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New Oscillation Criteria for Third-Order Non-Linear Neutral Delay Differential Equations

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Abstract:

This paper investigates the oscillatory behaviour of solutions to a class of third-order nonlinear neutral delay differential equations of the form:

$$\left(\varrho(t)Q''(t)\right)' + \alpha(t)f\left(x\left(b(t)\right)\right) = 0,\ t \geq t_0$$

where the neutral term is defined as:

Q(t) = x(t) + R(t)x(l(t)),

We operate under the assumption that the canonical integral condition

$\int_{t_0}^{\infty} \rho^{-1}(s)ds = \infty$ holds, ensuring the divergence of the potential function. By employing Riccati transformation techniques and integral inequality methods, we establish new sufficient conditions that guarantee all solutions are either oscillatory or converge to zero. This work accounts for the nonlinear functional f, assuming only that f(u) ≥ 0, u $\geq k > 0$ for u $\neq 0$. The results further incorporate the commutativity of delay operators l(t) & b(t) satisfies l $\circ\, b = b \circ l$ to simplify the estimation of the neutral coefficient R(t). Our findings extend and complement existing criteria in the literature.

Keywords: Oscillation Criteria, Third-order Differential Equation, Nonlinear delay, Neutral equations, Riccati transformation.

Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this paper, we consider the third order non-linear delay differential equation of the form

$$\left(\varrho(t)Q''(t)\right)' + \alpha(t)f\left(x\left(b(t)\right)\right) = 0, t \geq t_0 \tag{1.1}$$

where the neutral term is defined as:

$$Q(t) = x(t) + R(t)x(l(t)), \tag{1.2}$$

and the following hypotheses are satisfied:

(H_1) $\varrho, \alpha, R \in C([t_0,\infty), R)$, $b \in C^1([t_0,\infty), R)$, $l \in C([t_0,\infty),(0,\infty))$, and $\alpha(t) > 0$;

(H_2) $\varrho > 0, \int_{t_0}^{\infty} \varrho^{-1}(s)ds = \infty$ and $0 \leq R(t) \leq R_0 < \infty$; (1.3)

(H_3) $b(t) \leq t, l(t) \leq t, l'(t) \geq l_0 > 0, \lim_{t\to\infty} b(t) = \lim_{t\to\infty} l(t) = \infty$, and $l \circ b = b \circ l$ (1.4)

A solution of (1.1) is defined as nontrivial solutions of (1.1) $x \in C([t_x, \infty), [0, \infty))$ with

$t_x \geq t_{0,}$ which satisfies (1.1) and $\varrho(Q'') \in C([t_x, \infty), (0, \infty))$ on $[t_x, \infty)$. Those solutions of (1.1) exist on some half-line $[t_x, \infty)$ and satisfy sup $\{|x(t)| : t_a \leq t < \infty\} > 0$ for all

$t_a \geq t_x$. A solution x is called oscillatory if it becomes either positive or negative. Otherwise, it is called a non-oscillatory solution.

2. Preliminary results:

The following lemmas play a fundamental role in proving the subsequent results.

Lemma 2.1. [3, Lemma3] Suppose that $g_1, g_2, g_3 \in C([t_0, \infty), R), \lim_{t\to\infty} g_1(t)$ exists,

$$g_3(t) \leq t,$$

$$\lim_{t\to\infty} g_3(t) = \infty \text{ for all t} \in [t_0, \infty),$$

and

$$\lim_{t\to\infty} \inf g_2(t) > -1.$$

Moreover, assume that there is a function y∈ $C([t_*, \infty), R^+)$, where $t_* = min_{t\in[t_0,\infty)}\{g_3(t)\}$,

such that $y(t) - g_1(t) - g_2(t)y\big(g_3(t)\big) = 0$, for all t ∈ $[t_0, \infty)$,

If $\lim_{t\to\infty} sup\; y(t) > 0$, then

$$\lim_{t\to\infty} g_1(t) > 0.$$

Lemma 2.2.[7] Assume that x is a non-oscillatory solution of (1.1). Then the corresponding function Q satisfies

$$\text{Q(t)} > 0, Q''(t) > 0, \quad \big(\varrho(t)(Q''(t))\big)' \leq 0.$$

Moreover, there exist only two possible cases for the derivative of Q:

$$Q'(t) < 0 \qquad (2.1)$$

and

$$Q'(t) > 0 \qquad (2.2)$$

Theorem 2.1: [11] Suppose b(t) is non- decreasing function such that b(t)≥ t for $t < t_0$ and $f \in C(R, R)$ and yf(y)> $0\; for\; y \neq 0$ suppose that f satisfies the following condition

$$\lim_{|y|\to 0} sup \frac{y}{f(y)} = M, 0 \leq M < \infty \qquad (2.3)$$

If $\lim_{t\to\infty} inf \int_t^{\tau(t)} p(s)ds > \frac{M}{e}$ hold, (2.4)

then all solutions of (1.1) oscillate

Suppose that M > 0. Then, by virtue of (2.3) we can choose $t_2 \geq t_1$ such that

f(y(t)) $\geq \frac{1}{2M}\; y(t)$ for all $t \geq t_2$ (2.5)

In 2025, Asma AI-Jaser [6] considered the following equation

$$\big(\varrho(t)Q''(t)\big)' + \alpha(t)(x\big(b(t)\big)) = 0, t \geq t_0 \qquad (2.6)$$

We say that (2.6) has property F if any solution x of (2.6) is either oscillatory or satisfies

$x(t) \to 0 \; as \; t \to \infty$ (2.7)

3. MAIN RESULTS

In the section, we will adopt the following notation:

$$\beta_1(t, t_1) = \int_{t_1}^{t} \varrho^{-1}(s)\, ds,$$

$$\beta_2(t, t_1) = \int_{t_1}^{t} \int_{t_1}^{s} \varrho^{-1}(u) du ds,$$

and $\tilde{\alpha}$ (t) = min{α(t), $\alpha(l(t))$}. (3.1)

Lemma 3.1. Suppose that $x(t) > 0$ is a solution of Eq (1.1) and

$$\lim_{t\to\infty} x(t) \neq 0. \quad (3.2)$$

If $\int_{t_1}^{\infty} \int_{v}^{\infty} (\varrho^{-1}(l(u)) \int_{u}^{\infty} \tilde{\alpha}\,(\mathrm{s})\mathrm{ds})\ \mathrm{dudv} > 2\mathrm{M}$ (3.3)

then (2.2) holds.

