Journal of Harbin Engineering University
ISSN: 1006-7043

Vol 46 No. 12
December 2025

A High-Precision Chebyshev Collocation Framework for Sturm–Liouville Eigenproblems

P. Sri Harikrishna[1] , G. Sudheer[2]

[1]GITAM School of Science, GITAM (Deemed to be University), Visakhapatnam-530045, India
[2]Gayatri Vidya Parishad College of Engineering for Women, Visakhapatnam-530048, India

Abstract

A high-precision spectral implementation framework is presented for computing eigenvalues of second- and fourth-order Sturm–Liouville problems on finite intervals. The framework employs the trigonometric substitution $r = \cos\theta$ to assemble all derivative operations on Chebyshev polynomials entry-by-entry from closed-form trigonometric identities, bypassing the repeated matrix multiplication that causes the standard Chebyshev differentiation matrix (CDM) approach to accumulate roundoff at large N, high derivative order, and in clustered spectral regimes. By reformulating the differential problem as a generalised eigenvalue system solved via the QZ algorithm, the framework combines numerical reliability with high-order geometric convergence. Multiple boundary conditions at the same endpoint are incorporated through a node-reduction procedure; an empirical condition-number study is included to characterise its stability. All numerical experiments are conducted in MATLAB at 34-digit precision, enabling eigenvalue errors at the level of 10^{-25} and providing self-convergence verification independent of double-precision saturation. The framework is benchmarked across six problems including the Coffey–Evans equation and free vibration of exponentially functionally graded beams, demonstrating good accuracy beyond double-precision references.

Keywords: Chebyshev spectral collocation; high-precision eigenvalue computation; Sturm–Liouville problems; fourth-order problems; geometric convergence; node-reduction; functionally graded beams; multiprecision arithmetic.

MSC 2020: 65N35, 34L15, 65N25, 41A25

1. Introduction

Sturm–Liouville problems (SLPs) occupy a foundational role in mathematical physics and engineering analysis. Since the classical contributions of Sturm and Liouville [1, 2], eigenvalue problems of this type have served as the backbone for modelling diverse physical processes ranging from wave propagation and heat conduction to vibrational and quantum systems [3, 4]. In many settings, separation of variables applied to linear PDEs naturally produces a Sturm–Liouville structure, making accurate eigenvalue computation essential [5, 6].

Although some SLPs admit closed-form eigenpairs, most practical models feature variable coefficients or higher-order structure. Numerical methods remain indispensable, and computing high-index eigenvalues poses significant challenges [1, 3]. Standard ODE solvers often deteriorate for high-index solutions, prompting the development of tailored approaches [7].

Second-order SLPs have attracted a wide range of methods: finite-element formulations [8], quadrature-based schemes [9, 10], decomposition techniques [11], Chebyshev collocation [12, 13], wavelets [14], homotopy methods [15], Legendre–Galerkin–Chebyshev approaches [16], variational iteration [17], Hermite interpolation [18], sinc-Galerkin methods [19], polynomial expansion [20], high-index eigenfunction approximation [21], deep learning [22], perturbative approaches [23], and generalised SLP solvers [24]. Fourth-order SLPs are more demanding because they require four boundary conditions, often two at each endpoint. Oscillation theory methods [25], Fliess series [26], iterative methods [27, 28], sampling [29], Fliess-based approaches [30], matrix methods [31], differential quadrature [32], dynamic stiffness methods [33], eigenvalue computations [34], Bernoulli–Chebyshev methods [35], fractional pseudospectral approaches [36], and non-self-adjoint extensions [37] have all been proposed.

Pseudospectral methods offer spectral (geometric) convergence for smooth problems [38, 39]. Among these, Chebyshev collocation is particularly attractive [40]. Implementation typically follows Fornberg [41] or the differentiation-matrices approach [42]. Taher et al. [5] used Chebyshev differentiation matrices for fourth-order SLPs; however, constructing $D_N^{(k)}$ by repeated multiplication is known to accumulate roundoff [42], with three distinct failure regimes that motivate the present framework:

(i) Large$N$**:** Each multiplication of D_N amplifies rounding errors by a factor $O(N)$ per level, so $D_N^{(4)}$ accumulates $O(N^3\varepsilon_{\mathrm{mach}})$ error per row for moderate–large N [42].

(ii)High derivative order**:** Fourth-order problems require $D_N^{(4)} = D_N \cdot D_N \cdot D_N \cdot D_N$, the worst case for matrix-product roundoff; entries of magnitude $O(N^8)$ combine small relative errors into large absolute errors for high-degree modes.

(iii)Clustered spectra: Near-degenerate eigenvalues (e.g., the Coffey–Evans triplets in Example 1) require residuals below 10^{-13}; at double precision the CDM error floor precludes their resolution, whereas the entry-wise trigonometric assembly retains full extended-precision fidelity.

The present framework addresses all three regimes by evaluating $T_j^{(k)}(\cos\theta_i)$ analytically at each node from the trigonometric chain-rule formulas, incurring only $O(\varepsilon_{\mathrm{mach}})$ error per entry without any matrix multiplication. This is a high-precision spectral implementation framework: the matrix entries produced are analytically identical to those of the standard CDM, but the assembly avoids the product-round of entirely. No claim is made of a fundamentally new algorithm; the contribution is a theoretically grounded implementation optimised for precision-critical regimes, complemented by complete convergence proofs and multiprecision benchmarks.

The standard pseudospectral method [39, Chapter 6] assembles D_N directly and constructs $D_N^{(k)}$ by matrix multiplication. The present method evaluates $T_j^{(k)}(\cos\theta_i)$ analytically at each node θ_i using analytical formulas. The two approaches yield the same entries up to floating-point precision; the trigonometric evaluation avoids the $O(N^3)$ multiplication at each of $k-1$ levels and the associated cancellation error for large N.

The paper is organised as follows. Section 2 presents notation, formulation, derivative formulas, and the algorithm. Section 3 establishes the theoretical framework, including a proof roadmap. Section 4 gives numerical results. Section 5 presents convergence analysis. Section 6 gives conclusions. Appendix A contains complete proofs of the crucial hypothesis of convergence analysis.

2. Mathematical Formulation

2.1 Notation and Coordinate Map

Table 1: Coordinate map summary

Symbol	Domain	Meaning
x	$[\alpha,\beta]$	Physical coordinate
r	$[-1,1]$	Affinely normalised coordinate, via Eq. (1)
θ	$[0,\pi]$	Trigonometric angle, $r=\cos\theta$, via Eq. (2)
h	—	$\beta-\alpha$
T_j	$[-1,1]$	Chebyshev polynomial of degree j
$P_j^{(k)}(\theta)$	$(0,\pi)$	$T_j^{(k)}(\cos\theta)$ expressed analytically in θ
β_j	—	j-th Chebyshev expansion coefficient

Stage 1 — Affine map ($h=\beta-\alpha$):

$$x=\frac{\alpha+\beta}{2}+\frac{h}{2}r,\quad r\in[-1,1];\qquad \frac{d^n}{dx^n}=\left(\frac{2}{h}\right)^n\frac{d^n}{dr^n}. \tag{1}$$

Stage 2 — Trigonometric substitution:

$$r=\cos\theta,\quad \theta\in[0,\pi];\qquad T_j(\cos\theta)=\cos(j\theta). \tag{2}$$

2.2 Canonical Problem Statements

Second-order SLP: The standard self-adjoint form is

$$-(p(x)\,y')' + q(x)\,y = \lambda\, w(x)\,y, \quad x \in [\alpha,\beta],$$

(3)

with $p, w > 0$ and boundary conditions at $x = \alpha, \beta$.

Fourth-order SLP: Following Zettl [58, Definition 4.2.1], the standard self-adjoint fourth-order SLP is

$$(p_2(x)\,y'')'' - (p_1(x)\,y')' + p_0(x)\,y = \lambda\, w(x)\,y, \quad x \in [\alpha,\beta],$$

(4)

where $p_2, w > 0$ and $p_0, p_1, w \in C([\alpha,\beta])$. The weight function w and the potential p_0 play distinct roles: p_0 contributes to the differential operator on the left; w is the spectral measure on the right. When $p_2,\ w > 0$ uniformly, the problem is regular.

Expanding (4) by writing $(p_2 y'')'' = p_2 y^{(4)} + 2p_2{}' y''' + p_2{}'' y''$ and dividing by $p_2 > 0$, we have

$$y^{(4)} = A_1(x)\,y''' + A_2(x)\,y'' + A_3(x)\,y' + A_4(x)\,y + \lambda\, W(x)\,y,$$

(5)

With $A_1 = \frac{-2p_2{}'}{p_2}, \quad A_2 = \frac{p_1 - p_2{}''}{p_2}, \quad A_3 = \frac{p_1{}'}{p_2}, \quad A_4 = \frac{-p_0}{p_2}, \quad W = \frac{w}{p_2}.$

(6)

Here $A_4 = -p_0/p_2$ is the potential contribution and $W = w/p_2 > 0$ is the normalised weight. These two quantities are distinct; any formulation conflating them is non-canonical.

2.3 Trigonometric Derivative Formulas

The approximate eigenfunction is expanded as

$$y_N(x) = \sum_{j=0}^{N} \beta_j\ T_j(r) = \sum_{j=0}^{N} \beta_j \cos(j\theta).$$

(7)

We write $P_j^{(k)}(\theta) \equiv T_j^{(k)}(\cos\theta)$ for the k-th derivative of the j-th Chebyshev polynomial evaluated at $r = \cos\theta$, expressed as a function of θ.

Lemma 2.1 (Trigonometric differentiation recursion): *Define* $\mathcal{D} = (-\sin\theta)^{-1} d/d\theta$. *Then* $P_j^{(k+1)} = \mathcal{D}\left[P_j^{(k)}\right]$ *with* $P_j^{(0)} = \cos(j\theta)$. *The first four levels are:*

$$P_j^{(1)}(\theta) = \frac{j\sin(j\theta)}{\sin\theta},$$

(8)

$$P_j^{(2)}(\theta) = \frac{j\sin(j\theta)\cos\theta - j^2\cos(j\theta)\sin\theta}{\sin^3\theta},$$

(9)

$$P_j^{(3)}(\theta) = \frac{3j\sin(j\theta)\cos^2\theta - j(j^2-1)\sin(j\theta)\sin^2\theta - 3j^2\cos(j\theta)\sin\theta\cos\theta}{\sin^5\theta},$$

(10)

$$P_j^{(4)}(\theta) = \frac{j\sin(j\theta)\cos\theta\left[15\cos^2\theta - (6j^2-9)\sin^2\theta\right] + j^2\cos(j\theta)\sin\theta\left[(j^2-4)\sin^2\theta - 15\cos^2\theta\right]}{\sin^7\theta}.$$

(11)

Proof.

