Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 47 No. 01
January 2026

Research on Vehicle Vibration with 1/4 Car Model using Lagrange II equation

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Abstract

Research on vehicle vibration using the 1/4 car model based on the Newton-Euler method has been widely applied. However, in many cases, vehicle vibrations are influenced by suspension structural parameters such as the length of control arms and the force transmission path of the elastic and damping components, which are not considered in the Newton-Euler method. To evaluate the impact of these factors on vehicle vibration, the 1/4 car vibration model needs to incorporate the structural parameters of the suspension system. This study formulates the differential equations of the suspension system dynamics using the Lagrange II equation. The research investigates the effects of parameters including the inclination angle of the spring-damper axis, the mounting positions of the spring axis on the control arm and the vehicle body, spring stiffness, and damping coefficient on vehicle vibrations in the horizontal plane when traversing uneven road surfaces. The research results demonstrate the reliability of the theoretical model.

Keywords: Independent suspension, double wishbone suspension, ride comfort, vehicle vibration, spring, damper.

1. Introduction

In the process of designing and optimizing suspension for automobiles, vibration research is a key factor to ensure smoothness, safety and stability when the vehicle is in motion. In order to analyze the oscillations effectively, it is necessary to build a simple mathematical model that accurately reflects the kinetic properties of the system [1]. Among the commonly used models, the 1/4 car model is considered a suitable option for studying vertical oscillations, because it reduces the computational complexity and retains the necessary accuracy in the oscillation analysis of the suspension system [2].

When establishing motion differential equations for mechanical systems such as a 1/4 car model, there are two basic methods that are commonly used, the Newton-Euler method and the Lagrange method type II. The Newton-Euler method, based on Newton's II law, conducts a direct analysis of each force acting on each individual mass in the system. This method allows for a clearer visualization of the forces acting, but often becomes cumbersome when the system has many degrees of freedom or complex constraints [3,4,5,6].

In contrast, the Lagrange method type II allows the establishment of dynamic equations in a general way through the kinetic energy and potential energy of the system. This greatly simplifies the modeling process, which is convenient when building mechanical systems with multiple degrees of freedom. The Lagrange method not only simplifies the modeling process, but also clearly demonstrates the kinetic energy, potential energy, and dissipation in the oscillatory system [3,4,5,6].

With the goal of improving the smoothness and vibration stability of the car, the parameters that affect the vibration of the vehicle such as spring stiffness, damping drag coefficient, tire stiffness, and the placement of the elastic and damping elements are clearly analyzed. Through the establishment of the dynamic differential equation of the double-arm independent suspension system by the Lagrange method and the analysis of the equation system by Matlab - simulink software, the results of this study form the basis for the improvement of the suspension system design.

2. Building a dynamic model of dual arm independent suspension

2.1. Proposed model tives

- The road surface is considered absolutely hard, there is no deformation when the vehicle is moving.

- Only consider the elasticity of the tire, ignoring the damping force for simplicity in the calculation process.

- Only consider the normal jet force of the line acting on the wheel, ignoring the horizontal and vertical forces.

- The mechanical parts in the system are absolutely hard, not deformed when working,

- Roller wheels do not slip on the pavement, ensuring continuity of movement.

- Because the displacement of the outer end of the lower suspension (where it is caught with the wheel rotation cam) is relatively small, it is considered that the outer end of the lower suspension is stationary when the wheel passes through the bump.

- See the weight of the wheel concentrated at the center of the wheel.

- Assume the weight of the upper suspension, lower suspension, and push bar concentrated at the link point between the push bar and the lower horizontal strike.

- Assume the mass of the entire triangular pass concentrated at the junction between the pushbar and the triangular pass.

- Assume the mass of the spring bar and the damper concentrated at the junction between the triangular pass and the spring bar.