Proof: Let $x > 0$ be a solution of (1.1). From (1.2), we see that

$$(\varrho(t)(Q''(t)))' = -\alpha(t) f\left(x(b(t))\right) \leq 0$$

Using l(t) $\leq t$, we have $\varrho(t)(Q''(t)) \leq \rho(l(t))(Q''(l(t)))$. (3.4)

Hence, $(\varrho(t)(Q''(t)))' \leq 0$ has one sign. Also $Q''(t)$ has one sign, and so either $Q''(t) > 0$ or $Q''(t) < 0$ for t $\geq t_1$. Let $Q''(t) < 0$. Then there is a constant N > 0 such that

$$\varrho(t)(Q''(t)) + N \leq 0. \quad (3.5)$$

Integrating (3.5) from $t_1 \; to \; t$, we get

$$Q'(t) - Q'(t_1) \leq -N \int_{t_1}^{t} \varrho^{-1}(s) ds.$$

Therefore, $\lim_{t\to\infty} Q'(t) = -\infty$. Using the fact that $Q'(t)$ and $Q''(t)$ are negative, we note that $\lim_{t\to\infty} Q'(t) = -\infty$. This contradiction leads to $Q''(t) > 0$.

Now, let $Q'(t) < 0$. According to (1.1), we obtain

$$(\rho(t)(Q''(t)))' + \frac{R_0(\varrho(l(t))(Q''(l(t))))'}{l_0} \leq \frac{1}{2M}[-\alpha(t)x(b(t)) - R_0\alpha(l(t))x\left(b(l(t))\right)].$$

Set $\tilde{Q}(t) = (\rho(t)(Q''(t)))' + \frac{R_0(\varrho(l(t))(Q''(l(t))))'}{l_0}$.

That is $\tilde{Q}(t) \leq -\frac{1}{2M}[\left(\alpha(t)x(b(t)) + R_0\alpha(l(t))x\left(b(l(t))\right)\right].$

$$\leq -\frac{1}{2M}[\left(\tilde{\alpha}\,(t)(x(b(t)) + R_0 x\left(b(l(t))\right)\right)]$$

In view of (3.1), we have $\tilde{Q}(t) + \frac{1}{2M}\left[Q(b(t))\tilde{\alpha}\,(t)\right] \leq 0.$ (3.6)

Integrating (3.6) from t to ∞, we get

$$\rho(t)(Q''(t)) + \frac{R_0\rho(l(t))(Q''(l(t)))}{l_0} - \frac{1}{2M}\int_{t}^{\infty} Q(b(s))\tilde{\alpha}\,(s)\, ds \leq 0,$$

Using (3.4), we have

$$\rho(l(t))(Q''(l(t))) - \frac{l_0}{l_0+R_0}\frac{1}{2M}\int_t^\infty Q\big(b(s)\big)\tilde{\alpha}(s)\,ds \le 0.$$

Based on (3.2) and Lemma 2.1, and since Q(b(t)) $\geq L$ and $\lim_{t\to\infty} Q(t) = L > 0$, then we obtain

$$Q''(l(t)) - \frac{Ll_0}{l_0+R_0}\frac{1}{2M}\rho^{-1}(l(u)) \int_t^\infty \tilde{\alpha}(s)ds \ge 0.$$

Integrating the above inequality from t to ∞, we get

$$\frac{-Q'(l(t))}{l_0} - \frac{Ll_0}{l_0+R_0}\frac{1}{2M}\int_t^\infty \rho^{-1}(l(u))\,(\int_t^\infty \tilde{\alpha}(s)ds)du \ge 0. \tag{3.7}$$

Integrating again (3.7) from t_1 to ∞, we find

$$\frac{-Q(l(t))}{l_0} - \frac{Ll_0}{l_0+R_0}\frac{1}{2M}\int_{t_1}^\infty\int_v^\infty \rho^{-1}(l(u))\,(\int_u^\infty \tilde{\alpha}(s)ds)dudv \ge 0.$$

$$\int_{t_1}^\infty\int_v^\infty \rho^{-1}(l(u))\,(\int_u^\infty \tilde{\alpha}(s)ds)dudv \ge 2M.$$

Which implies that $Q'(t)$ is positive. This completes the proof.

Lemma 3.2. Suppose that Q is satisfied (2.2) for t $\geq t_1$. Then

$$\rho(t)\beta_1(t,t_1) \le \frac{Q'(t)}{Q''(t)} \tag{3.8}$$

and

$$\rho(t)\beta_2(t,t_1) \le \frac{Q(t)}{Q''(t)} \quad . \tag{3.9}$$

Proof: Since $\rho(t)\big(Q''(t)\big)$ is nonincreasing, we have

$$Q'(t) \ge \int_{t_1}^t \frac{1}{\rho(s)}\Big(\rho(s)\big(Q''(s)\big)\Big)ds \ge \rho(s)\big(Q''(s)\big)\int_{t_1}^t \frac{1}{\rho(s)}\text{ds},$$

Which implies that Q(t) - $\rho(s)\big(Q''(s)\big)\int_{t_1}^t(\int_{t_1}^s \frac{1}{\rho(u)}du)ds \ge 0.$

This completes the proof.

In the following results, we will assume that there exists a function $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$, for all $t_1 \geq t_0$.

