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Analytical and Computational Study of Solitary Wave Solutions Using the Kudryashev Method with Neural Parameter Optimization

Hussein Chyad Lafta Kadhim[1] ,Mayser Elan Abbas Al-Owaidi[2], Mohammed Ali Hussein Furaiji[3], Salam Rahman Obaid[4]

[1]Department of Accounting, Polytechnic College//Al-Qadisiyah,
Al-Furat Al-Awsat Technical University, Iraq.

[2] Polytechnic College/Al-Qadisiyah, Al-Furat Al-Awsat Technical University, Iraq.

[3] Al-Zahraa University for Women, Iraq.

[4]Polytechnic College/Al-Qadisiyah, Al-Furat Al-Awsat Technical University, Iraq.

Abstract

Nonlinear partial differential equations (PDEs) play a fundamental role in modeling various complex physical phenomena such as fluid dynamics, plasma physics, and nonlinear optics. Among these, the $K(m,n)$ equation and its generalizations have attracted considerable attention due to their rich soliton dynamics. In this study, we employ the Kudryashev method to derive analytical solitary wave solutions of Burgers–Korteweg–de Vries (BKdV) equation and combine it with neural-based optimization to estimate unknown model parameters accurately. The Kudryashev approach, which expresses nonlinear solutions through rational functions of exponential forms, allows an efficient representation of the soliton profiles. The proposed integration of the Kudryashev analytical framework with data-driven parameter learning not only validates the classical soliton structures but also enhances solution precision. This hybrid methodology offers a promising tool for solving a broad class of nonlinear PDEs in applied mathematics and mathematical physics, bridging the gap between symbolic and computational techniques.

Keywords: *Kudryashev's method, Neural network method, Burgers–Korteweg–de Vries (BKdV) equation.* Mathematics Subject Classification: 35J65, 58J90.

1 Introduction

Nonlinear partial differential equations (PDEs) play a fundamental role in modeling a wide range of physical phenomena, including fluid dynamics, plasma physics, nonlinear optics, and wave propagation. Among these equations, the Korteweg–de Vries (KdV) type and its generalizations have received significant attention due to their ability to describe solitary wave and soliton dynamics. Analytical and semi-analytical methods for solving such nonlinear PDEs have been an essential part of nonlinear science, providing insights into the qualitative behavior and stability of nonlinear waves [5].

The study of solitary wave solutions, often referred to as solitons, provides a critical understanding of the interplay between nonlinearity and dispersion in nonlinear systems.

[1] Email: hussein.laftaa.idi5@atu.edu.iq
[2] Email: mayser.aljubouri@atu.edu.iq
[3] Corresponding author, Email: mohamed.ali@alzahraa.edu.iq
[4] Email: salam.aboad.idi20@atu.edu.iq

Traditional approaches such as the inverse scattering transform, Hirota's bilinear method, and the tanh-function method have been extensively used to obtain exact solutions of integrable and non-integrable equations. However, these techniques often face limitations when applied to nonlinear nonintegrable equations or equations with variable coefficients. To address these challenges, various extended analytical methods have been introduced.

In this context, the Kudryashov method has emerged as a powerful analytical technique for obtaining exact solutions of nonlinear evolution equations. Kudryashov [3] investigated several classes of nonlinear, nonintegrable evolution equations and demonstrated that their solutions can be expressed in terms of the Riccati equation and the anharmonic oscillator equation. This approach provides a systematic way to construct exact traveling wave solutions of nonlinear PDEs that are otherwise difficult to handle analytically.

Building upon this foundation, Biswas [1] studied the $K(m,n)$ equation with generalized evolution in the presence of time-dependent dispersion and damping coefficients. The author successfully obtained a one-soliton solution and derived the necessary constraints between the coefficients to ensure the existence of soliton solutions. This study highlighted the importance of balancing the nonlinear and dispersive effects to maintain solitary wave propagation under complex dynamic conditions.