2.2. Development a survey model

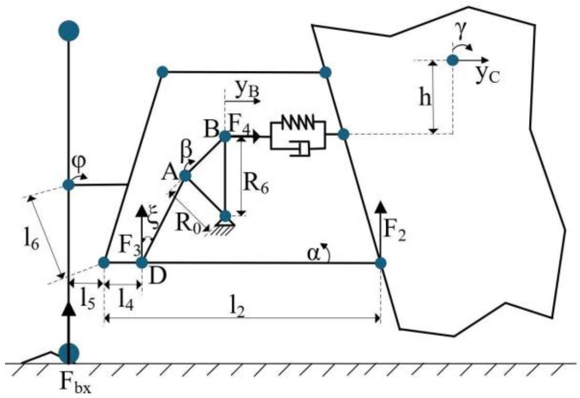

Figure 1. General dynamics model on 2-stroke guided type independent suspension

Table 1. Reference symbols and parameters

Symbol	Value	Unit
l_5	0.15	m
l_6	0.21	m
h	0.5	m
l_2	0.42	m
l_w	0.330	m
l_4	0.12	m
ζ	15	degree
R_0	0.12	m
R_6	0.15	m
b_w	0.185	m
b_v	0.9	m
l_v	1.6	m
m_w	6.5	kg
k_w	190000	N/m
m_2	15	kg
m_3	0.8	kg
m_4	13	kg
m_5	530	kg
k_{lx}	125370	N/m
c_{lx}	6090	Ns/m

The equation of motion of a car written in terms of deductible coordinates is introduced from the Lagrange equation of type 2 in the general form [3,4,5,6]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial \Pi}{\partial q_i}+\frac{\partial \Phi}{\partial \dot{q}_i}=Q_i^* \qquad (1)$$

Where T, Π, Φ are the kinetic energy, potential energy and loss functions of the mechnical system, respectively;   is the force of deduction that cannot correspond to the deductive coordinates of qi [3,4,5,6].

❖ Determine the kinetic energy of the dynamics system:

- In the deductible coordinate system φ:

$$T_1=\frac{1}{2}.J_w.\dot{\varphi}^2=\frac{1}{2}.\left(b_w^2+l_w^2\right).\frac{m_w}{12}.\dot{\varphi}^2$$

Where b_w, l_w, are wheel width, wheel height, respectively; m_w - the mass of the wheel.

The derivative of kinetic energy according to the coordinates of deterioration φ:

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\varphi}}\right)=\left(b_w^2+l_w^2\right).\frac{m_w}{12}.\ddot{\varphi}\,;\frac{\delta T}{\delta \varphi}=0 \qquad (2)$$

- In the deductible coordinate system α:

$$T_2=\frac{1}{2}.J_2.\dot{\alpha}^2=\frac{1}{2}.\frac{1}{3}.m_2.l_4^2.\dot{\alpha}^2$$

Where m_2 is the concentrated volume of the upper suspension, the lower suspension and the thrust bar.

The derivative of kinetic energy according to the coordinates of deterioration α:

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\alpha}}\right)=\frac{1}{3}.m_2.l_4^2.\ddot{\alpha}\,;\frac{\delta T}{\delta \alpha}=0 \qquad (3)$$

- In the deductible coordinate system β:

$$T_3=\frac{1}{2}.J_3.\dot{\beta}^2+\frac{1}{2}.J_4.\dot{\beta}^2=\frac{1}{2}.\frac{1}{18}.m_3.R_0^2.\dot{\beta}^2+\frac{1}{2}.\frac{1}{3}.m_4.R_6^2.\dot{\beta}^2$$

Where m_3 is the concentrated volume of the triangle pass.

The derivative of kinetic energy according to the coordinates of deterioration β :

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\beta}}\right)=\left(\frac{1}{3}.m_3.R_0^2+\frac{1}{18}.m_4.R_6^2\right).\ddot{\beta};\frac{\delta T}{\delta \beta}=0 \quad (4)$$

- In the deductible coordinate system y_B :

$$T_4=\frac{1}{2}.m_4.\dot{y}_B^2$$

Where m_4 is the concentrated mass of the spring rod.

The derivative of kinetic energy according to the coordinates of deterioration y_B :

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{y}_B}\right)=m_4.\ddot{y}_B;\frac{\delta T}{\delta y_B}=0 \quad (5)$$

- In the deductible coordinate system y_C :

$$T_5=\frac{1}{2}.m_5.\dot{y}_C^2$$

Where m_5 is the volume of the body.

The derivative of kinetic energy according to the coordinates of deterioration y_C :

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{y}_C}\right)=m_5.\ddot{y}_C;\frac{\delta T}{\delta y_C}=0 \quad (6)$$

- In the deductible coordinate system γ :

$$T_6=\frac{1}{2}.J_4.\dot{\gamma}^2=\frac{1}{2}.\left(b_v^2+l_v^2\right).\frac{m_5}{12}.\dot{\gamma}^2$$

Where b_v, l_v are respectively the width and length of the car body.