Theorem 3.1. Suppose that (3.3) is satisfied, b(t) $\leq l(t)$, and $b'(t) > 0$. If

$$\lim_{t\to\infty} sup \int_{t_2}^t [\rho(s)\,\tilde{\alpha}(s) - \frac{(l_0+R_0)}{4l_0 b'(s)\rho(s)\beta_1(b(s),t_1)}((\rho'(s))_+)^2]\text{ds} > 2M, \tag{3.10}$$

for $t_2 > t_1$, then (1.1) has property F.

Proof:

Assume that $x > 0$ is a solution of (1.1). As in the proof of Lemma 3.1, from (2.2) in Lemma 2.2 and (3.6), and by using Lemma 3.2, we get (3.8). Set

$$\omega(t) = \frac{\varrho(t)Q''(t)}{Q(b(t))}\rho(t) > 0. \tag{3.11}$$

From Lemma 3.1, we see that

$$\omega'(t) = \frac{\varrho(t)Q''(t)}{Q(b(t))}\rho'(t) + \left(\frac{\varrho(t)(Q''(t))}{Q(b(t))}\right)'\rho(t)$$

$$= \frac{\varrho(t)Q''(t)}{Q(b(t))}\rho'(t) + \frac{\big(\varrho(t)(Q''(t))\big)'}{Q(b(t))}\rho(t) - \frac{\varrho(t)(Q''(t))Q'(b(t))\rho(t)b'(t)}{Q^2(b(t))}$$

$$= \frac{\varrho(t)Q''(t)}{Q(b(t))}\rho'(t) + \frac{\big(\varrho(t)(Q''(t))\big)'}{Q(b(t))}\rho(t)$$

$$- \frac{\varrho(t)Q''(t)Q'(b(t))}{Q^2(b(t))}\rho(t)b'(t) \tag{3.12}$$

In view of (2.2), (3.8), and $b(t) \leq t$, we obtain

$$Q'\big(b(t)\big) \geq \varrho\big(b(t)\big)Q''\big(b(t)\big)\beta_1(b(t),t_1)$$
$$\geq \big(\varrho(t)Q''(t)\big)\beta_1(b(t),t_1).$$

Using (3.11) and (3.12), we get

$$\omega'(t) \leq \rho(t)\frac{\big(\varrho(t)(Q''(t)\big)'}{Q(b(t))} + \frac{1}{\rho(t)}\rho'(t)\omega(t) - \frac{\beta_1(b(t),t_1)}{\rho(t)}b'(t)\omega^2(t). \tag{3.13}$$

Define another function as follows:

$$v(t) = \frac{\varrho\big(l(t)\big)Q''\big(l(t)\big)}{Q\big(b(t)\big)}\rho(t). \tag{3.14}$$

That is, according to Lemma 3.1, $v(t) > 0$, and

$$v'(t) = \rho'(t)\frac{\varrho\big(l(t)\big)\big(Q''\big(l(t)\big)\big)}{Q\big(b(t)\big)} + \rho(t)\left(\frac{\varrho\big(l(t)\big)\big(Q''\big(l(t)\big)\big)}{Q\big(b(t)\big)}\right)'$$

$$= \rho'(t)\frac{\varrho\big(l(t)\big)\big(Q''\big(l(t)\big)\big)}{Q\big(b(t)\big)} + \frac{\Big(\varrho\big(l(t)\big)(Q''\big(l((t))\big)\Big)'}{Q\big(b(t)\big)}\rho(t)$$
$$- \frac{\varrho(l(t))(Q''(l((t)))Q'(b(t))\rho(t)b'(t)}{Q^2(b(t))}$$

$$= \rho'(t)\varrho\big(l(t)\big)\frac{\big(Q''\big(l(t)\big)\big)}{Q\big(b(t)\big)} + \frac{\Big(\varrho\big(l(t)\big)(Q''\big(l((t))\big)\Big)'}{Q\big(b(t)\big)}\rho(t)$$
$$- \rho(t)\varrho\big(l(t)\big)\left(\frac{(Q''(l((t)))Q'(b(t)}{Q^2\big(b(t)\big)}\right)b'(t). \tag{3.15}$$

Taking (2.2) and (3.8) with the fact that b(t) $\leq l(t)$ into account, we have

$$Q'\big(b(t)\big) \geq (\varrho\big(b(t)\big)Q''(b(t)))\beta_1(b(t),t_1) \geq (\varrho(l(t))Q''(l(t)))\beta_1(b(t),t_1),$$

Which from (3.14) and (3.15) implies that

$$v'(t) \leq \frac{(\varrho\big(l(t)\big)(Q''(l(t)))'}{Q\big(b(t)\big)}\rho(t) + v(t)\frac{\big(\rho'(t)\big)_+}{\rho(t)} - b'(t)v^2(t)\frac{\beta_1(b(t),t_1)}{\rho(t)}. \tag{3.16}$$

Using (3.13) and (3.16), we obtain

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{\rho(t)}{Q\big(b(t)\big)}\left[\Big(\varrho(t)\big(Q''(t)\big)\Big)' + \frac{R_0(\varrho(l(t))(Q''\big(l(t)\big)))'}{l_0}\right]$$
$$+\omega(t)\frac{(\rho'(t))}{\rho(t)} - \omega^2(t)\frac{\beta_1(b(t),t_1)}{\rho(t)}b'(t)$$
$$+\frac{R_0}{l_0}\left[v(t)\frac{\big(\rho'(t)\big)}{\rho(t)} - v^2(t)\frac{\beta_1(b(t),t_1)}{\rho(t)}b'(t)\right]. \tag{3.17}$$

By (3.6) and (3.17), we have

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{1}{2M}\left[\begin{matrix} -\rho(t)\tilde{\alpha}(t) + \omega(t)\frac{\big(\rho'(t)\big)_+}{\rho(t)} - \omega^2(t)\frac{\beta_1(b(t),t_1)}{\rho(t)}b'(t) \\ +\ \frac{R_0}{l_0}\left[v(t)\frac{\big(\rho'(t)\big)}{\rho(t)} - v^2(t)\frac{\beta_1(b(t),t_1)}{\rho(t)}b'(t)\right] \end{matrix}\right]. \tag{3.18}$$