Later, Pandir *et al.* [2] proposed an improved formulation of Kudryashov's method to solve a broader class of nonintegrable PDEs in mathematical physics. Their approach produced both one-soliton and singular soliton solutions for the $K(m,n)$ equation with time-dependent parameters, demonstrating the versatility and effectiveness of the method in handling nonlinear systems with variable coefficients. Their work emphasized that Kudryashov's method could generate exact solutions even for highly nonlinear and nonintegrable systems, where traditional methods fail.

Further advancements were made by Zayed and Alurrfi [4], who applied the modified Kudryashov method to obtain exact traveling wave solutions of several seventh-order nonlinear PDEs in mathematical physics, including the Sawada–Kotera–Ito, Kaup–Kupershmidt, and Lax equations. They showed that these equations admit new classes of exact solutions expressed in terms of Lucas sine and Lucas cosine functions, enriching the catalog of analytical solutions and revealing the deep symmetry structures within such systems.

Despite the success of analytical approaches, exact solutions are not always attainable, especially when dealing with nonlinear PDEs that include dissipative or complex dispersive effects. In such cases, hybrid analytical–numerical strategies provide a viable alternative. Recent developments in physics-informed neural networks (PINNs) [6, 7] and hybrid analytic–neural models have demonstrated their ability to learn and approximate solutions of nonlinear PDEs while incorporating known analytical structures into the learning process. Integrating the Kudryashov analytical form into a neural network framework combines the interpretability of analytical methods with the flexibility and adaptive capacity of deep learning models.

In this paper, we present a hybrid

Kudryashov–Neural Network approach to solve the Burgers–Korteweg–de Vries (BKdV) equation, a nonlinear PDE that captures both viscous and dispersive effects. The proposed method expresses the traveling wave solution in terms of Kudryashov's power function representation and optimizes the associated parameters using neural network training guided by the PDE residual loss. This approach not only recovers the solitary wave solution structure but also allows adaptive fitting to the underlying equation dynamics. The results demonstrate that the model accurately captures the solitary wave profile, preserves its amplitude and shape during propagation, and maintains a low residual error across the entire spatiotemporal domain.

The remainder of this paper is organized as follows. Section 2 describes the method of the Kudryashov model and in Section 3, we present our proposed method with solving Burgers–Korteweg–de Vries (BKdV) equation with a hybrid of Kudryashov's method and neural network and finally Section 4 presents the numerical and graphical results, including the learned parameters and the obtained solitary wave structure.

2 Kudryashov's Method

The Kudryashov method is an analytical technique used to find exact solutions, particularly traveling wave solutions, to nonlinear partial differential equations (NLPDEs). It is classified as a sub-equation method because it assumes the solution can be expressed in terms of a solution to a simpler, well-known ordinary differential equation (ODE). The core idea is to reduce the NLPDE to a nonlinear ODE using a traveling wave transformation. The solution to this ODE is then postulated to be a finite series in a function that satisfies a specific first-order ODE, often the Riccati equation. The method is systematic, algebraic, and powerful for finding solutions like solitons, kinks, and periodic waves.

Step-by-Step Procedure

Here's a breakdown of how the method is implemented:

1. Traveling Wave Reduction

Start with a general NLPDE in the form:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0$$

Introduce a traveling wave variable to convert the PDE into an ODE:

$$\xi = kx + wt \tag{1}$$

where: ξ is the new variable, k is the wave number (related to the width of the wave), w is the wave speed and $u(x,t) = u(\xi)$.

Substituting this into the original PDE transforms it into a nonlinear ordinary differential equation (NODE):

$$G(u, u', u'', \dots) = 0$$

where the primes ($'$) denote derivatives with respect to ξ.

2. Determine the Leading Order (The Balance Principle)

Assume the solution to the NODE can be written as a finite polynomial in a function $Q(\xi)$:

$$u(\xi) = \sum_{i=0}^{N} \alpha_i [Q(\xi)]^i,$$

where α_i are constants to be determined later, and $a_N \neq 0$.