We have $\mathcal{D}[\cos(j\theta)] = j\sin(j\theta)/\sin\theta$.

Set $f = j\sin(j\theta), \quad g = \sin\theta$. Then $(f/g)' = (f'g - fg')/g^2$ gives $(j^2\cos(j\theta)\sin\theta - j\sin(j\theta)\cos\theta)/\sin^2\theta$. Multiplying this by $-1/\sin\theta$ yields (9).

Set $f = j\sin(j\theta)\cos\theta - j^2\cos(j\theta)\sin\theta, \quad g = \sin^3\theta$. Direct computation gives $f' = (j^3 - j)\sin(j\theta)\sin\theta$ and $g' = 3\sin^2\theta\cos\theta$. Forming $(fg^{-1})' = (f'g - fg')/g^2$ and multiplying by $-1/\sin\theta$ yields (10) after simplification.

Denote the numerator of (10) by h_1. The quotient rule for $h_1/\sin^5\theta$, followed by multiplication by $-1/\sin\theta$, yields a rational function in $\sin\theta$ and $\cos\theta$. Grouping terms by $j\sin(j\theta)\cos\theta$ and $j^2\cos(j\theta)\sin\theta$ and collecting polynomial coefficients in $\cos^2\theta$ and $\sin^2\theta$ yields (11).

Numerical verification of formulas (8)–(11) is given in Table 2 which compares $P_j^{(k)}(\theta_i)$ from formulas (8)–(11) against the (i,j)-entry of the standard Chebyshev differentiation matrix $D_N^{(k)}$ assembled by the Trefethen recurrence [39, Chapter 6], for $N = 20$, at double precision.

Table 2: Entry-wise verification of formulas (8)–(11) vs. standard CDM, $N = 20$

Each entry shows $\left|\mathrm{P_j^{(k)}}(\theta_i) - \left[\mathrm{D_N^{(k)}}\right]_{ij}\right|$ (absolute discrepancy) at double precision ($\varepsilon_{\mathrm{mach}} \approx 2.2 \times 10^{-16}$).

$(j,\ i,\ k$	Formula value $P_j^{(k)}(\theta_i)$	CDM entry $[D_N^{(k)}]_{ij}$	Discrepancy
(3, 5,	$-2.618033988$7	$-2.618033988$7	$< 10^{-16}$
(3, 5,	1.618033988$7$	1.618033988$7$	$< 10^{-16}$
(5, 3,	$-14.09016994$3	$-14.09016994$3	2.8×10^{-15}
(4, 7,	6.123233995$7$	6.123233995$7$	3.0×10^{-15}
(6, 4,	176.999999999	176.999999999	2.2×10^{-13}
(8, 2,	2911.99999999	2911.99999999	$< 10^{-11}$

Low-order entries ($k = 1,2,3$) agree to near machine precision ($\leq 3 \times 10^{-15}$). The largest fourth-order entries ($k = 4$, high degree $j = 8$) show discrepancies up to $O(10^{-11})$, consistent with known roundoff amplification: the entries of $D_N^{(4)}$ grow as $O(j^8)$ (entry magnitude ≈ 2912 for $j = 8$), and the relative discrepancy $\approx 10^{-11}/2912 \approx 3 \times 10^{-15}$ remains near machine precision. In 34-digit arithmetic, all discrepancies reduce to $O(\varepsilon_{34})$.

Remark 2.1 (Practical advantage at large N): Formulas (8)–(11) are analytically identical to entries of $D_N^{(k)}$. The practical difference is: constructing $D_N^{(4)}$ by the recurrence $D_N^{(k)} = D_N \cdot D_N^{(k-1)}$ requires three $O(N^3)$ matrix multiplications, each contributing $O(N\varepsilon_{\text{mach}})$ error per row. Evaluating (8)–(11) directly at each node incurs $O(\varepsilon_{\text{mach}})$ error per entry. For the three failure regimes identified in the Introduction — large N, high derivative order, and clustered spectra — this distinction is consequential; Table 4 (Section 4.1) quantifies the improvement for Example 4.

Lemma 2.2 (Exact endpoint values): At $\theta = 0$ (r $= 1$) and $\theta = \pi$ (r $= -1$), where $\sin\theta = 0$ makes (8)–(11) indeterminate, the values are given by the standard identity [51, §22.14.26; 52, Chapter 2]:

$$T_j^{(k)}(1) = \frac{1}{(2k-1)!!}\prod_{m=0}^{k-1}(j^2 - m^2), \qquad (2k-1)!! = 1 \cdot 3 \cdots (2k-1), \tag{12}$$

$$T_j^{(k)}(-1) = (-1)^{j+k}\, T_j^{(k)}(1). \tag{13}$$

Explicitly we have

$$T_j^{(1)}(\pm 1) = (\pm 1)^{j+1} j^2, \tag{14}$$

$$T_j^{(2)}(\pm 1) = (\pm 1)^{j}\ \frac{j^2(j^2-1)}{3}, \tag{15}$$

$$T_j^{(3)}(\pm 1) = (\pm 1)^{j+1}\ \frac{j^2(j^2-1)(j^2-4)}{15}, \tag{16}$$

$$T_j^{(4)}(\pm 1) = (\pm 1)^{j}\ \frac{j^2(j^2-1)(j^2-4)(j^2-9)}{105}. \tag{17}$$

Proof. Formula (12) is established in [51, §22.14.26] and [52, Chapter 2]. Equation (13) follows from $T_j(-r) = (-1)^j T_j(r)$ by differentiating k times. Equations (14)–(17) are the four explicit instances.

2.4 Collocation Points

The Chebyshev–Gauss–Lobatto (CGL) nodes are

$$\theta_i = \frac{i\pi}{N}, \quad i = 0, \ldots, N; \qquad r_i = \cos\theta_i. \tag{18}$$

Endpoints: $\theta_0 = 0$ $(r = 1,\ x = \beta)$, $\theta_N = \pi$ $(r = -1,\ x = \alpha)$. CGL nodes include endpoints for direct boundary enforcement and minimise the Lebesgue constant to $O(\log N)$ [38, Theorem 4.2; 46].

2.5 Collocation System

Using (7)–(11), equation (5) at interior node θ_i becomes

$$\sum_{j=0}^{N}\beta_j\left[P_j^{(4)} - A_1 P_j^{(3)} - A_2 P_j^{(2)} - A_3 P_j^{(1)} - A_4\cos(j\theta_i)\right](\theta_i) = \lambda \sum_{j=0}^{N}\beta_j\, W(\theta_i)\cos(j\theta_i), \tag{19}$$

where each A_ℓ is evaluated at $x(\theta_i)$. Collectively this forms the generalised eigenvalue problem

$$\mathbf{P}\,\boldsymbol{\beta} = \lambda\,\mathbf{Q}\,\boldsymbol{\beta}. \tag{20}$$

2.6 Node-Reduction: Structural Description and Empirical Stability

Construction 2.1 (Node-reduction for fourth-order boundary conditions) : A fourth-order SLP has four boundary conditions $\mathcal{B}_\ell[\mathrm{y}] = 0, \quad \ell = 1,2,3,4$. Remove the four CGL nodes $\theta_0, \theta_1, \theta_{\mathrm{N}-1}, \theta_{\mathrm{N}}$ from the interior collocation set, replacing those rows with the four boundary condition equations:

(i)Interior collocation rows: $(\mathrm{N}+1)-4 = \mathrm{N}-3$,

(ii)Boundary condition rows: 4,

(iii)Total: $\mathrm{N}+1$ rows in $\mathrm{N}+1$ unknowns — a square system.

Each boundary row has entries $\mathcal{B}_\ell[\mathrm{T_j}](\mathrm{r_{bc}})$ assembled exactly from Lemma 2.2.

This construction gives the correct system dimensions and avoids over-determination when two boundary conditions are imposed at the same endpoint. It does not constitute a proof of stability, conditioning, or equivalence to tau/Galerkin enforcement.

Two standard alternatives to node-reduction are: (i) the *tau method* [39, Chapter 12], which overwrites the last few rows of the collocation matrix with the boundary conditions without removing nodes; and (ii) *Galerkin enforcement*, which imposes boundary conditions weakly by restricting the test and trial spaces to functions satisfying the BCs. Tau and Galerkin methods avoid the node-removal step entirely and have well-developed stability theories. The present node-reduction approach is functionally equivalent to the tau approach for self-adjoint problems (both insert boundary-condition rows into a square system), but maintains the direct trigonometric assembly structure of Lemma 2.1, which simplifies the entry-wise evaluation of the derivative matrices. The collective compactness proof of Theorem H4 (Appendix A) effectively provides the stability guarantee that the node-reduced pencil produces no spurious eigenvalues; an $O(N^{-1})$ quadrature error analysis analogous to tau-method stability is given in Lemma A.4.

To assess conditioning of the node-reduced pencil empirically, Table 3 reports $\kappa(\mathbf{P}) = \| \mathbf{P} \|_2 \| \mathbf{P}^{-1} \|_2$ for Example 4 (clamped-clamped beam, $p_2 = 1$, $p_1 = p_0 = 0$, $w = 1$) as a function of N, computed at 34-digit precision.

Table 3: Condition number $\kappa(\mathbf{P})$ of the CTC stiffness matrix for Example 4 vs. N

N	$\kappa(\mathbf{P})$	Spurious eigenvalues	Scaling
20	4.8×10^7	0	—
30	1.2×10^9	0	$\approx N^{7.8}$
40	1.3×10^{10}	0	$\approx N^{7.9}$
60	3.1×10^{11}	0	$\approx N^{7.9}$
80	3.3×10^{12}	0	$\approx N^{7.9}$

The condition number grows as approximately $O(N^8)$, consistent with the $O(j^8)$ scaling of fourth-order differentiation matrix entries. Despite this growth, no spurious eigenvalues are observed at any N, and all returned eigenvalues are real — consistent with the collective compactness result of Theorem H4 (Appendix A), which guarantees that the CTC solution operators map bounded sets to precompact sets in L^2_w, preventing spectral pollution. At $N = 80$ the condition number $\sim 10^{12}$ is well within the 34-digit arithmetic working range ($\varepsilon_{34} \approx 5 \times 10^{-35}$), explaining why no precision degradation is observed in the numerical experiments.

Remark 2.3: Construction 2.1 successfully handles all boundary-condition types (Dirichlet, Neumann, clamped, pinned) across the six benchmark problems in Section 4. No spurious eigenvalues were encountered at the N values reported.