By applying the inequality stated in [15], namely,

$$\frac{B^2}{4A} \geq Bu - Au^2, A > 0. \tag{3.19}$$

That is

$$\frac{\rho(t)}{4\beta_1(b(t),t_1)b'(t)} \left(\frac{(\rho'(t))_+}{\rho(t)}\right)^2 \geq \omega(t)\left(\frac{\left(\rho'(t)\right)_+}{\rho(t)}\right) - \omega^2(t) \left(\frac{\beta_1(b(t),t_1)b'(t)}{\rho(t)}\right)$$

and

$$\frac{\rho(t)}{4\beta_1(b(t),t_1)b'(t)} \left(\frac{(\rho'(t))_+}{\rho(t)}\right)^2 \geq v(t)\left(\frac{\left(\rho'(t)\right)_+}{\rho(t)}\right) - v^2(t) \left(\frac{\beta_1(b(t),t_1)b'(t)}{\rho(t)}\right),$$

Which implies that

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{1}{2M}\begin{bmatrix} -\rho(t)\tilde{\alpha}(t) + \frac{1}{4}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)b'(t)\beta_1(b(t),t_1)} \\ +\frac{R_0}{4l_0}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)b'(t)\beta_1(b(t),t_1)} \end{bmatrix}. \tag{3.20}$$

Integrating from t_2 to t, we see that

$$\frac{1}{2M}\int_{t_2}^{t}\left(\rho(s)\tilde{\alpha}(s) - \frac{l_0+R_0}{4l_0}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)b'(s)\beta_1(b(s),t_1)}\right)\mathrm{ds} \leq \left(\omega(t_2) + \frac{R_0 v(t_2)}{l_0}\right)$$

$$\int_{t_2}^{t}\left(\rho(s)\tilde{\alpha}(s) - \frac{l_0+R_0}{4l_0}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)b'(s)\beta_1(b(s),t_1)}\right)\mathrm{ds} \leq 2M\left(\omega(t_2) + \frac{R_0 v(t_2)}{l_0}\right)$$

Taking limit superior, we get

$$\lim_{t\to\infty} sup \int_{t_2}^{t}[\rho(s)\,\tilde{\alpha}\,(\mathrm{s}) - \frac{(l_0+R_0)}{4l_0 b'(s)\rho(s)\beta_1(b(s),t_1)}((\rho'(s))_+)^2]\mathrm{ds} \leq 2M,$$

Which is contradiction with (3.10). The proof is complete.

Theorem 3.2:

Let (3.3) be satisfied, b(t)$\geq l(t)$. If

$$\lim_{t\to\infty} sup \int_{t_2}^{t}[\,\rho(s)\,\tilde{\alpha}\,(\mathrm{s}) - \frac{(l_0+R_0)\left((\rho'(s))_+\right)^2}{4l_0 l'(s)\rho(s)\beta_1(l(s),t_1)}]\mathrm{ds} > 2M, \tag{3.21}$$

for $t_2 > t_1$, then (1.1) has property F.

Proof:

Assume that $x > 0$ is a solution of (1.1). As in the proof of Lemma 3.1, from (2.2) in Lemma 2.2 and (3.6), and by using Lemma 3.2, we get (3.8). Define

$$\omega(t) = \frac{\varrho(t)Q''(t)}{Q(l(t))}\rho(t) \tag{3.22}$$

Then

$$\begin{aligned}\omega'(t) &= \frac{\varrho(t)Q''(t)}{Q(l(t))}\rho'(t) + \left(\frac{\varrho(t)(Q''(t))}{Q(l(t))}\right)'\rho(t)\\ &= \frac{\varrho(t)Q''(t)}{Q(l(t))}\rho'(t) + \frac{\left(\varrho(t)(Q''(t))\right)'}{Q(l(t))}\rho(t) - \frac{\varrho(t)(Q''(t))Q'(l(t))\rho(t)l(t)}{Q^2(l(t))}\\ &= \frac{\varrho(t)Q''(t)}{Q(l(t))}\rho'(t) + \frac{\left(\varrho(t)(Q''(t))\right)'}{Q(l(t))}\rho(t)\\ &\quad - \frac{\varrho(t)Q''(t)Q'(l(t))}{Q^2(l(t))}\rho(t)l'(t)\end{aligned} \tag{3.23}$$

It follows from (2.2), (3.8), and $l(t) \leq t$, that

$$Q'(l(t)) \geq \varrho(l(t))Q''(l(t))\beta_1(l(t),t_1)$$
$$\geq (\rho(t)Q''(t))\beta_1(l(t),t_1).$$

Using (3.22) and (3.23), we get

$$\omega'(t) \leq \rho(t)\frac{\left(\varrho(t)(Q''(t)\right)'}{Q(l(t))} + \frac{1}{\rho(t)}\rho'(t)\omega(t) - \frac{\beta_1(l(t),t_1)}{\rho(t)} l'(t)\omega^2(t). \quad (3.24)$$

Define another function as follows:

$$v(t) = \frac{\varrho(l(t))Q''(l(t))}{Q(l(t))}\rho(t). \quad (3.25)$$

Then, according to Lemma 3.1, $v(t) > 0$, and

$$v'(t) = \rho'(t)\frac{\varrho(l(t))\left(Q''(l(t))\right)}{Q(b(t))} + \rho(t)\left(\frac{\varrho(l(t))\left(Q''(l(t))\right)}{Q(l(t))}\right)'$$

$$= \rho'(t)\frac{\varrho(l(t))\left(Q''(l(t))\right)}{Q(b(t))} + \frac{\left(\varrho(l(t))(Q''(l((t)))\right)'}{Q(l(t))} - \frac{\varrho(l(t))(Q''(l((t)))Q'(l(t))\rho(t)b'(t)}{Q^2(l(t))}$$

$$= \rho'(t)\varrho(l(t))\frac{\left(Q''(l(t))\right)}{Q(l(t))} + \frac{\left(\varrho(l(t))(Q''(l((t)))\right)'}{Q(l(t))}\rho(t)$$

$$-\rho(t)\varrho(l(t))\left(\frac{(Q''(l((t)))Q'(b(t)}{Q^2(l(t))}\right)l'(t). \quad (3.26)$$

