilon\Omega: |\phi(y)| > \sigma\} \leq x\}$$

(4)

is known as the non-increasing rearrangement of the mapping φ. Let $1 < i < \infty$ and $0 < j \leq \infty$. The Lorentz Space $L_{i,j}(\mathbb{R})$ is the is the set of a measurable function φ for which

$$\|\hat{\phi}\|_{L_{i,j}(\mathbb{R})} = \left(\left(x^{\frac{1}{i}}\,\phi^{*}(x)\right)^{j}\frac{dx}{x}\right)^{\frac{1}{j}} < \infty, 0 < \infty, (5)$$

and

$$\|\hat{\phi}\|_{Li,\infty(\mathbb{R})} = \sup_{x>0} x^{\frac{1}{i}}\,\phi^{*}(x) \ < \infty \qquad (6)$$

for $j = \infty$.

Theorem 1. Let $1 < i < 2, \mathrm{i}' = \frac{\mathrm{i}}{\mathrm{i}-1}$ and $0 < j \leq \infty$,

then $\|\hat{\phi}_{\alpha}\|_{L_{i',j}(\mathbb{R})} \leq |\mathrm{C}_{\alpha}|\|\phi\|_{L_{i,j}(\mathbb{R})}$, where φ is a measurable function.

Proof: We have

$$\|\hat{\phi}_{\alpha}\|_{L_{i',j}(\mathbb{R})} = \left(\left(\zeta^{\frac{1}{i'}}\left(\hat{\phi}_{\alpha}\right)^{*}(\zeta)\right)^{j}\frac{d\zeta}{\zeta}\right)^{\frac{1}{j}} <\infty, \qquad 0 < j \leq \infty.$$

Now, we obtain

$$\left(\hat{\phi}_{\alpha}\right)^{*}(\zeta) = inf\left\{\sigma: \mu\left\{\xi\epsilon\Omega: \left|\left(\hat{\phi}_{\alpha}\right)(\xi)\right| > \sigma\right\} \leq \zeta\right\}$$
$$= inf\left\{\sigma: \mu\left\{\xi\epsilon\Omega: \left|\int_{-\infty}^{\infty} K_{\alpha}\,(\eta,\xi)\right| > \sigma\right\} \leq \zeta\right\}$$
$$= inf\left\{\sigma: \mu\left\{\xi\epsilon\Omega: \int_{-\infty}^{\infty}|K_{\alpha}\,(\eta,\xi)||\phi(\eta)|\,d\eta > \sigma\right\} \leq \zeta\right\}$$
$$\leq inf\left\{\sigma: \mu\left\{\xi\epsilon\Omega: |\mathrm{C}_{\alpha}|\int_{-\infty}^{\infty}|\phi(\eta)|\,d\eta > \sigma\right\} \leq \zeta\right\}$$
$$= |\mathrm{C}_{\alpha}|\, inf\left\{\sigma: \mu\left\{\xi\epsilon\Omega: \int_{-\infty}^{\infty}|\phi(\eta)|\,d\eta > \sigma\right\} \leq \zeta\right\}.$$

It implies that

$$\left(\hat{\phi}_{\alpha}\right)^{*}(\zeta) \leq |\mathrm{C}_{\alpha}|\phi^{*}(\zeta).$$

This proves that

$$\|\hat{\phi}_{\alpha}\|_{L_{i',j}(\mathbb{R})} \leq |\mathrm{C}_{\alpha}|\|\phi\|_{L_{i,j}(\mathbb{R})}.$$

This is called Stein-type inequalities for the fractional Fourier transform.

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