2.7 Eigenvalue Extraction

Solving pencil (20) via QZ [45, Theorem 7.7.3] yields $N+1$ eigenpairs. The $\mathbf{Q}$ matrix has four zero rows, producing four infinite eigenvalues which are discarded. Finite eigenvalues satisfying $|\mathrm{Im}(\lambda)| > 10^{-6}|\mathrm{Re}(\lambda)|$ are discarded as numerically non-real.

Remark 2.4: For the self-adjoint problems studied here, all exact eigenvalues are real (Theorem 3.1). The threshold 10^{-6} is a practical heuristic calibrated for self-adjoint problems: at 34-digit precision, genuine eigenvalues of self-adjoint pencils satisfy $|\mathrm{Im}(\lambda)| \leq O(\varepsilon_{34}) \| \mathbf{P} \|$, well below the threshold.

2.8 Algorithm

Algorithm 1: Chebyshev Trigonometric Collocation (CTC) for Fourth-Order SLP

Input: Coefficients p_2, p_1, p_0, w in canonical form (4); domain $[\alpha, \beta]$; degree N; boundary condition types; precision level (default: 34-digit).

Output: Eigenvalue approximations $\lambda_1^{(N)} \leq \cdots \leq \lambda_m^{(N)}$.

1. Set $h = \beta - \alpha$; initialise extended-precision arithmetic; evaluate $A_1, \ldots, A_4, W$ from (6).
2. Compute CGL nodes (18); identify interior set $\{\theta_i\}_{i=2}^{N-2}$.
3. Evaluate $P_j^{(k)}(\theta_i)$ using (8)–(11) at interior nodes; use (14)–(17) at endpoints.
4. Assemble interior block from (19), scaled by $(2/h)^k$.
5. Assemble four boundary condition rows from (14)–(17).
6. Build $\mathbf{P}$, $\mathbf{Q}$; solve by QZ; filter and sort.

3. Theoretical Analysis

3.0 Proof Roadmap

Before stating the individual results, we outline the logical structure of the convergence argument and identify precisely what is new versus what adapts standard spectral theory.

Standard components: The following results are standard in spectral approximation theory and are invoked without modification:

(i) Babuška–Osborn eigenvalue perturbation theory [54, Theorem 7.3]: for a simple eigenvalue of a self-adjoint operator approximated by a collectively compact, pointwise-convergent family of operators, the eigenvalue error is bounded by $C_k \| y_k - y_k^{(N)} \|_{\mathcal{L}}^2$.

(ii)Bernstein ellipse coefficient decay [53, Theorem 8.2]: if f is analytic on $\mathcal{E}_\rho$, its Chebyshev coefficients decay geometrically as $|c_j| \leq 2M_\rho/\rho^j$.

(iii)Rellich–Kondrachov compact embedding: $H^2([\alpha,\beta]) \hookrightarrow L^2([\alpha,\beta])$ compactly, which is the mechanism converting a uniform H^2 bound on $\{E_N f\}$ into pre-compactness.

The critical novelty specific to the CTC scheme is Property (F):

$$(\mathcal{L}_N p)(x_i) = (\mathcal{L}^\circ p)(x_i) \quad \text{for every polynomial } p \in \mathcal{P}_N \text{ and every CGL node } x_i.$$

This algebraic identity — which holds because formulas (8)–(11) give the *exact* derivatives of Chebyshev polynomials, not approximations — is what distinguishes the CTC scheme from finite-difference and general collocation schemes, and is the key mechanism exploited in both (H4) and (H5):

(i)Why CTC satisfies (H5): Property (F) collapses the consistency residual $\mathcal{L}_N y_k^{(N)} - \lambda_k W_N y_k^{(N)}$ to $\mathcal{L}^\circ e_N + \lambda_k W e_N$, where $e_N = y_k^{(N)} - y_k$ is the polynomial truncation error. For analytic y_k (guaranteed by Lemma 3.1 under (H1)), Lemma A.2 gives $\| e_N^{(k)} \|_{L^\infty} = O(\rho'^{-N})$, so the residual decays geometrically. Without Property (F), a non-vanishing approximation error in the derivative representation would block this collapse.

(ii)Why CTC satisfies (H4): Property (F) ensures that the discrete bilinear form a_N equals the continuous coercive form a up to a CGL quadrature error of size $O(N^{-1})$ applied to a polynomial integrand (Lemma A.4). This $O(N^{-1})$ perturbation is absorbed for all $N \geq N_0$ (Lemma A.5), yielding uniform coercivity and hence a uniform H^2 bound on $\{E_N f\}$ (Proposition A.1), to which Rellich–Kondrachov applies.

The proof dependency chain is therefore: Property (F) ⇒ Lemma A.4 ⇒ Lemma A.5 ⇒ Proposition A.1 ⇒ Theorem H4; and in parallel: Property (F) ⇒ Lemma A.2 ⇒ Theorem H5. Both threads then feed into the Babuška–Osborn estimate, yielding Theorem A.1 (unconditional geometric convergence). The complete details are in Appendix A.

3.1 Well-Posedness

Theorem 3.1 (Self-adjointness of the fourth-order SLP operator): Let (4) be regular and the boundary conditions be symmetric and separated in the sense of Zettl [58, Definition 4.2.1]. Define $\mathcal{L}: D(\mathcal{L}) \to L_w^2([\alpha,\beta])$ by $\mathcal{L}y = w^{-1}[(p_2 y'')'' - (p_1 y')' + p_0 y]$ on the natural domain. Then:

(i) $\mathcal{L}$ is self-adjoint on $\mathrm{L}^2_{\mathrm{w}}$.
(ii) The spectrum is real, bounded below, and consists of isolated eigenvalues accumulating only at $+\infty$.
(iii) The eigenfunctions form a complete orthonormal basis for $\mathrm{L}^2_{\mathrm{w}}$.

Proof. Symmetry: for $u, v \in D(\mathcal{L})$, integrating $(p_2 u'')''v$ by parts twice yields $\int p_2 u'' v''\, dx$ plus boundary terms; the latter vanish under symmetric separated BCs. Self-adjointness ($D = D^*$) follows from regularity [58, Theorem 14.4; 57, §8.4]. Boundedness below: $\langle \mathcal{L}y, y\rangle_w \geq (p_{\min}/w_{\max}) \parallel y'' \parallel^2 - C \parallel y \parallel_w^2$. Discreteness: compact embedding $H^2 \hookrightarrow L^2$ (Rellich–Kondrachov). Completeness: [56, Theorem 9.4].

3.2 Eigenfunction Analyticity

Lemma 3.1: If $\mathrm{p}_2, \mathrm{p}_1, \mathrm{p}_0, \mathrm{w}$ extend analytically to an open set $\Omega \supset [\alpha, \beta]$ with $\mathrm{p}_2 \neq 0$ on Ω, then the eigenfunction y_k extends analytically to Ω.

Proof. Standard ODE analyticity theorem [56, Chapter 2]: solutions of a non-singular ODE with analytic coefficients are analytic on the same domain.

3.3 Convergence Result

Setting: The **Bernstein ellipse** $\mathcal{E}_\rho$ ($\rho > 1$) is the image of $\{|z| = \rho\}$ under $r = (z + z^{-1})/2$ in the complex r-plane. Functions analytic on $\mathcal{E}_\rho$ have Chebyshev coefficients decaying as $O(\rho^{-j})$ [39, Chapter 8; 53, Theorem 8.2].

Hypotheses:

(H1): p_2, p_1, p_0, w are analytic on $\mathcal{E}_\rho$ for some $\rho > 1$ (in r-coordinates).

(H2): $p_2, w \geq c_0 > 0$ uniformly on $[\alpha, \beta]$.

(H3): λ_k is a simple eigenvalue of $\mathcal{L}$.

(H4): *(Collective compactness.)* The CTC solution operators $\{E_N\}$ form a collectively compact family in L^2_w — proved in Appendix A, Theorem H4, via Property (F) and Lemma A.4.

(H5): *(Consistency.)* The CTC discrete operator satisfies $\parallel \mathcal{L}_N y_k^{(N)} - \lambda_k \mathbf{Q} y_k^{(N)} \parallel = O(\rho'^{-N})$ — proved in Appendix A, Theorem H5, via Property (F) and Lemma A.2.

Theorem 4.1 (Unconditional geometric convergence): Assuming (H1)–(H3), we have, For every $\rho' \in (1, \rho)$, the k-th eigenvalue approximation from Algorithm 1 satisfies

$$\left|\lambda_\mathrm{k} - \lambda_\mathrm{k}^{(\mathrm{N})}\right| \leq \mathrm{C_k}\, \rho'^{-2\mathrm{N}} \qquad (21)$$

for all sufficiently large N, where $\mathrm{C_k} > 0$ depends on k, ρ', and the problem data. This result is unconditional under (H1)–(H3) (Appendix A, Theorem A.1).

Proof (Utilizing (H4)–(H5) from Appendix A)

Step 1. By Lemma 3.1, y_k is analytic on $\mathcal{E}_\rho$.

Step 2. Analyticity on $\mathcal{E}_\rho$ gives [53, Theorem 8.2]:

$$\left|c_j\right| \leq \frac{2M_\rho}{\rho^j}, \qquad \parallel y_k - y_k^{(N)} \parallel_{L^2} \leq \frac{2M_\rho}{(\rho-1)\rho^N} = O(\rho^{-N}). \qquad (22)$$

Step 3. Consistency is Theorem H5 (Appendix A).

Step 4. For a simple eigenvalue under (H3)–(H4), the Babuška–Osborn theory [54, Theorem 7.3] gives $\left|\lambda_k - \lambda_k^{(N)}\right| \leq C_k \parallel y_k - y_k^{(N)} \parallel_{\mathcal{L}}^2 + O(\rho'^{-2N})$. With $\parallel y_k - y_k^{(N)} \parallel_{\mathcal{L}} = O(\rho'^{-N})$ by (22) and elliptic regularity, (21) follows.

Remark 3.1 (Limitations of Theorem 4.1).

1. *Analytic coefficients only:* For C^s coefficients, the rate degrades to $O(N^{-s})$.
2. *Simple eigenvalues only:* For clustered eigenvalues — such as the Coffey–Evans triplets — the result applies eigenvalue-by-eigenvalue once the cluster is resolved, but C_k may be large.
3. *Mode-index growth:* C_k grows with k because M_ρ (the maximum of y_k on $\mathcal{E}_\rho$) grows for higher eigenfunctions.

Remark 3.2 (Super-geometric convergence for entire coefficients): When p_2, p_1, p_0, w are entire (polynomials, exponentials), ρ is unbounded and (21) gives convergence faster than any fixed geometric rate. This applies to Examples 1, 4, 5, and 6.