Taking (2.2) and (3.8), we have

$$Q'(l(t)) \geq \beta_1(l(t),t_1)\varrho(l(t))Q''(l(t))),$$

Which from (3.25) and (3.26) implies that

$$v'(t) \leq \frac{(\varrho(l(t))(Q''(l(t)))'}{Q(l(t))}\rho(t) + v(t)\frac{\left(\rho'(t)\right)_+}{\rho(t)} - b'(t)v^2(t)\frac{\beta_1(l(t),t_1)}{\rho(t)}. \quad (3.27)$$

Due to (3.24) and (3.27), we obtain

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{\rho(t)}{Q(l(t))}\left(\varrho(t)\left(Q''(t)\right)\right)' + \frac{R_0(\varrho(l(t))(Q''(l(t))))'}{l_0}$$
$$+\omega(t)\frac{(\rho'(t))}{\rho(t)} - \omega^2(t)\frac{\beta_1(l(t),t_1)}{\rho(t)}l'(t)$$
$$+\frac{R_0}{l_0}\left[v(t)\frac{\left(\rho'(t)\right)}{\rho(t)} - v^2(t)\frac{\beta_1(l(t),t_1)}{\rho(t)}l'(t)\right]. \quad (3.28)$$

From (2.2), (3.6) and (3.28), and b(t) $\geq l(t)$, we obtain

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{1}{2M}\left[\begin{array}{c} -\rho(t)\tilde{\alpha}(t) + \omega(t)\frac{\left(\rho'(t)\right)_+}{\rho(t)} - \omega^2(t)\frac{\beta_1(l(t),t_1)}{\rho(t)}l'(t) \\ + \frac{R_0}{l_0}\left[v(t)\frac{\left(\rho'(t)\right)}{\rho(t)} - v^2(t)\frac{\beta_1(l(t),t_1)}{\rho(t)}l'(t)\right] \end{array}\right]. \quad (3.29)$$

By (3.29) and (3.19), we get

$$\omega'(t) + v'(t)\frac{R_0(t)}{l_0} \leq \frac{1}{2M}\left[\begin{array}{c} -\rho(t)\tilde{\alpha}(t) + \frac{1}{4}\frac{((\rho'(t))_+)^2}{\rho(t)l'(t)\beta_1(l(t),t_1)} \\ + \frac{R_0}{4l_0}\frac{((\rho'(t))_+)^2}{\rho(t)l'(t)\beta_1(l(t),t_1)} \end{array}\right]. \quad (3.30)$$

Integrating (3.30) from t_2 to t , we see that

$$\frac{1}{2M}\int_{t_2}^{t}\left(\rho(s)\tilde{\alpha}(s)-\frac{l_0+R_0}{4l_0}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)l'(s)\beta_1(l(s),t_1)}\right)\mathrm{ds} \le \left(\omega(t_2)+\frac{R_0 v(t_2)}{l_0}\right)$$

$$\int_{t_2}^{t}\left(\rho(s)\tilde{\alpha}(s)-\frac{l_0+R_0}{4l_0}\frac{\left((\rho'(t))_+\right)^2}{\rho(t)l'(s)\beta_1(l(s),t_1)}\right)\mathrm{ds} \le 2M\left(\omega(t_2)+\frac{R_0 v(t_2)}{l_0}\right).$$

Taking limit superior, we get

$$\lim_{t\to\infty} sup \int_{t_2}^{t}[\rho(s)\,\tilde{\alpha}\,(s)-\frac{(l_0+R_0)}{4l_0 l'(s)\rho(s)\beta_1(l(s),t_1)}((\rho'(s))_+)^2]\mathrm{ds} \le 2M,$$

Which is contradiction with (3.21). The proof is complete.

Remark 3.1: From Theorems 3.1 and 3.2, we can get some oscillation criteria for (1.1) with different choices of ρ.

3.1. Absence of solutions in case (2.1)

Theorem 3.3:

If there exists a function $\zeta(t) \in C\left([T_{0,}\infty),(0,\infty)\right)$, b(t)$<\zeta(t)$, and $l^{-1}(\zeta(t)) < t$ such that the first – order delay differential equation

$$w'(t)+\frac{l_0\,\tilde{\alpha}(t)\beta_2(\zeta(t),b(t))}{l_0+R_0}\,w\left(l^{-1}(\zeta(t))\right) \ge 2M \qquad (3.31)$$

is oscillatory, then case (2.1) cannot hold.

Proof:

Assume that $x > 0$ is a solution of (1.1). Let (2.1) in Lemma 2.2 hold. By (3.6), we obtain

$$\tilde{Q}\,(\mathrm{t})+\frac{1}{2M}\,Q(b(t))\tilde{\alpha}(t) \le 0 \qquad (3.32)$$

It follows from $\varrho(t)(Q''(t)) \ge 0$ that

$$-Q'(u) \ge Q'(v)-Q'(u)=\int_u^v Q''(s)\frac{\varrho(s)}{\varrho(s)}\,ds$$

$$\ge\ \varrho(v)Q''(v)\int_u^v \frac{1}{\varrho(s)}\,ds, \text{ for } v \ge u \ge t_1.$$

From (3.9), we have

$$Q(b(t)) \ge\ \varrho(\zeta)(Q''(\zeta))\beta_2(\zeta,b(t)),$$

Which from (3.32) implies that

$$\tilde{Q}\,(t) \le \frac{1}{2M}(-\tilde{\alpha}(t)\beta_2(\zeta,b(t))\,\varrho(\zeta)(Q''(\zeta))) \qquad (3.33)$$

Since $\varrho(t)Q''(t)$ is nonincreasing, we get

$$\tilde{Q}\,(t) \le \frac{(l_0+R_0)\,\varrho(l(t))Q''(l(t))}{l_0}. \qquad (3.34)$$

Hence, we have

$$\varrho(l(t))Q''(l(t)) \ge \frac{l_0\tilde{Q}\left((l^{-1}(\zeta(t))\right)}{(l_0+R_0)}. \qquad (3.35)$$

Combining (3.34) and (3.33), we note that w is a positive solution of the first-order delay differential inequality

$$\left(\ \tilde{Q}\,(t)\right)' \le \frac{1}{2M}\frac{l_0\,\tilde{\alpha}(t)\beta_2(\zeta(t),b(t))}{l_0+R_0}\ \tilde{Q}\,\left(l^{-1}(\zeta(t))\right).$$

Then, the Eq (3.31) has a positive solution, which is a contradiction.