The derivative of kinetic energy according to the coordinates of deterioration γ :

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\gamma}}\right)=\left(b_v^2+l_v^2\right).\frac{m_5}{12}.\ddot{\gamma};\frac{\delta T}{\delta \gamma}=0 \quad (7)$$

❖ Determination of the potential energy of the dynamics system

- In the deductible coordinate system φ :

$$\Pi_1=m_w.g.l_6.\sin\varphi+\frac{1}{2}.k_w.\left(h_{mm}-l_6.\sin\varphi\right)^2$$

Where: k_w - the elastic coefficient of the wheel; h_{mm} – the height of bumpy.

The derivative of the potential energy according to the coordinated of deterioration φ :

$$\frac{\delta \Pi}{\delta \varphi}=m_w.g.l_6.\cos\varphi-k_w.h_{mm}.l_6.\cos\varphi+k_w.l_6^2.\sin\varphi \quad (8)$$

- In the deductible coordinate system α :

$$\Pi_2=m_2.g.l_4.\sin\alpha$$

The derivative of the potential energy according to the coordinated of deterioration α :

$$\frac{\delta \Pi}{\delta \alpha}=m_2.l_{4.}\cos\alpha$$

- In the deductible coordinate system β :

$$\Pi_3=m_3.g.R_0.\sin\beta-m_4.g.R_6.\sin\beta$$

The derivative of the potential energy according to the coordinated of deterioration β :

$$\frac{\delta \Pi}{\delta \beta}=\left(m_3.g.R_0-m_4.g.R_6\right).\cos\beta \quad (10)$$

- In the deductible coordinate system y_B :

$$\Pi_4=\frac{1}{2}k_{lx}.y_B^2$$

Where k_{lx} is the spring hardness coefficient.

The derivative of the potential energy according to the coordinated of deterioration y_B :

$$\frac{\delta \Pi}{\delta y_B}=k_{lx}.y_B \quad (11)$$

- In the deductible coordinate system y_C :

$$\Pi_5=0$$

The derivative of the potential energy according to the coordinated of deterioration y_C :

$$\frac{\delta \Pi}{\delta y_C}=0 \quad (12)$$

- In the deductible coordinate system γ :

$$\Pi_6=-m_5.g.h.\cos\gamma$$

The derivative of the potential energy according to the coordinated of deterioration γ :

$$\frac{\delta \Pi}{\delta \gamma}=m_5.h.\sin\gamma \quad (13)$$

❖ Determination of the loss of the dynamics system: The loss of the system is determined through the loss of suspension damping [3,4,5,6,7].

- In the deductible coordinate system y_C :

$$\phi=\frac{1}{2}.c_{lx}.\dot{y}_C^2$$

Where c_{lx} is the damping drag coefficient.

The derivative of the potential energy according to the coordinated of deterioration γ :

Potential energy derivative according to the coordinates of deterioration y_C :

$$\frac{\delta \phi}{\delta \dot{y}_B} = c_{lx}.\dot{y}_B \qquad (14)$$

Broad deterioration forces of the dynamics system include $Q_\varphi, Q_\alpha, Q_\beta, Q_{yB}, Q_{yC}, Q_\gamma$ and they are calculated based on principle of possible movement [3,4,5,6], thus:

- $Q_\varphi = \frac{\delta A}{\delta \varphi} = m_v.l_6.\ddot{\varphi}.l_5 = F_{bx} \times l_5 \left(N.m\right)$

Where m_v is the 1/4 of the car mass.

- $Q_\alpha = \frac{\delta A}{\delta a} = F_{bx} \times l_5 \left(N.m\right)$
- $Q_\beta = \frac{\delta A}{\delta \beta} = \frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0 \left(N.m\right)$
- $Q_{y_B} = \frac{\delta A}{\delta y_B} = \frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6} \left(N\right)$
- $Q_{y_C} = \frac{\delta A}{\delta y_C} = \frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6} - k_{lx}.y_B - c_{lx}.\dot{y}_B \left(N\right)$
- $Q_\gamma = \frac{\delta A}{\delta \gamma} = \left(\frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6} - k_{lx}.y_B - c_{lx}.\dot{y}_B \right).h \left(N.m\right)$

Where δA - Possible movement; δy_C - body displacement in the Oy direction; δy_B - spring displacement in the Oy direction; $\delta\varphi, \delta\alpha, \delta\beta, \delta\gamma$ - displacement of the car body corresponding to the angle: $\varphi, \alpha, \beta, \gamma$; F_{bx} - the agitated force from the road surface applied to the wheels.