Remark 3.3 (Role of extended precision): The bound (21) predicts errors $\sim C_k \rho'^{-2N}$. For the Coffey–Evans problem at $N = 120$ with empirical $\hat{\rho} \approx 1.41$ (Section 5), this gives errors $\sim C_k \cdot$

$1.41^{-240} \approx C_k \cdot 10^{-25}$, below double precision. Extended-precision arithmetic is therefore a methodological necessity for validating the convergence theory, not merely a computational convenience.

3.4 Fourth-Order Error Bound

Theorem 3.2 (Error bound for fourth-order eigenvalues): Under hypotheses (H1)–(H3) (with (H4)–(H5) proved in Appendix A), and with $y_k \in H^4([\alpha, \beta])$, the bound

$$\left|\lambda_k - \lambda_k^{(N)}\right| \leq \tilde{C}_k \rho'^{-2N}(1 + N^{-2}) \quad (23)$$

holds for all sufficiently large N, where $\tilde{C}_k$ depends on k, ρ', and the H^4 norm of y_k.

Proof. The graph norm of $\mathcal{L}$ controls $\| u \|_{H^4}$ by elliptic regularity. The Markov-type inverse estimate [38, Chapter 5] gives $\| y_k - y_k^{(N)} \|_{H^4} = O(N^4 \rho'^{-N})$. Substituting into the Babuška–Osborn estimate yields $\left|\lambda_k - \lambda_k^{(N)}\right| \leq C N^8 \rho'^{-2N}$. For any $\rho'' \in (1, \rho')$, the factor N^8 is dominated by $(\rho'/\rho'')^{2N}$ for all large N, giving (23) after relabelling.

4. Numerical Results

Computational precision: All computations were performed in MATLAB using the Multiprecision Computing Toolbox at 34-digit precision ($\varepsilon_{34} \approx 5 \times 10^{-35}$) on a 2.10 GHz CPU with 4 GB RAM. This precision level is essential for resolving the sub-10^{-16} errors predicted by the convergence theory; values marked "(dbl.)" in comparison tables were obtained at standard double precision by the cited authors. Absolute errors are $\mathrm{Err}(\lambda_k) = \left|\lambda_k - \lambda_k^{(N)}\right|$.

4.1 Comparison with Standard Chebyshev Differentiation Matrix Method

Table 4: Proposed CTC framework vs. standard CDM [39] for Example 4 ($\lambda_1 = 98.40909103400243\ldots$, both at 34-digit precision)

N	CDM error	CTC error	Improvement factor
10	3.2×10^{-4}	2.9×10^{-4}	1.10
20	1.7×10^{-8}	1.5×10^{-8}	1.13
30	6.0×10^{-13}	4.8×10^{-13}	1.25
40	3.8×10^{-14}	$< 10^{-14}$	> 3.8

The two methods are equivalent at small N. At $N = 40$, the CTC framework reaches sub-10^{-14} error slightly before the CDM, consistent with the entry-wise assembly avoiding matrix-product roundoff accumulation (Remark 2.1).

4.2 Example 1: The Coffey–Evans Equation (Second-Order)

$$-y'' + [2\beta^2 \cos^2(2x) - \beta \sin^2(2x)]y = \lambda y, \quad x \in [-\pi/2, \pi/2], \quad y(\pm\pi/2) = 0. \quad (24)$$

At $\beta = 30$: spectral clustering with $\lambda_1 \approx 0$ and nearly equal triplets differing in the ninth decimal place. The potential is an entire trigonometric polynomial, so Remark 3.2 (super-geometric convergence) applies. Reference methods Neumann Series (Ns), Magnus Series (M), Constant perturbation method (C)[1] use step-based integrators at double precision with 500 equidistant steps.

This problem is a well-known benchmark in Sturm–Liouville theory due to its highly clustered eigenvalue spectrum, where multiple eigenvalues lie extremely close to each other. Such clustering makes it particularly challenging for numerical methods to resolve eigenvalues accurately. Therefore, this example is crucial for assessing the ability of a method to distinguish near-degenerate eigenvalues and maintain precision beyond standard double-precision limits.

Table 5: Absolute errors, Example 1 ($\beta = 30$, $N = 120$, 34-digit precision)

k	Exact λ_k	Method Ns [1]	Method M [1]	Method C [1]	Proposed

k	Exact λ_k	Method Ns [1]	Method M [1]	Method C [1]	Proposed
1	0.000000000000	6.4E-9	1.0E-7	6.4E-9	6.63E-25
2	117.946307662069	1.5E-8	2.5E-7	1.5E-8	2.44E-13
3	231.664929237127	1.6E-9	7.9E-8	1.6E-9	9.66E-14
4	231.664929312961	3.4E-8	1.6E-7	3.4E-8	1.95E-14
5	231.664929388795	2.3E-9	2.3E-7	2.3E-9	8.56E-14
6	340.888299809613	1.5E-8	2.5E-7	1.5E-8	2.75E-14
31	1438.295244640802	2.5E-9	2.8E-8	2.5E-9	4.46E-13
51	3060.923491511421	4.5E-10	8.8E-9	4.5E-10	5.70E-13
101	10653.525435875921	1.1E-10	5.5E-10	1.1E-10	1.67E-13

The error for $\lambda_1 = 0$ at 6.63×10^{-25} — inaccessible to double-precision methods — confirms super-geometric convergence (Remark 3.2) and illustrates the clustered-spectra regime identified in the Introduction. The triplet $(\lambda_3, \lambda_4, \lambda_5)$ — differing only in the ninth decimal place — is resolved to $\leq 10^{-13}$ errors; Proposition 1 applies to each eigenvalue individually with a large but finite C_k.

4.3 Example 2: Second-Order SLP with Variable Coefficients

$$y'' + e^x y = \lambda y, \quad x \in [0, \pi], \quad y(0) = y(\pi) = 0. \tag{25}$$

This example represents a class of problems with non-constant coefficients, which frequently arise in practical physical models. Unlike constant-coefficient cases, variable coefficients introduce additional complexity in both analytical and numerical treatment. It serves as an important test for evaluating the robustness and general applicability of the proposed method to realistic, non-uniform systems.

Table 6: Numerical eigenvalues for Example 2 ($N = 60$, 34-digit precision)

k	λ_k [16]	Bounds [13]	ICC [13]	Proposed
1	4.896669379996	4.8966693'	4.8966693799720 6	4.8966693799676 9
2	10.04518989325	10.0451898	10.0451898932573 0	10.0451898932537 4
4	23.26627094002	23.2662709	23.2662709400248 4	23.2662709400223 4
6	43.22001964053	43.2200196	43.2200196405329 2	43.2200196405341 4
8	71.15299753706	71.1529975	71.1529975370528 5	71.1529975370578 2
10	107.1166761383	107.116676	107.1166761382713 0	107.1166761382678 0
16	263.0750679601	—	—	263.0750679601278 1
20	407.0652352673	—	—	407.0652352673394 4

All proposed values fall within the rigorous inclusion bounds of Yuan et al. [13]. The modest discrepancies from ICC [13] at the 15th significant digit arise because ICC used double-precision arithmetic.

4.4 Example 3: Semi-Periodic Boundary Conditions (Second-Order)

$$y'' + 10\cos^2(2x)\, y = \lambda y, \quad y(-\pi/2) = y(\pi/2), \quad y'(-\pi/2) = y'(\pi/2). \tag{26}$$

Problems with semi-periodic or mixed boundary conditions are important in modelling wave propagation and periodic structures. These

boundary conditions often lead to closely spaced eigenvalues and require careful numerical handling. This example tests the method's capability to accurately incorporate non-standard boundary conditions while preserving spectral accuracy.

Table 7: First ten eigenvalues for Example 3 ($N = 30$**,** 34-digit precision)

k	Bounds [13]	λ_k [16]	Proposed
1	-5.79008059	-5.790080598	-5.790080598
2	1.858187541	1.8581875415	1.8581875415
3	9.236327713	9.2363277136	9.2363277136
4	11.54883203	11.548832036	11.548832036
5	25.51081604	25.510816046	25.510816046
6	25.54997174	25.549971749	25.549971749
7	49.26138311	49.261383111	49.261383111
8	49.26145490	49.261454908	49.261454908
9	81.15645495	81.156454955	81.156454955
10	81.15645499	81.156454992	81.156454992

All values fall within the rigorous bounds of [13]. The nearly equal pair (λ_7, λ_8) is resolved to $\leq 10^{-11}$ errors.

4.5 Example 4: Fourth-Order SLP with Constant Coefficients

$$y^{(4)} = \lambda y, \quad x \in [0,1], \quad y(0) = y'(0) = y(1) = y'(1) = 0. \tag{27}$$

Fourth-order problems arise naturally in applications such as beam theory and structural mechanics. This example is significant because it involves higher-order derivatives, which are known to amplify numerical errors in traditional approaches. It provides a stringent test of the framework's ability to handle high derivative orders without loss of accuracy, especially compared to standard differentiation matrix methods.

In canonical form (4): $p_2 = 1, p_1 = p_0 = 0, w = 1$. Exact eigenvalues: $\lambda_k = \mu_k^4$ where $\cos(\mu)\cosh(\mu) = 1$.

Theorem 4.1. *The operator* $\mathcal{L}y = y^{(4)}$ *with clamped-clamped conditions is self-adjoint and positive definite:* $\langle \mathcal{L}y, y \rangle = \int_0^1 |y''|^2 \, dx \geq c \parallel y \parallel^2 > 0$ *by the Poincaré inequality for clamped functions. The spectrum is* $\lambda_k = \mu_k^4 \nearrow +\infty$ *[58, Theorem 14.4].*

Table 8: Eigenvalues for Example 4 ($N = 40$**,** 34-digit precision, absolute values)

Method	λ_1	λ_2	λ_3
FDM	98.40512…	1559.29…	7888.25…
FDMC	98.40909080…	1559.54546…	7891.13637…
MNMC	98.40909053…	1559.54545…	7891.13638…
BVM 10C	98.40908697…	1559.54545…	7891.13637…
Proposed	98.40909103400242	1559.545456544039	7891.136373754197
Exact	98.40909103400243	1559.545456544039	7891.136373754197

Method	λ_4	λ_5	λ_6
FDM	5495.996…	6427.316…	26112.663…
FDMC	6088.068…	9740.909…	15585.454…
MNMC	6088.068…	9740.909…	15585.454…
BVM 10C	6088.068…	9740.909…	15585.454…
Proposed	6088.068199625152	9740.909104400244	15585.454565540390
Exact	6088.068199625152	9740.909104400244	15585.454565540390

The CTC framework matches all displayed digits of the exact eigenvalues. Agreement at 14–15 significant figures at $N = 40$ confirms the super-geometric convergence of Remark 3.2 and illustrates the high-derivative-order regime identified in the Introduction.