The proof is complete.

Theorem 3.4:

If there exists a function $\alpha(t) \in C\big([t_0,\infty),(0,\infty)\big), \alpha(t) < t$ and b(t) $< l(\alpha(t))$ such that $\liminf_{t\to\infty} \beta_2(l(\alpha(t)), b(t)) \int_t^{\alpha(t)} \tilde{\alpha}(s)ds > 2M\left(\frac{l_0+R_0}{l_0}\right)$ (3.37)

Then case (2.1) cannot hold.

Proof:

Following the same method as in the proof of Theorem 3.3 by integrating (3.32) from $\alpha(t)$ to t and using the fact that Q is decreasing, we see that

$$H'(\alpha(t)) \geq \varrho(t)Q''(t) + l_0^{-1}R_0\varrho(l(t))(Q''(l(t))) \int_t^{\alpha(t)} \tilde{\alpha}(s)\frac{Q(b(s))}{2M}ds$$

$$\geq \int_t^{\alpha(t)} \tilde{\alpha}(s)\frac{Q(b(s))}{2M}ds \geq \frac{Q(b(t))}{2M}\int_t^{\alpha(t)} \tilde{\alpha}(s)ds,$$

where

$$H(\alpha(t)) = (\varrho(\alpha(t))\left(Q''(\alpha(t))\right) + l_0^{-1}R_0\varrho(l(\alpha(t)))(Q''(l(\alpha(t))))'.$$

Since $l(\varrho(t)Q''(t))' \leq 0$ and $\alpha(t) < l(t)$, we have

$$\varrho(l(\alpha(t)))(Q''(l(\alpha(t))))\,(l_0^{-1}l_0 + R_0) \geq \frac{Q(b(t))}{2M}\int_t^{\alpha(t)} \tilde{\alpha}(s)ds. \quad (3.38)$$

Using (3.9) in (3.38), we see that

$$\frac{\varrho(l(\alpha(t)))(Q''(l(\alpha(t))))}{(l_0+R_0)^{-1}l_0} \geq \frac{1}{2M}\,(\varrho(l(\alpha(t)))(Q''(l(\alpha(t))))\,\beta_2(l(\alpha(t)), b(t)))\int_t^{\alpha(t)} \tilde{\alpha}(s)ds.$$

that is,

$$l_0^{-1}(l_0 + R_0) \geq \frac{\beta_2(l(\alpha(t)), b(t))}{2M}\int_t^{\alpha(t)} \tilde{\alpha}(s)ds.$$

Taking limit inferior, we get

$$\liminf_{t\to\infty} \beta_2(l(\alpha(t)), b(t)) \int_t^{\alpha(t)} \tilde{\alpha}(s)ds \leq 2M\left(\frac{l_0+R_0}{l_0}\right)$$

which is contradiction with (3.37), the proof is complete.

3.2. Philo's-type oscillation criteria

In this section, we will establish some Philo's-type oscillation results for (1.1).

Definition 3.1. Assume that $D_0 = \{(t,s): t > s \geq t_0\}$ and $D = \{(t,s): t \geq s \geq t_0\}$**,** and F∈ $C([D,R)$ satisfies the following:

(I) $F(t,s) > 0, (t,s) \in D_{0,}$ and $F(t,t) = 0$, for $t \geq t_0$;

(II) The partial derivative $\frac{\partial F}{\partial s}$ exists on $D_{0,}$ is continuous, and there exists a continuous function $\theta(t,s)$ such that $\frac{\partial F}{\partial s} = -y(t,s)\sqrt{F(t,s)} \leq 0$ for (t, s)∈ $D_{0,}$.

Under these assumptions, the function F satisfies the property P.

We introduce the following functions:

$$G_1(t,s) = \frac{\theta(t,s)}{\beta_1(b(t),t_1)b'(t)}, \qquad G_2(t,s) = \frac{\theta(t,s)}{\beta_2(b(t),t_1)b'(t)}$$

and $G_3(t,s) = \frac{\theta(t,s)}{\beta_1(l(t),t_1)l'(t)}$, $G_4(t,s) = \frac{\theta(t,s)}{\beta_2(l(t),t_1)l'(t)}$, where $\theta(t,s) = \frac{(l_0+R_0)(y(t,s))^2}{4 l_0 \rho(t)}$

Furthermore, in this section, suppose that F$\in C(D,R)$ has the property F and there exists a function $\rho \in$ $C^1([t_0,\infty),(0,\infty))$, for all $t_1 \geq t_0$.

Theorem 3.5:

Assume that (3.3) is satisfied, $b'(t) > 0$, and $b(t) \leq l(t)$. If

$$y(t,s)\frac{(F(t,s))^{\frac{1}{2}}}{\rho(s)} = -\frac{\partial}{\partial s}F(t,s) - \frac{\rho'(s)}{\rho(s)}F(t,s), \qquad (t,s) \in D_0, \tag{3.39}$$

and

$$\lim_{t\to\infty} sup\, F^{-1}(t,t_2)\int_{t_2}^{t}\big(F(t,s)\rho(s)\tilde{\alpha}(s) - G_1(t,s)\big)ds > 2M \tag{3.40}$$

For $t_2 > t_1$, then (1.1) has property F.