Replace the kinetic energy derivatives T from equations (2), (3), (4), (5), (6), (7); deriving the potential energy function Π from equations (8), (9), (10), (11), (12), (13); the dissipation function Φ from equation (14) into the general equation (1), the system of dynamical differential equations of the mechanism is obtained according to each system of deductible coordinates:

$$\left[\left(b_w^2 + l_w^2\right).\frac{m_w}{12} - m_v.l_6.l_5 \right]\ddot{\varphi} + m_w.g.l_6.\cos\varphi$$
$$-k_w.h_{mm}.l_6.\cos\varphi + k_w.l_6^2.\sin\varphi = 0$$

$$\frac{1}{3}.m_2.l_4^2.\ddot{\alpha} + m_2.l_4.\cos\alpha = F_{bx}.l_5$$

$$\left(\frac{1}{3}.m_3.R_0^2 + \frac{1}{18}.m_4.R_6^2\right).\ddot{\beta} + \left(m_3.g.R_0 - m_4.g.R_6\right).\cos\beta$$
$$= \frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0$$

$$m_4.\ddot{y}_B + k_{lx}.y_B + c_{lx}.\dot{y}_B = \frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6}$$

$$m_5.\ddot{y}_C = \frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6} - k_{lx}.y_B - c_{lx}.\dot{y}_B$$

$$\left(b_v^2 + l_v^2\right).\frac{m_5}{12}.\ddot{\gamma} + m_5.h.\sin\gamma =$$
$$\left(\frac{\frac{F_{bx}.l_5}{l_4}.\cos\xi.R_0}{R_6} - k_{lx}.y_B - c_{lx}.\dot{y}_B \right).h$$

3. Investigation and evaluation of results

After establishing the differential equation system of motion of the double-arm independent suspension system by the Lagrange type II method, the oscillation characteristics of the system are investigated and analyzed by numerical method through Matlab/Simulink software. Conduct a case study of the wheel passing through the sine pattern bump and the stepped bump with the variable push bar placement angle. The study is not only to evaluate the vibration of the car under the influence of road conditions, but also to evaluate the effectiveness of design parameters such as spring stiffness, damping coefficient, or the geometry of suspension components.

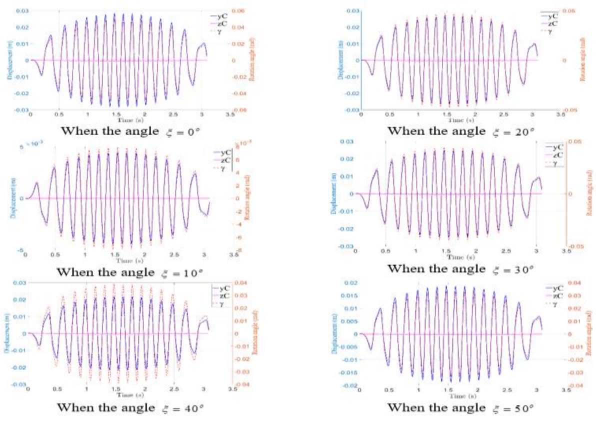

Figure 2. The survey results fluctuate vertically, horizontally, and the angle of rotation of the car body when the wheels pass through the sine pattern.

Figure 2 presents the simulation results of body oscillation when passing through a sinusoidal bump road, with the survey quantities including: body rotation angle (γ), vertical displacement (z_c) and horizontal displacement (y_c). The survey results show that when the tilt angle of the push bar is 00 (0 rad), the body rotation angle is 2,860 (0.05 rad), the horizontal oscillation is 0.028m, and the vertical oscillation is almost 0. When changing the tilt angle of the push bar gradually increasing to 500 (0.87 rad), the body rotation angle decreases to 1,710 (0.03 rad), the horizontal oscillation is 0.018m, the vertical oscillation is almost 0 but the amplitude of the decrease is smaller than the original.