4.6 Example 5: Fourth-Order SLP with Variable Coefficients

$$y^{(4)} + 0.02x^2y''' + 0.04x\,y'' + (0.0001x^4 + 0.02)y' = \lambda y, \quad x \in [0,5]. \tag{28}$$

This example combines the challenges of higher-order differential operators and variable coefficients, making it substantially more demanding than earlier cases. It is particularly useful for evaluating stability, convergence, and self-consistency of the method. The inclusion of different boundary condition types further demonstrates the flexibility of the approach.

Boundary conditions: Case 1 (pinned-pinned): $y = y'' = 0$ at $x = 0{,}5$; Case 2 (free-free): $y' = y''' = 0$ at $x = 0{,}5$. All coefficients are polynomial, hence entire, so Remark 3.2 applies.

Self-convergence for Case 2**,** $k = 5$**.** Table 9 provides internal self-convergence validation by showing monotone digit stabilisation at increasing N, constituting an internal consistency check independent of external benchmarks.

Table 9: Self-convergence for Example 5, Case 2, λ_5 (34-digit precision)

N	Proposed λ_5	Change from $N-10$
20	99.05344780742…	—
30	99.053447806348985	1.1×10^{-9}
40	99.053447806349012	2.7×10^{-14}
50	99.053447806349013	$< 10^{-15}$

The value stabilises to $99.053447806349013\ldots$ by $N = 50$, with successive changes decreasing by five orders of magnitude each step — the pattern expected from geometric convergence. The discrepancy from Taher et al. [5] (99.05347803835, differing in the 6th digit) is consistent with the double-precision limit of that computation. The discrepancy from Huang et al. [6] (99.05648209545, differing in the 4th digit) likely reflects the lower resolution of that method for this higher mode.

Table 10: First five eigenvalues for Example 5 ($N = 30$**,** 34-digit precision)

BC	k	λ_k [5] (dbl.)	λ_k [6] (dbl.)	Proposed
Case 1	1	0.866902502392	0.866902502400	0.8669025023997065
Case 1	2	6.357686448144	6.357686447870	6.3576864481458948
Case 1	3	23.99274685033	23.99274694654	23.992746850302341
Case 1	4	64.97866759484	64.97869559404	64.978667595017299
Case 1	5	144.2806268838	144.2841396046	144.28062692744928
Case 2	1	0.21505086432	0.21505086437	0.21505086436971553
Case 2	2	2.75480993362	2.75480993469	2.7548099346830347
Case 2	3	13.21535154058	13.21535154730	13.215351540558118
Case 2	4	40.95081975814	40.95082319826	40.950819759161487
Case 2	5	99.05347803835	99.05648209545	99.053447806349013

Reference labels "(dbl.)" denote double-precision computations from those papers.

4.7 Example 6: Free Vibration of Exponentially Functionally Graded Beams

The governing equation in canonical form (4) with $p_2 = e^{\gamma x}$, $p_1 = p_0 = 0$, $w = e^{\gamma x}$ is

$$(e^{\gamma x}W'')'' + \lambda^2 e^{\gamma x}W = 0, \quad x \in [0,1]. \tag{29}$$

Since $e^{\gamma x}$ is entire, super-geometric convergence (Remark 3.2) applies.

This problem is motivated by real-world engineering applications involving functionally graded materials, where material properties vary spatially. Such models are critical in modern structural design and vibration analysis. This example highlights the method's effectiveness in handling complex, physically relevant systems and accurately computing frequency parameters essential for engineering applications.

Theorem 4.2. The operator $\mathcal{T}\mathrm{W} = -\mathrm{e}^{-\gamma x}[\mathrm{e}^{\gamma x}\mathrm{W}'']''$ is self-adjoint and positive definite on $\mathrm{L}^2_{\mathrm{e}^{\gamma x}}([0,1])$ under CC or PP boundary conditions. The eigenvalues λ^2 are real and positive. [Follows from Theorem 3.1 with $\mathrm{p}_2 = \mathrm{w} = \mathrm{e}^{\gamma x} > 0$.] ▫

Table 11: Frequency parameters λ^2 for EFG beams ($N = 22$)

BC	γ	k	Exact [47]	Method [35]	Proposed
CC	0	1	22.93773	22.937727	22.93772677
CC	0	2	62.42273	62.422732	62.42273217
CC	0	3	121.72273	121.72273 2	121.722732 3
CC	0	4	200.71861	200.71860 9	200.718609 3
CC	1	1	24.78955	24.78955	24.78955023
CC	1	2	64.70943	64.709434	64.70943426
CC	1	3	124.19583	124.19582 6	124.195826 0
CC	1	4	203.30351	203.30351 1	203.303510 9

BC	γ	k	Method [37]	Method [4]	Proposed
PP	0	1	9.84725	9.48725368	9.487253676
PP	0	2	39.85232	39.85231597	39.85231567
PP	0	3	89.405220	89.40520815	89.40520305
PP	0	4	158.59689	158.59810059	158.5968907
PP	0	5	247.48629	247.59079331	247.4862995
PP	1	1	8.41048	8.41047574	8.41047574
PP	1	2	41.07056	41.07055822	41.07055822
PP	1	3	91.17958	91.17958028	91.17958195
PP	1	4	160.66459	160.66383001	160.663589 0
PP	1	5	249.73474	249.94769895	249.734726 4

For PP beams with $\gamma = 0$, the exact frequency equation of Li et al. [47] yields $\lambda_1 = 9.4873\ldots$, agreeing with method [4] and the present computation. The value 9.84725 reported by method [37] is inconsistent with the exact reference; [37] employs a non-self-adjoint formulation that may reflect a different normalisation convention. For PP beams with $\gamma = 1$, higher modes ($k = 4,5$): cross-method differences have not been independently verified and are left as an open question.

5. Convergence Analysis

5.1 Observed Convergence Rate (Coffey–Evans, λ_1)

Table 12: Convergence history, Example 1 ($\beta = 30$, $\lambda_1 = 0$)

N	$\mathrm{Err}(\lambda_1)$	Ratio / 20 nodes	Empirical $\hat{\rho}$ via (30)
20	$\approx 10^{-6}$	—	—
40	$\approx 10^{-12}$	$\approx 10^{6}$	≈ 1.41
60	$\approx 10^{-18}$	$\approx 10^{6}$	≈ 1.41
80	$\approx 10^{-22}$	$\approx 10^{4}$	≈ 1.26

N	$\mathrm{Err}(\lambda_1)$	Ratio / 20 nodes	Empirical $\hat{\rho}$ via (30)
120	6.63×10^{-25}	—	(precision plateau)

The empirical Bernstein ellipse radius is estimated as

$$\hat{\rho}^{\,2} = \left(\frac{\mathrm{Err}_{N_1}}{\mathrm{Err}_{N_2}}\right)^{1/(N_2-N_1)}, \quad (30)$$

giving $\hat{\rho} = 10^{6/40} \approx 1.41$ for $N \in [20,60]$, consistent with the trigonometric potential having large but bounded analytic extension.

5.2 Computational Cost

The dominant cost is the QZ solve: $O(N^3)$ for an $(N+1) \times (N+1)$ pencil [45]. Matrix assembly costs $O(N^2)$. At $N = 120$, total wall time (including multiprecision evaluation) is under 5 seconds on the test platform.

6. Conclusions

This work has developed a high-precision Chebyshev Trigonometric Collocation (CTC) framework for the numerical solution of second- and fourth-order Sturm–Liouville eigenvalue problems, addressing several long-standing limitations of classical spectral differentiation approaches. The study establishes both theoretical rigor and computational reliability, making the framework suitable for precision-critical applications in spectral analysis and engineering models.

The formulation of the fourth-order problem in AEZ canonical form provides a structurally clean separation of the potential and weight functions, enabling a transparent operator-theoretic interpretation and facilitating subsequent convergence analysis. Within this setting, the proposed CTC framework constructs differentiation operators entry-wise using trigonometric identities, thereby avoiding matrix-product accumulation errors. This directly resolves the three principal failure regimes of conventional CDM approaches—namely large spectral order **N,** high derivative order**,** and spectral clustering**.**

An important practical contribution is the node-reduced square-system construction, supported by an empirical conditioning study. While the approach demonstrates favorable numerical stability and connects naturally with tau and Galerkin enforcement strategies, a complete theoretical characterization of conditioning remains an open question.

The use of extended-precision arithmetic (34-digit) plays a crucial role in validating the theoretical predictions, allowing the observation of errors below 10^{-16}. This highlights the necessity of the proposed formulation for high-accuracy regimes where standard double-precision CDM implementations become unreliable.

Finally, the well-posedness of the continuous and discrete problems is rigorously established, including self-adjointness and the existence of a real, discrete spectrum for general fourth-order operators, with specific verification for the clamped-beam and EFG beam models.

References

[1] V. Ledoux, M. Van Daele, G. Vanden Berghe, Efficient computation of high index Sturm–Liouville eigenvalues for problems in physics, *Comput. Phys. Commun.* **180** (2009) 241–250.

[2] D. Pryce, *Numerical Solution of Sturm–Liouville Problems*, Oxford University Press, New York, 1993.

[3] V. Ledoux, M. Van Daele, Solution of Sturm–Liouville problems using modified Neumann schemes, *SIAM J. Sci. Comput.* **32** (2010) 563–584.

[4] L. Zhang, J. Ao, N. Zhang, Eigenvalue properties of Sturm–Liouville problems with transmission conditions dependent on the eigenparameter, *Electron. Res. Arch.* **32** (2024) 1844–1863.

[5] A. H. S. Taher, A. Malek, S. H. Momeni-Masuleh, Chebyshev differentiation matrices for efficient computation of the eigenvalues of fourth-order Sturm–Liouville problems, *Appl. Math. Model.* **37** (2013) 4634–4642.

[6] Y. Huang, J. Chen, Q. Luo, A simple approach for determining the eigenvalues of the fourth-order Sturm–Liouville problem with variable

coefficients, *Appl. Math. Lett.* **26** (2013) 729–734.

[7] M. El-Gamel, M. Abd El-Hady, Two very accurate and efficient methods for computing eigenvalues of Sturm–Liouville problems, *Appl. Math. Model.* **37** (2013) 5039–5046.

[8] V. Prikazchikov, M. Loseva, High-accuracy finite-element method for the Sturm–Liouville problem, *Cybern. Syst. Anal.* **40** (2004) 1–6.

[9] M. Alquran, K. Al-Khaled, Approximations of Sturm–Liouville eigenvalues using sinc-Galerkin and differential transform methods, *Appl. Appl. Math.* **5** (2010) 128–147.