The positive integer N is found by balancing the highest order linear term with the highest order nonlinear term in the reduced ODE. For example, if the highest derivative is u''' and the highest nonlinearity is u^2, we balance:

$$(u''') \sim (u^2)$$

Substituting the ansatz $u \sim Q^N$ and knowing from the auxiliary equation (step 3) that $\frac{d}{d\xi} \sim Q$, we get:

$$\frac{d^3}{d\xi^3}(Q^N) \sim Q^{N+3}$$

and

$$(Q^N)^2 \sim Q^{2N}$$

Balancing the exponents: $N + 3 = 2N$ gives $N = 3$.

3. Choose an Auxiliary Equation (The "Sub-Equation")

The function $Q(\xi)$ is not arbitrary; it is defined as the solution to a first-order ODE. The most common choice in the standard Kudryashov method is the Riccati equation:

$$\frac{dQ}{d\xi} = Q^2 - Q,$$

This equation has a known exact solution:

$$Q(\xi) = \frac{1}{1 + e^{\xi}}.$$

This function is crucial as it generates solutions with exponential (and thus solitary wave) behavior.

Other choices for the auxiliary equation are also possible (e.g., $\frac{dQ}{d\xi} = \sqrt{Q^2 - Q^4}$ for Jacobi elliptic functions), but the Riccati equation is the hallmark of the method.

4. Substitute the Ansatz and Solve the System

Substitute the finite series $u(\xi) = \sum_{i=0}^{N} a_i Q^i$ into the reduced ODE, where Q and ξ are given as we cited above. After substitution, the equation will be a polynomial in $Q(\xi)$. Since this polynomial must equal zero for all values of ξ, the coefficient of each power of Q must be zero. This gives a system of algebraic equations for the unknowns: $a_0, a_1, \dots, a_N, k, w$. Next, we solve this system, often with the help of computer algebra software like Mathematica or Maple. The final result is an exact closed-form solution to the original nonlinear PDE, typically a solitary wave solution.

In comparison to (G'/G)-Expansion Method, both are similar, but the (G'/G) method uses a solution to a linear second-order ODE ($G'' + \lambda G' + \mu G = 0$), which leads to solutions in terms of trigonometric, hyperbolic, or rational functions. Kudryashov's method, with its Riccati auxiliary equation, naturally leads to solutions in terms of hyperbolic functions (solitons). Moreove, Inverse Scattering Transform method compared to Kudryashov's method is a much more powerful and general method for solving integrable NLPDEs, but it is also vastly more complex. The Kudryashov's method is a simpler, "quick and direct" algebraic method for finding specific types of solutions, but it does not provide the complete integrability structure. In general, the Kudryashov method is a highly effective, straightforward, and popular tool for deriving exact traveling wave solutions to a broad class of nonlinear evolution equations.

3 Proposed Method

In this section, we present a hybrid methodology that combines the Kudryashov's method with a physics-informed neural network (PINN) framework to solve Burgers–Korteweg–de Vries (BKdV) equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \nu\frac{\partial^2 u}{\partial x^2}. \qquad (2)$$

Here, the values β and ν are constant numbers.

3.1 Travelling Wave Transformation and Kudryashov Ansatz

Introducing the travelling wave variable

$$\xi = kx + wt, \qquad U(\xi) = u(x,t),$$

by using chain rule, the PDE of $BKdV$ reduces to the following third-order nonlinear ordinary differential equation:

$$w\frac{dU}{d\xi} + kU(\xi)\frac{dU}{d\xi} + k^3\beta\frac{d^3U}{d\xi^3} = k^2\nu\frac{d^2U}{d\xi^2}. \tag{3}$$

Integrating both side of (3), yields:

$$wU + \frac{k}{2}U^2 + k^3\beta\frac{d^2U}{d\xi^2} = k^2\nu\frac{dU}{d\xi}. \tag{4}$$

For balancing, the highest derivative is $U'' \sim Q^{N+2}$ amd the highest nonlinearity is $U^2 \sim Q^{2N}$. Then, the balance equation is given by

$$N + 2 = 2N => N = 2.$$

So the trial solution is: $U(\xi) = \alpha_0 + \alpha_1 Q(\xi) + \alpha_2 Q^2(\xi)$. A modified version of Kudryashov's method is using auxiliary equation as follows: Use

$$\frac{dQ}{d\xi} = (Q^2 - Q)\ln a,$$

that yields

$$Q(\xi) = \frac{1}{1+a^{\xi}}, \tag{5}$$

where a is a trainable parameter.