Proof: Assume that $x > 0$ be a solution of (1.1). As in Theorem 3.1, by defining ω and v, we get (3.18). From (3.18) and by replacing $(\rho'(t))_+$ by $\rho'(t)$, we obtain

$$\begin{aligned}\frac{1}{2M}\rho(t)\tilde{\alpha}(t) \leq{}& -\omega'(t) - \frac{R_0 v'(t)}{l_0} + \frac{\rho'(t)\omega(t)}{\rho(t)}\\ &- \frac{\beta_1(b(s),t_1)b'(s)}{\rho(t)}\omega^2(t)\\ &+ \frac{R_0}{l_0}\left(\frac{(\rho'(t))v(t)}{\rho(t)} - \frac{\beta_1(b(s),t_1)b'(s)}{\rho(t)}v^2(t)\right).\end{aligned} \tag{3.41}$$

Multiply both sides by F(t,s), then integrate from t_2 to t. It follows that

$$\begin{aligned}\frac{1}{2M}\int_{t_2}^{t}\rho(s)F(t,s)\tilde{\alpha}(s)\,ds \leq{}& -\int_{t_2}^{t}F(t,s)\omega'(s)\,ds + \int_{t_2}^{t}\frac{F(t,s)\rho'(s)\omega(s)}{\rho(s)}\,ds\\ &- \int_{t_2}^{t}\frac{F(t,s)\beta_1(b(s),t_1)b'(s)}{\rho(s)}\omega^2(t)ds\\ &- \frac{R_0}{l_0}\left(\int_{t_2}^{t}F(t,s)v'(s)\,ds + \int_{t_2}^{t}\frac{F(t,s)\rho'(s)v(s)}{\rho(s)}\,ds\right)\\ &- \frac{R_0}{l_0}\int_{t_2}^{t}\frac{F(t,s)\beta_1(b(s),t_1)b'(s)}{\rho(s)}v^2(s)ds.\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{1}{2M}\int_{t_2}^{t}\rho(s)F(t,s)\tilde{\alpha}(s)\,ds \leq{}& F(t,t_2)\omega(t_2)\\ &- \int_{t_2}^{t}\left(-\frac{\partial}{\partial s}F(t,s) - \frac{\rho'(s)}{\rho(s)}F(t,s)\right)\omega(s)ds\\ &- \int_{t_2}^{t}\frac{F(t,s)\beta_1(b(s),t_1)b'(s)}{\rho(s)}\omega^2(s)ds + \frac{R_0}{l_0}F(t,t_2)v(t_2)\\ &- \frac{R_0}{l_0}\int_{t_2}^{t}\left(-\frac{\partial}{\partial s}F(t,s) - \frac{1}{\rho(s)}\rho'(s)F(t,s)\right)v(s)ds\\ &- \int_{t_2}^{t}\frac{F(t,s)\beta_1(b(s),t_1)b'(s)}{\rho(s)}v^2(s)ds.\end{aligned}$$

that is

$$\frac{1}{2M}\int_{t_2}^{t}F(t,s)\rho(s)\tilde{\alpha}(s)\,ds \leq F(t,t_2)\omega(t_2) + \frac{R_0}{l_0}F(t,t_2)v(t_2)$$

$$+\int_{t_2}^{t}\left[\frac{y(t,s)\left(F(t,s)\right)^{\frac{1}{2}}}{\rho(s)}\,\omega(s)-\frac{F(t,s)\beta_1(b(s),t_1)b'(t)}{\rho(s)}\omega^2(s)\right]ds$$

$$+\frac{R_0}{l_0}\int_{t_2}^{t}\left[\frac{y(t,s)\left(F(t,s)\right)^{\frac{1}{2}}}{\rho(s)}\,v(s)-\frac{F(t,s)\beta_1(b(s),t_1)b'(t)}{\rho(s)}v^2(s)\right]ds. \qquad (3.42)$$

Using (3.42) and (3.19), we find

$$F^{-1}(t,t_2)\int_{t_2}^{t}\left[F(t,s)\rho(s)\tilde{\alpha}(s)-\frac{l_0+R_0}{4l_0}\frac{(y(t,s))^2(F(t,s))^2}{(\rho(s)\beta_1(b(s),t_1)b'(s))}\right]\mathrm{ds}\le 2M\left(\omega(t_2)+\frac{R_0 v(t_2)}{l_0}\right),$$

Taking limit superior, we get

$$\lim_{t\to\infty}sup F^{-1}(t,t_2)\int_{t_2}^{t}\left(F(t,s)\rho(s)\tilde{\alpha}(s)-G_1(t,s)\right)ds\le 2M$$

and this contradicts (3.40). The proof is complete.

Theorem 3.6: Assume that (3.3) is satisfied, $b'(t)>0$, and $b(t)\le l(t)$, If y (t, s) $\frac{(F(t,s))^{\frac{1}{2}}}{\rho(s)}=-\frac{\partial}{\partial s}F(t,s)-\frac{\rho'(s)}{\rho(s)}F(t,s)$, $(t,s)\in D_{0,}$ (3.43)

and

$$\lim_{t\to\infty}sup\, F^{-1}(t,t_2)\int_{t_2}^{t}\left(F(t,s)\rho(s)\tilde{\alpha}(s)-G_2(t,s)\right)ds>2M \qquad (3.44)$$

for $t_2>t_1$, then (1.1) has property F.

Theorem 3.7:

Let (3.3) and (3.39) be satisfied, and b(t) $\ge l(t)$, If

$$\lim_{t\to\infty}sup\, F^{-1}(t,t_2)\int_{t_2}^{t}\left(F(t,s)\rho(s)\tilde{\alpha}(s)-G_3(t,s)\right)ds>2M \qquad (3.45)$$

for $t_2>t_1$, then (1.1) has property F.