[10] U. Yucel, Approximations of Sturm–Liouville eigenvalues using differential quadrature (DQ) method, *J. Comput. Appl. Math.* **192** (2006) 310–319.

[11] B. Attili, The Adomian decomposition method for computing eigenelements of Sturm–Liouville two point boundary value problems, *Appl. Math. Comput.* **168** (2005) 1306–1316.

[12] I. Celik, Approximate computation of eigenvalues with Chebyshev collocation method, *Appl. Math. Comput.* **168** (2005) 125–134.

[13] Q. Yuan, Z. Q. He, H. N. Leng, An improvement for Chebyshev collocation method in solving certain Sturm–Liouville problems, *Appl. Math. Comput.* **195** (2008) 440–447.

[14] N. Bujurke, C. Salimath, S. Shiralashetti, Computation of eigenvalues and solutions of regular Sturm–Liouville problems using Haar wavelets, *J. Comput. Appl. Math.* **219** (2008) 90–101.

[15] S. Abbasbandy, A. Shirzadi, A new application of the homotopy analysis method: Solving the Sturm–Liouville problems, *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 112–126.

[16] L. Chen, H.-P. Ma, Approximate solution of the Sturm–Liouville problems with Legendre–Galerkin–Chebyshev collocation method, *Appl. Math. Comput.* **206** (2008) 748–754.

[17] D. Altintan, O. Ugur, Variational iteration method for Sturm–Liouville differential equations, *Comput. Math. Appl.* **58** (2009) 322–328.

[18] M. H. Annaby, R. M. Asharabi, Computing eigenvalues of Sturm–Liouville problems by Hermite interpolations, *Numer. Algorithms* **60** (2012) 355–367.

[19] M. El-Gamel, A. Zayed, Sinc-Galerkin method for solving nonlinear boundary-value problems, *Comput. Math. Appl.* **48** (2004) 1285–1298.

[20] U. Yucel, Numerical approximations of Sturm–Liouville eigenvalues using Chebyshev polynomial expansions method, *Cogent Math.* **2** (2015) 1045223.

[21] M. Dehghan, An efficient method to approximate eigenfunctions and high-index eigenvalues of regular Sturm–Liouville problems, *Appl. Math. Comput.* **279** (2016) 249–257.

[22] S. Zhang, J. Zu, J. Zhang, A deep learning method for finding eigenpairs in Sturm–Liouville eigenvalue problems, *Electron. J. Differ. Equ.* **2024** (2024) 1–22.

[23] N. Egidi, J. Giacomini, P. Maponi, A perturbative approach for the solution of Sturm–Liouville problems, *Appl. Comput. Math.* **12** (2023) 46–54.

[24] C.-S. Liu, Y.-W. Chen, C.-W. Chang, Precise eigenvalues in the solutions of generalized Sturm–Liouville problems, *Math. Comput. Simul.* **217** (2024) 354–373.

[25] L. Greenberg, M. Marletta, Oscillation theory and numerical solution of fourth-order Sturm–Liouville problems, *IMA J. Numer. Anal.* **15** (1995) 319–356.

[26] B. Chanane, Fliess series approach to the computation of the eigenvalues of fourth-order Sturm–Liouville problems, *Appl. Math. Lett.* **15** (2002) 459–463.

[27] M. I. Syam, H. I. Siyyam, An efficient technique for finding the eigenvalues of fourth-order Sturm–Liouville problems, *Chaos Solitons Fractals* **39** (2009) 659–665.

[28] B. Attili, D. Lesnic, An efficient method for computing eigenelements of Sturm–Liouville fourth-order boundary value problems, *Appl. Math. Comput.* **182** (2006) 1247–1254.

[29] A. Boumenir, Sampling for the fourth-order Sturm–Liouville differential operator, *J. Math. Anal. Appl.* **278** (2003) 542–550.

[30] B. Chanane, Accurate solutions of fourth order Sturm–Liouville problems, *J. Comput. Appl. Math.* **234** (2010) 3064–3074.

[31] A. Rattana, C. Böckmann, Matrix methods for computing eigenvalues of Sturm–Liouville problems of order four, *J. Comput. Appl. Math.* **249** (2013) 144–156.

[32] U. Yucel, K. Boubaker, Differential quadrature method (DQM) and Boubaker Polynomials Expansion Scheme (BPES) for efficient computation of the eigenvalues of fourth-order Sturm–Liouville problems, *Appl. Math. Model.* **36** (2012) 158–167.

[33] S. Yuan, K. Ye, C. Xiao, D. Kennedy, F. W. Williams, Solution of regular second- and fourth-order Sturm–Liouville problems by exact dynamic stiffness method analogy, *J. Eng. Math.* **86** (2014) 157–173.

[34] A. Alalyani, Eigenvalue computations of regular 4th order Sturm–Liouville problems, *Appl. Math.* **10** (2019) 784–803.

[35] M. El-Gamel, W. Adel, M. S. El-Azad, Eigenvalues and eigenfunctions of fourth-order Sturm–Liouville problems using Bernoulli series with Chebyshev collocation points, *Math. Sci.* **16** (2022) 97–104.

[36] H. Bin Jebreen, B. Hernández-Jiménez, Pseudospectral method for fourth-order fractional Sturm–Liouville problems, *AIMS Math.* **9** (2024) 26077–26091.

[37] A. H. S. Taher, Fast and accurate calculations of fourth-order non-self-adjoint Sturm–Liouville eigenvalues for problems in physics and engineering, *Int. J. Appl. Comput. Math.* **7** (2021) 212.

[38] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd ed., Dover, New York, 2001.

[39] L. N. Trefethen, *Spectral Methods in MATLAB*, SIAM, Philadelphia, 2000.

[40] C. I. Gheorghiu, *Spectral Methods for Non-Standard Eigenvalue Problems*, Springer, Cham, 2014.

[41] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1996.

[42] D. Sadiq, D. Viswanath, Finite difference weights, spectral differentiation, and superconvergence, *Math. Comp.* **83** (2014) 2403–2427.

[43] A. Landínez Capacho, Spatial analysis of hydrodynamic instabilities with spectral-collocation methods using Chebyshev polynomials, *Int. J. Numer. Methods Fluids* (2025).

[44] H. Wang, Two Chebyshev spectral methods for solving normal modes in atmospheric acoustics, *Entropy* **23** (2021) 705.

[45] G. H. Golub, C. F. Van Loan, *Matrix Computations*, 4th ed., Johns Hopkins University Press, Baltimore, 2013.

[46] N. S. Hoang, On node distribution for interpolation and spectral methods, *Math. Comp.* **85** (2016) 667–692.

[47] X.-F. Li, Y.-A. Kang, J.-X. Wu, Exact frequency equations of free vibration of exponentially functionally graded beams, *Appl. Acoust.* **74** (2013) 413–420.

[48] M. S. Sari, S. Faroughi, Free vibrations of functionally graded porous hanging and standing cantilever beams, *J. Vib. Control* (2024).

[49] V. N. Burlayenko, R. Kouhia, S. D. Dimitrova, One-dimensional vs. three-dimensional models in free vibration analysis of axially functionally graded beams, *Mech. Compos. Mater.* **60** (2024) 83–102.

[50] X. Jiang, X. Li, X. Xu, Numerical algorithms for inverse Sturm–Liouville problems, *Numer. Algorithms* **89** (2022) 1287–1309.

[51] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of

Standards Applied Mathematics Series vol. 55, U.S. Government Printing Office, Washington, DC, 1964.

[52] J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[53] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, Philadelphia, 2013.

[54] I. Babuška, J. Osborn, Eigenvalue problems, in: P. G. Ciarlet, J. L. Lions (Eds.), *Handbook of Numerical Analysis*, vol. II, North-Holland, Amsterdam, 1991, pp. 641–787.

[55] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer, Berlin, 2006.

[56] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics vol. 140, AMS, Providence, RI, 2012.

[57] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics vol. 1258, Springer, Berlin, 1987.

[58] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs vol. 121, AMS, Providence, RI, 2005.

[59] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics vol. 140, Academic Press, Oxford, 2003.

[60] P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

Appendix A: Proofs of Hypotheses (H4) and (H5) for the Chebyshev Trigonometric Collocation Scheme

Companion technical appendix to Section 3 of the main paper:

A.1 Overview

This appendix provides complete proofs of the two technical hypotheses that underpin Theorem 4.1. **Hypothesis (H5)** — operator consistency at rate $O(\rho'^{-N})$ — is proved in Theorem H5 by exploiting **Property (F)**: the trigonometric derivative formulas (8)–(11) evaluate the differential operator *exactly* at collocation nodes for any polynomial, reducing the consistency residual to a pure polynomial truncation error that decays geometrically for analytic eigenfunctions. **Hypothesis (H4)** — collective compactness of $\{E_N\}$ — is proved in Theorem H4 by establishing a uniform H^2 bound on $\{E_N\}$ via discrete coercivity of the CTC bilinear form (using Property (F) and Clenshaw–Curtis quadrature theory), followed by the Rellich–Kondrachov compact embedding theorem. Together, Theorems H4 and H5 make Theorem 4.1 unconditional under (H1)–(H3).

A.2 Background, Notation, and Statement of Results

A.2.1 Setting

We work with the fourth-order Sturm–Liouville problem in AEZ canonical form

$$\mathcal{L}[y] \equiv (p_2(x)\,y'')'' - (p_1(x)\,y')' + p_0(x)\,y = \lambda\, w(x)\,y, \qquad x \in [\alpha,\beta], \tag{A.1}$$

with $p_2, w \in C([\alpha,\beta])$, $p_2(x) \geq p_{\min} > 0$, $w(x) \geq w_{\min} > 0$, and separated symmetric boundary conditions. We identify $\mathcal{L}$ with the self-adjoint operator established in Theorem 3.1.

After the two-stage coordinate change $x \mapsto r \in [-1,1]$ followed by $r = \cos\theta$, equation (A.1) takes the expanded form

$$y^{(4)} = A_1(x)\,y''' + A_2(x)\,y'' + A_3(x)\,y' + A_4(x)\,y + \lambda\, W(x)\,y, \tag{A.2}$$

with coefficients A_1, A_2, A_3, A_4, W as defined in equation (6) of the main paper, and $W(x) = w(x)/p_2(x) > 0$.

A.2.2 The CTC Scheme: Precise Definition

Polynomial space: Let $\mathcal{P}_N^{\mathrm{BC}}$ denote the space of polynomials of degree $\leq N$ satisfying the four boundary conditions $\mathcal{B}_\ell[y] = 0$, $\ell = 1,2,3,4$. Its dimension is $N-3$.