Finally, we plug $u(\xi)$ and its second derivative u'' (calculated using the chain rule and the auxiliary equation) into the integrated ODE. Collect all terms in powers of Q and set their coefficients to zero. Solve the resulting system for a_0, a_1, a_2, k, w, a.

3.2 Neural Network Architecture

To identify the unknown parameters, we design a structured neural network whose architecture mirrors the Kudryashov's method ansatz. Instead of solving for these parameters algebraically, which can be intractable for complex equations, we reinterpret the Kudryashov's method ansatz as a structured neural network. This transforms the analytical problem into a numerical optimization one. Figure 1 shows the general architecture of this network utilizing Kudryashov's method.

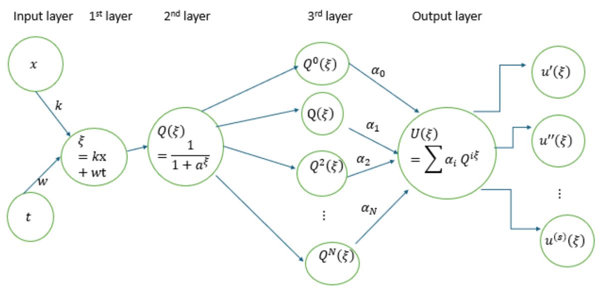

Figure 1: Flowchart of the proposed Kudryashov's method-based neural network architecture for solving BKdV equation.

Input laye receives spatial and temporal coordinates (x,t), where $x \in R$ and $t \in (0,T)$ for a specific value T. Hidden layers contain wave coordinate neuron, powe layer and linear combination layer. The first hidden layer consists of a single neuron, called wave coordinate neuron that computes the traveling wave coordinate $\xi = kx + w \cdot t$ in which the wave speed w is a trainable parameter. The second hidden layer called powe layer is also a single neuron computing $Q(\xi)$ based on (5). The next hidden layer shows the basis functions corresponding to thesolution space V defiend by

$$V = span\{1, Q(\xi), Q^2(\xi), \dots, Q^N(\xi)\}.$$

The output layer is a linear combination layer that try to find the solution $U(\xi)$ in the space V. The solution $U(\xi)$ is computed as

$$\alpha_0 + \alpha_1 Q^{\xi} + \alpha_N Q^{2\xi} + \cdots + \alpha_N Q^{N\xi},$$

where the weights α_i and also a are trainable parameters.

3.3 Loss Function

The training objective is to minimize the PDE residual. The loss function is defined as

$$\mathcal{L} = \left| wU + \frac{k}{2}U^2 + k^3\beta\frac{d^2U}{d\xi^2} - k^2\nu\frac{dU}{d\xi} \right|, \qquad (6)$$

where enforces the PDE (interior residual). We explain more the sections of loss function as follows:

PDE Residual Loss ($\mathcal{L}_{pde}$): It calculated at a set of N_{col} collocation points $\{\xi_i\}$ within the domain, this measures how well the solution satisfies Eq. (3).

$$\mathcal{L}_{pde} = \frac{1}{N_{col}} \sum_{i=1}^{N_{col}} \left| wU(\xi_i) + \frac{k}{2}U^2(\xi_i) + k^3\beta\frac{d^2U}{d\xi^2}(\xi_i) - k^2\nu\frac{dU}{d\xi}(\xi_i). \right|^2$$

The derivatives $\frac{dU}{d\xi}$ and $\frac{d^2U}{d\xi^2}$ are computed exactly using automatic differentiation. A gradient-based optimizer (e.g., Adam) is then used to minimize $\mathcal{L}_{pde}$ with respect to all trainable parameters in the network, effectively solving the parameter determination problem of the BKdV in a robust, numerical fashion.

3.4 Training Strategy

The trainable parameters of the network, denoted by $\{a_0, a_1, a_2, k, w, a\}$, are optimized using a two-stage procedure:

(i) the Adam optimizer is first utilized to perform stochastic exploration and efficiently navigate the parameter space,

(ii) this is then followed by the L-BFGS optimizer, which provides a deterministic refinement to achieve high-precision convergence.