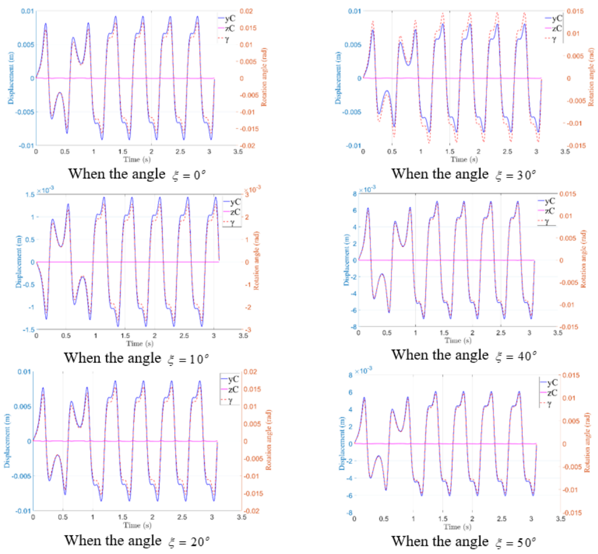

Figure 3. The survey results fluctuate vertically, horizontally, and the angle of rotation of the car body when the wheels pass through the step pattern.

Figure 3 presents the simulation results of body oscillation when passing through a stepped bumpy road surface, with the survey quantities including: body rotation angle (γ), vertical displacement (z_c) and horizontal displacement (y_c). The survey results show that when the tilt angle of the push bar is 00 (0 rad), the body rotation angle is 1,030 (0.018 rad), the horizontal oscillation is 0.008m, and the vertical oscillation is almost 0. When changing the tilt angle of the push bar gradually increased to 500 (0.87 rad), the body rotation angle decreased to 0.570 (0.01 rad), the horizontal oscillation was 0.006m, the vertical oscillation was almost 0 but the amplitude of the decrease was smaller than the original.

The results represent each case of changing the placement angle of the push rod in the suspension system. From the graphs, it is clear that the oscillation of the body depends greatly on the geometric layout of the suspension, namely the angle of inclination of the thrusters. As this angle increases, the amplitudes of the oscillations of all three surveyed quantities tend to decrease significantly, indicating a marked improvement in damping and body oscillation stability. For the angle of rotation γ and vertical oscillation (z_c), it can be seen that the values of these two quantities are very small. When the angle value is increased, the rotating oscillating thrust bar of the body has a smaller amplitude and is quickly suppressed to the balance value, which demonstrates the effective sway resistance of the suspension. At the same time, the value of vertical oscillation is almost equal to "0", indicating a significant improvement in the smoothness of the car. However, the horizontal oscillation (y_c) although the amplitude also decreases when changing the thrust bar placement angle, the decrease is not as large as the other two quantities. This indicates that horizontal oscillation still exists and can cause horizontal slippage or body instability.

4. Conclusion

Through the process of researching, building a model and surveying the oscillation of the dual arm independent suspension system by the Lagrange type II method, the paper has achieved theoretically remarkable results. The construction of the dynamics model allows for an accurate description of the oscillating movements of the car body when subjected to stimulation from the road surface. The use of the extended deduction coordinate system and the energy method to establish the differential equation has simplified the problem while retaining the main factors affecting the oscillation of the suspension system. Through simulation with Matlab/Simulink software, the results obtained show that the element in terms of the layout of the suspension elements – especially the placement angle of the thrust bar – plays an extremely important role in adjusting the vibration characteristics of the vehicle. When changing the tilt angle of the thrust bar, kinematic indicators such as body rotation oscillation, horizontal and vertical displacement all change significantly. Simulation results have shown that with the appropriate thrust bar placement angle values, the suspension system can significantly minimize fluctuations, contributing to improving smoothness, stability and safety when the vehicle moves on rough terrain. In particular, the attenuation of the amplitude of the two body rotation angles and vertical oscillations proves to be effective in damping and suppressing unwanted vibrations from the road surface. Although horizontal oscillations still exist, they also show a downward trend when the tilt angle improves, opening up the next research direction in further optimizing structural elements and materials to completely suppress horizontal oscillations. From the results obtained, it can be affirmed that optimizing the suspension geometry, especially through the arrangement of elements in the suspension, is an effective approach to improve the quality of the movement of the car. The simulation results serve as an important reference in selecting the right configuration parameters for the suspension system, thereby better serving the design and optimization of modern automotive suspension systems, contributing to improving product quality and user experience.

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