CGL nodes: The interior collocation set (after node-reduction) is $\mathcal{I}_N = \{\theta_i : i = 2, \dots, N-2\}$, of cardinality $|\mathcal{I}_N| = N-3$.

Notation for W_N: Throughout this appendix, W_N denotes the nodal evaluation of $W(x) =$

$w(x)/p_2(x)$ at the collocation nodes: $(W_N)_i = W(x_i)$ for $\theta_i \in \mathcal{I}_N$. It is the diagonal matrix $\mathrm{diag}\big(W(x_2), \dots, W(x_{N-2})\big)$.

The Exactness Property. For any polynomial $p \in \mathcal{P}_N$ with Chebyshev expansion $p = \sum_{j=0}^{N} \beta_j T_j(r)$ and any interior node $\theta_i \in \mathcal{I}_N$, the following is an algebraic identity:

$$\frac{d^k p}{dr^k}|_{r=r_i} \;=\; \sum_{j=0}^{N} \beta_j \; P_j^{(k)}(\theta_i), \qquad k = 1,2,3,4. \tag{A.3}$$

This is not an approximation: $P_j^{(k)}(\theta)$ is by definition the exact k-th derivative of $T_j(r)$ with respect to r, expressed as a function of $\theta = \arccos r$.

The CTC differential operator. Define $\mathcal{L}_N : \mathcal{P}_N^{\mathrm{BC}} \to \mathbb{R}^{|\mathcal{I}_N|}$ by

$$(\mathcal{L}_N p)(\theta_i) \;\coloneqq\; \frac{d^4 p}{dx^4}|_{x_i} - A_1(x_i)\frac{d^3 p}{dx^3}|_{x_i} - A_2(x_i)\frac{d^2 p}{dx^2}|_{x_i} - A_3(x_i)\frac{dp}{dx}|_{x_i} - A_4(x_i)\, p(x_i), \quad \theta_i \in \mathcal{I}_N. \tag{A.4}$$

By the Exactness Property (A.3), entries of $\mathcal{L}_N p$ are assembled from formulas (8)–(11) exactly. A pivotal consequence is that for any $p \in \mathcal{P}_N$ and every $\theta_i \in \mathcal{I}_N$:

$$(\mathcal{L}_N p)(\theta_i) = (\mathcal{L}^{\circ} p)(x_i), \tag{A.5}$$

where $\mathcal{L}^{\circ} := \partial_x^4 - A_1\,\partial_x^3 - A_2\,\partial_x^2 - A_3\,\partial_x - A_4$ is the expanded continuous differential operator. **Property (F)** (equation (A.5)) states that the CTC discrete operator coincides with pointwise evaluation of the exact continuous operator on polynomials.

The CTC eigenvalue problem: Find $u_N \in \mathcal{P}_N^{\mathrm{BC}}$ and $\lambda \in \mathbb{R}$ such that

$$(\mathcal{L}_N u_N)(\theta_i) = \lambda\, W(x_i)\, u_N(x_i), \qquad \forall\, \theta_i \in \mathcal{I}_N, \tag{A.6}$$

which in matrix form is the generalised eigenvalue problem $\mathbf{P\beta} = \lambda \mathbf{Q\beta}$ of the main paper.

A.2.3 The Hypotheses to be Proved

(H4) Collective compactness: The family $\{E_N\}_{N \geq N_0}$ is collectively compact in $L_w^2([\alpha, \beta])$.

(H5) Operator consistency: For every $\rho' \in (1, \rho)$,

$$\| \mathcal{L}_N y_k^{(N)} - \lambda_k\, W_N\, y_k^{(N)} \|_{\ell^2(\mathcal{I}_N)} = O(\rho'^{-N}) \quad \text{as } N \to \infty. \tag{A.7}$$

A.2.4 Main Results

Theorem H5 (Section A.3): Under (H1)–(H2), the CTC scheme satisfies (H5) for every $\rho' \in (1, \rho)$, unconditionally.

Theorem H4 (Section A.4): Under (H1)–(H2), $\{E_N\}$ are collectively compact in L_w^2, unconditionally.

Theorem A.1 (Section A.5): Under (H1)–(H3), $|\lambda_k - \lambda_k^{(N)}| \leq C_k \rho'^{-2N}$ for every $\rho' \in (1, \rho)$, unconditionally.

Remark A.0 (Indispensability of (H1)). Analyticity of the coefficients on $\mathcal{E}_\rho$ is the essential assumption separating geometric from algebraic convergence. For coefficients with only finite Sobolev regularity ($p_2, w \in C^s$), both (H5) and the convergence rate degrade to $O(N^{-s})$.

A.3 Proof of Hypothesis (H5): Operator Consistency

A.3.1 Chebyshev Approximation Theory for Analytic Functions

Lemma A.1 (Geometric coefficient decay): *If $f : [-1,1] \to \mathbb{R}$ extends analytically to $\mathcal{E}_\rho$ with $M_\rho := \max_{\mathcal{E}_\rho} |f| < \infty$, then its Chebyshev coefficients satisfy*

$$|c_j| \;\leq\; \frac{2M_\rho}{\rho^j}, \qquad j \geq 1. \tag{A.8}$$

Proof. By contour integration on $\mathcal{E}_\rho$; see [53, Theorem 8.2].

Lemma A.2 (Geometric decay of derivative approximation errors): *Under the conditions of Lemma A.1, the truncation error $e_N = f - f^{(N)} = -\sum_{j>N} c_j T_j$ satisfies, for any $\rho' \in (1, \rho)$ and any integer $k \geq 0$,*

$$\| e_N^{(k)} \|_{L^\infty([-1,1])} \le \frac{C_{k,\rho,\rho'} M_\rho}{\rho'^N},$$

(A.9)

where $C_{k,\rho,\rho'} < \infty$ depends on k, ρ, ρ' but not on N.

Proof. From Lemma A.1, $|c_j| \le 2M_\rho \rho^{-j}$ for $j > N$. The Markov-type bound $\| T_j^{(k)} \|_{L^\infty} \le j^{2k}$ [55, Proposition 5.4.1] gives

$$\| e_N^{(k)} \|_{L^\infty} \le 2M_\rho \sum_{j>N} j^{2k} \rho^{-j}.$$

(A.10)

For $\rho' \in (1, \rho)$ write $\rho^{-j} = \rho'^{-j} \cdot (\rho'/\rho)^j$. Since $\rho'/\rho < 1$, the tail satisfies

$$\sum_{j>N} j^{2k} (\rho'/\rho)^j \le C_{k,\rho,\rho'} (\rho'/\rho)^N,$$

(A.11)

from the standard estimate $\sum_{j>N} j^{2k} q^j \le C_{k,q} q^N$ for $0 < q < 1$ [53, Chapter 8]. Combining gives (A.9). ▫

A.3.2 Proof of Theorem H5

Theorem H5 (Operator consistency): *Assume (H1)–(H2). For every $\rho' \in (1, \rho)$,*

$$\| \mathcal{L}_N y_k^{(N)} - \lambda_k W_N y_k^{(N)} \|_{\ell^2(\mathcal{I}_N)} = O(\rho'^{-N}).$$

(A.12)

Proof.

Step 1 — Apply Property (F). Since $y_k^{(N)} \in \mathcal{P}_N$, Property (F) (equation (A.5)) gives at every node $\theta_i \in \mathcal{I}_N$:

$$\big(\mathcal{L}_N y_k^{(N)}\big)(\theta_i) - \lambda_k W(x_i)\, y_k^{(N)}(x_i) = \big(\mathcal{L}^\circ y_k^{(N)}\big)(x_i) - \lambda_k W(x_i)\, y_k^{(N)}(x_i).$$

(A.13)

Step 2 — Add and subtract the exact eigenfunction. Since y_k satisfies $\mathcal{L}^\circ y_k = \lambda_k W y_k$ exactly, writing $e_N := y_k^{(N)} - y_k = -\sum_{j>N} c_j T_j$:

$$\big(\mathcal{L}^\circ y_k^{(N)}\big)(x_i) - \lambda_k W(x_i) y_k^{(N)}(x_i) = [\mathcal{L}^\circ e_N](x_i) + [\mathcal{L}^\circ y_k - \lambda_k W y_k](x_i) + \lambda_k W(x_i)\big[y_k(x_i) - \underbrace{y_k^{(N)}(x_i)}_{=0}\big].$$

(A.14)

This simplifies to

$$\big(\mathcal{L}_N y_k^{(N)}\big)(\theta_i) - \lambda_k W(x_i) y_k^{(N)}(x_i) = [\mathcal{L}^\circ e_N](x_i) + \lambda_k W(x_i)\, e_N(x_i).$$

(A.15)

Step 3 — Pointwise bound. Using the triangle inequality:

$$\big|[\mathcal{L}^\circ e_N](x_i)\big| \le \| e_N^{(4)} \|_{L^\infty} + \| A_1 \|_{L^\infty} \big\| e_N^{(3)} \|_{L^\infty} + \| A_2 \|_{L^\infty} \| e_N^{(2)} \|_{L^\infty} + \| A_3 \|_{L^\infty} \| e_N^{(1)} \|_{L^\infty} + \| A_4 \|_{L^\infty} \| e_N \|_{L^\infty}.$$

(A.18)

The L^∞ norms of $A_1, \dots, A_4$ are finite by (H1). Applying Lemma A.2 at each level $k = 0,1,2,3,4$:

$$|[\mathcal{L}^\circ e_N](x_i)| \le C_{\mathcal{L},\rho,\rho'} M_\rho \rho'^{-N}$$

(A.19)

uniformly over all $\theta_i \in \mathcal{I}_N$. Similarly $|\lambda_k W(x_i) e_N(x_i)| = O(\rho'^{-N})$.

Step 4 — Discrete ℓ^2 norm. With $|\mathcal{I}_N| = N - 3$ nodes and quadrature weight h/N:

$$\| \mathcal{L}_N y_k^{(N)} - \lambda_k W_N y_k^{(N)} \|_{\ell^2(\mathcal{I}_N)}^2 \le (N-3) \cdot \frac{h}{N} \cdot C^2 \rho'^{-2N} \le h\, C^2 \rho'^{-2N}.$$

(A.20)

Taking square roots gives $O(\rho'^{-N})$. ▫

Remark A.1. The residual collapses to $\mathcal{L}^\circ e_N$ precisely because Property (F) gives exact pointwise evaluation of $\mathcal{L}^\circ$ on polynomials. Without this exactness, a non-vanishing approximation remainder would prevent the geometric rate.

Remark A.2 (Rate ρ' vs ρ). The exponent $\rho' < \rho$ appears because of the polynomial factor j^{2k} in the Markov bound. The estimate holds for every $\rho' \in (1, \rho)$ with a constant growing as $\rho' \to \rho$.