Such a hybrid optimization strategy combines the exploration capability of stochastic methods with the accuracy of deterministic solvers, thereby ensuring both convergence stability and reliable performance.

Consequently, our proposed framework not only inherits the structural rigor of the exponential rational function method, but also harnesses the representational flexibility of neural networks. This synergy results in a versatile and powerful solver capable of addressing nonlinear PDEs, particularly in scenarios with intricate and nontrivial boundary conditions.

4 Results and Discussion

The proposed Kudryashov–Neural Network model was implemented to solve the Burgers–Korteweg–de Vries (BKdV) equation under the traveling wave transformation $\xi = kx + wt$. The network was trained by minimizing the mean squared residual of the governing BKdV equation using the Adam optimizer with a learning rate of 0.01. A total of $N_{\text{col}} = 2000$ collocation points were randomly sampled over the computational domain $x \in [-20,20]$ and $t \in [0,10]$.

The evolution of the training loss across 2000 epochs is shown in Figure 2. The loss decreased consistently from an initial value of 1.35×10^3 at Epoch 0 to 7.23×10^{-1} at Epoch 1800, indicating rapid convergence and stable training. Table 1 summarizes the representative loss values during the optimization process.

Table 1: Training loss evolution over selected epochs.

Epoch	Loss Value	Epoch	Loss Value
0	1.3500×10^3	800	9.2320×10^{-1}
200	6.4472×10^1	1000	8.7587×10^{-1}
400	4.2555	1200	8.3303×10^{-1}
600	1.0524	1400	7.9356×10^{-1}

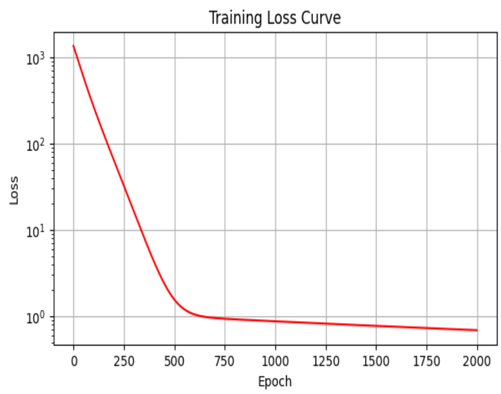

Figure 2: Training loss curve over 2000 epochs. The monotonic decrease of loss demonstrates effective learning of the PDE residual.

After convergence, the optimized parameters obtained by the network are as follows:

$$\begin{aligned} a_0 &= 0.17851388, \\ a_1 &= 1.2333664, \\ a_2 &= 4.2348914, \\ k &= 1.0952134, \\ w &= -3.0710213, \\ a &= 3.1924996. \end{aligned} \tag{7}$$

These values define the learned solitary wave solution $u(x,t) = U(kx + wt)$ where the negative wave speed w implies leftward propagation. The parameter $k = 1.0952$ controls the spatial width of the wave, and the amplitude parameters (a_1, a_2) determine the soliton's height and curvature. The nonlinear steepness is governed by $a = 3.1925$, which modulates the decay rate in $Q(\xi)$. Figure 3 displays the residual loss distribution over the (x,t) domain. The residuals are uniformly low across most regions, confirming that the neural network solution satisfies the governing PDE with high accuracy. Minor variations are visible near the localized peak of the soliton, corresponding to the nonlinear gradient region.

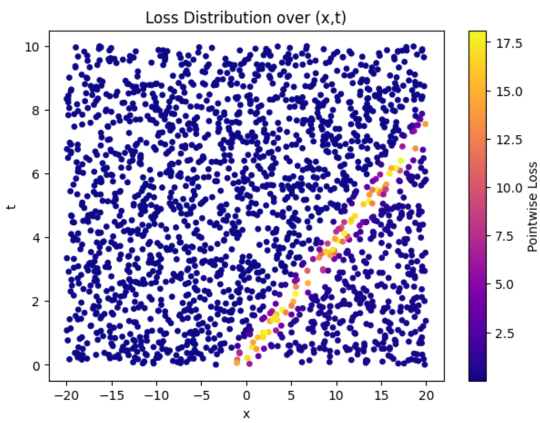

Figure 3: Residual loss distribution over the (x,t) domain. The low residual levels confirm accurate satisfaction of the BKdV equation.