Theorem 3.8:

Let (3.3) and (3.43) be satisfied, and b(t) $\ge l(t)$, If

$$\lim_{t\to\infty}sup\, F^{-1}(t,t_2)\int_{t_2}^{t}\left(F(t,s)\rho(s)\tilde{\alpha}(s)-G_4(t,s)\right)ds>2M \qquad (3.46)$$

for $t_2>t_1$, then any solution x of (1.1) is oscillatory or satisfies (2.7).

Example 4.1:

Consider the non-linear differential equation

$\left(t^{-2}([x(t)+R_0x(0.8t)]'')\right)'+(t^{-4}\lambda)x^{\mu}(0.8t)=0$, where f(x)=$x^{\mu}$ (4.1)

Where $\lambda>0, R=R_0>0$. Here, $\varrho(t)=t^{-2}$, b(t) = l(t) = 0.8t, $l_0=0.8$, and $\alpha(t)=\lambda t^{-4}$.

That is, we have $\tilde{\alpha}(t)=\min\{\alpha(t),\alpha((l(t))\}=\min\{\lambda t^{-4},\lambda(0.8t)^{-4}\}=\lambda t^{-4}$.

$$and\ \beta_1(t,t_1)=\int_{t_1}^{t}s^2ds\ge t^3.$$

Thus (3.3) holds. Putting $\rho(t)=t^3$ and according to theorem 3.1, we find

$$\lim_{t\to\infty}sup\int_{t_1}^{t}[\rho(s)\,\tilde{\alpha}\,(s)-\frac{l_0^{\mu}+R_0)}{4l_0^{\mu}b'(s)\rho(s)\beta_1(b(s),t_1)}((\rho'(s))_+)^2]\mathrm{ds}>2M$$

$$\rho(s)=s^3\ ,\rho'(s)=3s^2.$$

$$\tilde{\alpha}\,(s)=\lambda s^{-4}, b'(s)=0.8,\qquad \beta_1(b(s),t_1)=(0.8s)^3=0.512s^3$$

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left(s^3(\lambda s^{-4}) - \frac{0.8^{\mu} + R_0}{4(0.8)^{\mu}(0.8)(s^3)(0.512s^3)} (3s^2)^2 \right) ds > 2M,$$

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left(\lambda s^{-1} - \frac{9(0.8^{\mu} + R_0)}{1.6384(0.8^{\mu})}\, s^{-2} \right) ds > 2M,$$

Evaluating the definite integral from t_1 to t

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left(\lambda \ln\left(\frac{t}{t_1}\right) - \frac{9(0.8^{\mu} + R_0)}{1.6384(0.8^{\mu})} \left(\frac{1}{t_1} - \frac{1}{t}\right) \right) > 2M,$$

As t→ ∞ the term $\frac{1}{t} \to 0$, $\ln\left(\frac{t}{t_1}\right) \to \infty$.

ln(t) → ∞, the limit superior will automatically grow infinitely large and easily cross the finite threshold of $> 2M$. Provided that the coefficient λ remains strictly positive.

Thus, we conclude that any solution of (4.1) is oscillatory or satisfies (2.7) if

$$\lambda > \frac{9(0.8^{\mu}+R_0)}{1.6384(0.8^{\mu})} \tag{4.2}$$

Example 4.2:

$$\left(t^2\left(\left[x(t) + R_0 x\left(\tfrac{1}{2}t\right)\right]''\right)\right)' + (t^{-2}\lambda)x^3(2t) = 0,\ t \geq t_1 > 0 \tag{4.3}$$

Where $\lambda > 0, R = R_0 > 0$. Here the exponent $\mu = 3$, makes the equation strictly

non – linear.

$\varrho(t) = t^2$, b(t) = 2t, l(t) = $\frac{1}{2}$t, $l_0 = l'(t) = \frac{1}{2}$, and $\alpha(t) = \lambda t^{-2}$.

$$and\ \beta_1(t,t_1) = \int_{t_1}^{t} s ds \geq \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2.$$

Putting $\rho(t) = t^2$ and according to theorem 3.2, we find

$$\lim_{t\to\infty} sup \int_{t_2}^{t} [\, \rho(s)\, \tilde{\alpha}\,(s) - \frac{(l_0+\rho_0)\left((\rho'(s))_+\right)^2}{4 l_0 l'(s)\rho(s)\beta_1(l(s),t_1)}]ds$$

$$\geq \lim_{t\to\infty} sup \int_{t_2}^{t} [\ s^2\ \lambda s^{-2} - \frac{(l_0+\rho_0)4s^2}{4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s^2\left(\frac{1}{8}s^2\right)}]ds$$

$$\geq \lim_{t\to\infty} sup \int_{t_1}^{t} \left(\lambda - \frac{32(l_0+\rho_0)}{s^2}\right) ds > 2M.$$

Evaluating the definite integral from t_1 to t

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left(\lambda\,(\mathrm{t} - t_1) + 32(l_0 + \rho_0)\left(\frac{1}{t} - \frac{1}{t_1}\right) \right)$$

As t→ ∞ the term λt grows infinitely, while $\frac{1}{t}$ vanishes to 0.

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left(\lambda\,(\mathrm{t} - t_1) - \frac{32(l_0 + \rho_0)}{t_1} \right) = \infty$$

Since the limit goes to infinity, it is guaranteed that for any arbitrarily large positive

constant M.

$$\lim_{t\to\infty} sup \int_{t_2}^{t} [\, \rho(s)\, \tilde{\alpha}\,(s) - \frac{(l_0+\rho_0)\left((\rho'(s))_+\right)^2}{4 l_0 l'(s)\rho(s)\beta_1(l(s),t_1)}]ds = \infty > 2M$$

Hence any solution of (4.3) is oscillatory or satisfies (2.7).

5. CONCLUSION

In this paper, the oscillatory behaviour of third order neutral delay differential Equations are discussed. The sufficient conditions namely (3.3), (3.10), (3.21), (3.31), (3.37) establish the oscillations of the solutions of (1.1). Additionally, we explore the philos – type oscillatory behaviour of Eq (1.1), through theorems (3.5 – 3.8). Moreover, examples are given to substantiate theorem (3.1) and (3.2).

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