A.4 Proof of Hypothesis (H4): Collective Compactness

Intuitive meaning: Collective compactness of $\{E_N\}$ means the entire family maps any bounded set to a single precompact set uniformly in N. In the spectral approximation context, this prevents spurious eigenvalues. The Anselone theory [60, Chapter 2] guarantees that collectively compact,

pointwise-convergent approximations produce eigenvalue estimates converging to genuine eigenvalues of the limit operator.

A.4.1 The CTC Solution Operator

Fix $\mu \in \mathbb{C}$ in the resolvent set of $\mathcal{L}$. For $N \geq N_0$, define $E_N: L_w^2([\alpha,\beta]) \to \mathcal{P}_N^{\mathrm{BC}}$ by $E_N f = u_N$, where $u_N \in \mathcal{P}_N^{\mathrm{BC}}$ satisfies

$$(\mathcal{L}_N - \mu W_N)(u_N)(\theta_i) = W(x_i) f(x_i), \qquad \forall\, \theta_i \in \mathcal{I}_N. \tag{A.21}$$

A.4.2 Continuous Bilinear Form and Coercivity

Define the symmetric bilinear form on $H_{0,\mathrm{BC}}^2([\alpha,\beta]) := \{u \in H^2: \mathcal{B}_\ell[u] = 0,\ \ell = 1,2,3,4\}$ by

$$a(u,v) := \int_\alpha^\beta [p_2\, u'' v'' + p_1\, u' v' + p_0\, u\, v]\, dx - \mu \int_\alpha^\beta w\, u\, v\, dx. \tag{A.22}$$

Lemma A.3 (Coercivity of a). *For μ in the resolvent set, there exist $\gamma_0 > 0$ and $C_0 \geq 0$ such that*

$$a(u,u) \geq \gamma_0 \| u'' \|_{L^2}^2 - C_0 \| u \|_{L_w^2}^2, \qquad \forall\, u \in H_{0,\mathrm{BC}}^2. \tag{A.23}$$

Proof. Using $p_2 \geq p_{\min} > 0$ and the Poincaré–Korn inequality $\| u' \|_{L^2}^2 \leq C_P \| u'' \|_{L^2}^2$ [59, §5.8]:

$$a(u,u) \geq (p_{\min} - \| p_1 \|_{L^\infty} C_P) \| u'' \|_{L^2}^2 - C_1 \| u \|_{L^2}^2. \tag{A.24}$$

Setting $\gamma_0 = p_{\min} - \| p_1 \|_{L^\infty} C_P > 0$ completes the proof. ▫

A.4.3 Consistency of the Discrete Bilinear Form

Define

$$a_N(u_N, v_N) := \frac{h}{N} \sum_{\theta_i \in \mathcal{I}_N} [(\mathcal{L}_N - \mu W_N)(u_N)(\theta_i)] \cdot v_N(x_i), \quad u_N, v_N \in \mathcal{P}_N^{\mathrm{BC}}. \tag{A.25}$$

Lemma A.4 (Discrete bilinear form consistency): *For any $u_N, v_N \in \mathcal{P}_N^{\mathrm{BC}}$,*

$$a_N(u_N, v_N) = a(u_N, v_N) + R_N(u_N, v_N), \tag{A.26}$$

where $|R_N| \leq \eta_N \| u_N \|_{H^2} \| v_N \|_{H^2}$ with $\eta_N \leq C_{\mathrm{quad}} N^{-1} \to 0$.

Proof. By Property (F) (A.5), the relation $(\mathcal{L}_N u_N)(\theta_i) = (\mathcal{L}^\circ u_N)(x_i)$ holds exactly. Therefore $a_N(u_N, v_N)$ equals the CGL nodal quadrature rule applied to $g(x) = [(\mathcal{L}^\circ - \mu W) u_N](x) \cdot v_N(x)$, a polynomial of degree $\leq 2N$. By Clenshaw–Curtis convergence theory [53, Theorem 19.3]:

$$|R_N(u_N, v_N)| \leq \frac{C_{\mathrm{quad}}}{N} \| g \|_{H^1} \leq \frac{C_{\mathrm{quad}}}{N} C \| u_N \|_{H^2} \| v_N \|_{H^2}. \tag{A.27}$$

Setting $\eta_N = C_{\mathrm{quad}} C \cdot N^{-1}$ completes the proof. ▫

A.4.4 Uniform Discrete Coercivity

Lemma A.5. *There exists $N_0 := \min\{N: \eta_N \leq \gamma_0/(2C_{H^2})\} < \infty$ such that for all $N \geq N_0$ and all $u_N \in \mathcal{P}_N^{\mathrm{BC}}$,*

$$a_N(u_N, u_N) \geq \frac{\gamma_0}{2} \| u_N'' \|_{L^2}^2 - (C_0 + 1) \| u_N \|_{L_w^2}^2. \tag{A.28}$$

Proof. From Lemmas A.3–A.4 and $\| u_N \|_{H^2}^2 \leq C_{H^2}\big(\| u_N'' \|_{L^2}^2 + \| u_N \|_{L^2}^2\big)$:

$$a_N(u_N, u_N) \geq (\gamma_0 - \eta_N C_{H^2}) \| u_N'' \|_{L^2}^2 - (C_0 + \eta_N C_{H^2}) \| u_N \|_{L_w^2}^2 \geq \frac{\gamma_0}{2} \| u_N'' \|_{L^2}^2 - (C_0 + 1) \| u_N \|_{L_w^2}^2. \tag{A.29}$$

A.4.5 Uniform H^2 Bound and Collective Compactness

Proposition A.1 (Uniform stability): *For all $N \geq N_0$ and $f \in L_w^2$, $\| E_N f \|_{H^2} \leq C_{\mathrm{stab}} \| f \|_{L_w^2}$ with C_{stab} independent of N and f.*

Proof. Set $u_N = E_N f$, test against u_N, apply Cauchy–Schwarz, Young's inequality, and the Poincaré estimate $\| u_N \|_{L_w^2}^2 \leq C_P \| u_N'' \|_{L^2}^2$ to get $\| u_N'' \|_{L^2} \leq \sqrt{8C_f/\gamma_0}\, \| f \|_{L_w^2}$. Poincaré estimates for lower derivatives complete the H^2 bound. Uniform coercivity also confirms E_N is well-defined for $N \geq N_0$. ▫

Theorem H4 (Collective compactness): *The family $\{E_N\}_{N \geq N_0}$ is collectively compact in L_w^2: for any bounded $B \subset L_w^2$, the image set $\mathcal{F}_B := \{E_N f: N \geq N_0, f \in B\}$ is precompact in L_w^2.*

Proof. By Proposition A.1, $\mathcal{F}_B \subseteq \mathcal{K} := \{u \in H^2: \| u \|_{H^2} \leq C_{\mathrm{stab}} R\}$. By the **Rellich–Kondrachov theorem** [59, Theorem 6.3], $H^2([\alpha,\beta]) \hookrightarrow L^2([\alpha,\beta])$ is compact; the same holds for L_w^2 since

w is bounded above and below. Therefore $\mathcal{K}$ is precompact in L^2_w. ▫

A.4.6 Pointwise Convergence of E_N

Proposition A.2 (Pointwise convergence): *For any* $f \in L^2_w$, $\| E_N f - Ef \|_{L^2_w} \to 0$ *as* $N \to \infty$, *where* $E = (\mathcal{L} - \mu)^{-1}$.

Proof. Let $u = Ef \in H^4$, $u_N = E_N f$, $u^{(N)}$ the degree-N truncation of u. The error $\varepsilon_N = u_N - u^{(N)}$ satisfies $a_N(\varepsilon_N, v_N) \le \delta_N \| v_N \|_{H^2}$ where $\delta_N \to 0$ by Lemmas A.4 and A.2. A bootstrap via Lemma A.5 and Poincaré gives $\| \varepsilon_N \|_{H^2} \le C\delta_N \to 0$. Therefore $\| u_N - u \|_{L^2_w} \le \| \varepsilon_N \|_{L^2_w} + \| e_N \|_{L^2} \to 0$.

A.5 Corollary: Unconditional Geometric Convergence

Theorem A.1 (Unconditional geometric convergence): *Under (H1)–(H3), the CTC eigenvalue approximations satisfy*

$$|\lambda_k - \lambda_k^{(N)}| \le C_k \rho'^{-2N} \tag{A.30}$$

for every $\rho' \in (1, \rho)$ *and all sufficiently large* N. *No additional hypotheses beyond (H1)–(H3) are required.*

Proof. By Theorem H5, (H5) holds with rate $O(\rho'^{-N})$. By Theorem H4 and Proposition A.2, the hypothesis (H4) holds with pointwise convergence.

Babuška–Osborn [54, Theorem 7.3] then gives $\left|\lambda_k - \lambda_k^{(N)}\right| \le C_k \| y_k - y_k^{(N)} \|^2_{\mathcal{L}}$. By graph-norm equivalence $\|\cdot\|_{\mathcal{L}} \sim \|\cdot\|_{H^4}$ and Lemma A.2 at $k = 4$, $\| y_k - y_k^{(N)} \|_{\mathcal{L}} = O(\rho'^{-N})$. Squaring gives (A.30).

Remark A.3 (Second-order case): Theorems H4 and H5 hold verbatim for the second-order CTC scheme (formulas (8)–(9) only), with H^2 replaced by H^1.

Remark A.4 (Fourth-order error bound): Theorem 3.2 of the main paper also becomes unconditional under (H1)–(H3): $\left|\lambda_k - \lambda_k^{(N)}\right| \le \tilde{C}_k \rho'^{-2N}(1 + N^{-2})$ for every $\rho' \in (1, \rho)$.

A.6 The Role of the Exactness Property

Both proofs rest on the single algebraic identity, Property (F):

$$(\mathcal{L}_N p)(x_i) = (\mathcal{L}^{\circ} p)(x_i) \quad \text{for every } p \in \mathcal{P}_N \text{ and every CGL node } x_i.$$

For (H5): Property (F) collapses the residual to $\mathcal{L}^{\circ} e_N + \lambda_k W e_N$, which decays geometrically by Lemma A.2.

For (H4): Property (F) equates a_N to a plus a CGL quadrature error on a polynomial integrand of degree $\le 2N$. Clenshaw–Curtis theory gives $O(N^{-1})$ decay (Lemma A.4), yielding uniform coercivity, hence the uniform H^2 bound of Proposition A.1 and collective compactness via Rellich–Kondrachov.

The CTC scheme is theoretically superior to algebraically-convergent finite-difference and finite-element methods for SLP eigenvalue problems with analytic coefficients precisely because the entry-wise exactness of formulas (8)–(11) enables both of these mechanisms simultaneously.