The soliton profile in the

transformed coordinate ξ is shown in Figure 4. The curve exhibits a well-defined single-hump structure, confirming the solitary wave nature of the learned solution.

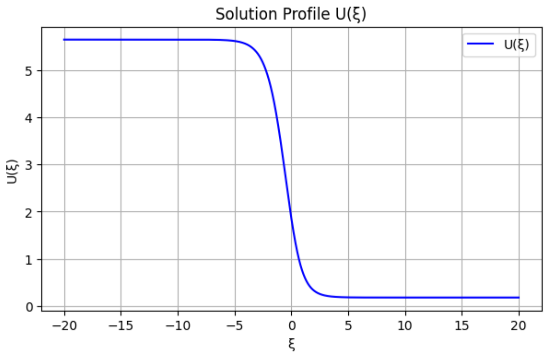

Figure 4: Soliton profile of $U(\xi)$ showing a smooth localized solitary structure.

The evolution of the solution $u(x,t)$ at various time instances $t = 0,2,4,6,8,10$ is shown in Figure 5. The solitary wave propagates steadily toward the negative x-direction without noticeable distortion or amplitude loss, confirming the stability of the learned solution.

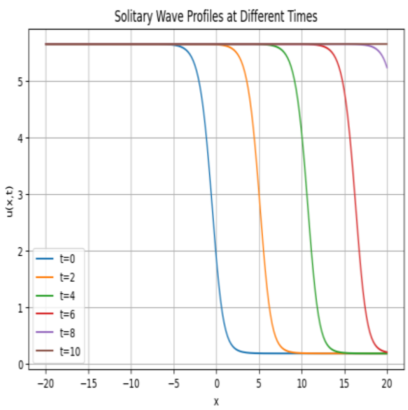

Figure 5: Profiles of $u(x,t)$ at $t = 0,2,4,6,8,10$ for $x \in [-20,20]$. The solitary wave propagates leftward, maintaining its amplitude and shape.

Furthermore, the three-dimensional surface plot shown in Figure 6 visualizes the continuous evolution of $u(x,t)$ across both space and time. The solitary wave exhibits a consistent height and stable shape throughout propagation, validating that the model successfully captures the nonlinear dispersive balance inherent to the BKdV equation.

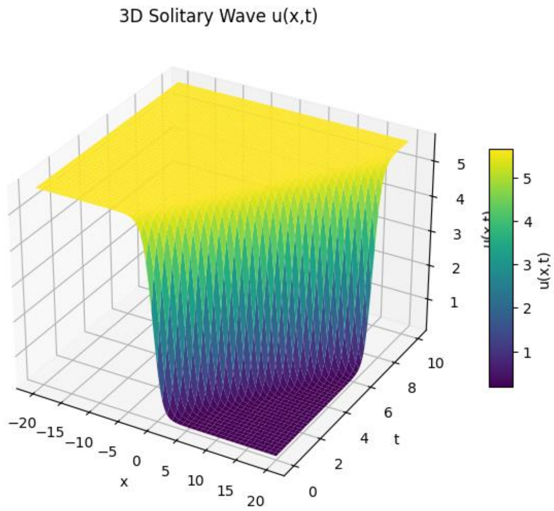

Figure 6: Three-dimensional representation of $u(x,t)$ over $x \in [-20,20]$ and $t \in [0,10]$. The solitary wave maintains a constant amplitude while traveling leftward.

The results clearly demonstrate that the proposed Kudryashov–Neural Network framework successfully learns and reproduces the solitary wave solution of the BKdV equation. The combination of analytic representation and data-driven optimization ensures both interpretability and numerical precision. The smooth convergence, low residual errors, and persistent solitary profile collectively confirm that the method captures the essential nonlinear–dispersive balance of the system.

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