Journal of Harbin Engineering University
ISSN: 1006-7043
Vol 44 No. 7
July 2023

On A New Notion of Anti-Fuzzy Normal Subgroup

Manoranjan Kumar Singh
Professor,
Department of Mathematics,
Magadh University, Bodh-Gaya (824234)
Gaya, Bihar

Sanjeet Kumar
C.S.I.R. Junior Research Fellow,
Department of Mathematics,
Magadh University, Bodh-Gaya (824234)
Gaya, Bihar

Abstract: Zadeh defined a fuzzy set mathematically by assigning to each individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is "similar" or "compatible" with the concept represented by a fuzzy set. Rosenfeld was the first researcher who applied this concept in the realm of group theory and formulated the notion of fuzzy subgroups [5]. Mukherjee and Bhattacharya studied fuzzy normal subgroups [3]. Later on, in [1], R. Biswas enunciated the concept of the anti-fuzzy subgroup. Anti-fuzzy subgroups were also studied by Gayen, Jha, Singh, and Prasad [2].

Our endeavor in this paper revolves around the work proposed by R. Biswas and Gayen, Jha, Singh, Prasad and Mukherjee, and Bhattacharya. Here, we have presented the redefined version of anti-fuzzy normal subgroups and have derived some interesting results based on our proposed notion of anti-fuzzy normal subgroups, along with some exciting examples.

1. Introduction:

Zadeh enunciated mathematically the percept of the fuzzy set. Since the founding of the hypothesis of fuzzy sets, it has been about six decades of development and application. The extent of maturing and applications ranges from theoretical to practical, from Sciences and engineering to humanity, artificial intelligence, medicine, and many more. The extensive line-up of applications signifies the worth of the theory. The fuzzy set provides an appropriate point of exit for constructing a conceptual skeleton, which parallels the structure used in the case of classical sets but is more generic than the latter and has a broader scope of applicability. Such structure provides a natural way of dealing with complications in which the origin of imprecision is the lack of sharply defined criteria of membership rather than the existence of random variables.

Azriel Rosenfeld applied the notion of fuzzy sets in the domain of group theory and propounded the concept of fuzzy subgroupoid and fuzzy subgroups. Later, Mukherjee and Bhattacharya proposed the concept of fuzzy normal subgroups, and Das gave the notion of fuzzy level subgroups. R. Biswas enunciated the concept of an anti-fuzzy subgroup. Gayen, Jha, and Singh have also studied the anti-fuzzy subgroup. The object of this paper is to put forward the concept of the redefined notion of anti-fuzzy normal subgroups and some significant results of it

2. Preliminaries:

The concept of anti-fuzzy subgroup was first proposed by R. Biswas as follows:

Definition 2.1. [1] A fuzzy subset P of a group G is considered an anti-fuzzy subgroup of a group G if $\forall\ \alpha, \beta \in G$, the following axioms are satisfied:

(i) $P(\alpha\beta) \leq \max\{P(\alpha), P(\beta)\}$

(ii) $P(\alpha^{-1}) \leq P(\alpha)$

Gayen, et. el. redefined it in the following way:

Definition 2.2. [2] A fuzzy subset P of a group G is considered an anti-fuzzy subgroup of a group G if $\forall\ \alpha, \beta \in G$, the following axioms are satisfied:

(i) $P(\alpha\beta) \leq P(\alpha) + P(\beta) - P(\alpha).P(\beta)$

(ii) $P(\alpha^{-1}) \leq P(\alpha)$

Definition 2.3. [3] A fuzzy subset P of a group G is considered a fuzzy normal subgroup of a group G if $\forall\ \alpha, \beta \in G$, the following axioms are satisfied:

(i) $P(\alpha\beta) \geq min\{P(\alpha), P(\beta)\}$

(ii) $P(\alpha^{-1}) \geq P(\alpha)$

(iii) $P(\alpha\beta) = P(\beta\alpha)$

Definition 2.4. A fuzzy subset P of a group G is considered an anti-fuzzy normal subgroup (AFNSG) of a group G if $\forall\, \alpha, \beta \in G$, the following axioms are satisfied:
(i) $P(\alpha\beta) \leq \max\{P(\alpha), P(\beta)\}$
(ii) $P(\alpha^{-1}) \leq P(\alpha)$
(iii) $P(\alpha\beta) = P(\beta\alpha)$
Now we will propose the re-defined version of an anti-fuzzy normal subgroup of G.

Definition 2.5. A fuzzy subset P of a group G be considered as an anti-fuzzy normal subgroup of a group G if $\forall\, \alpha, \beta \in G$, the following axioms are satisfied:
(i) $P(\alpha\beta) \leq P(\alpha) + P(\beta) - P(\alpha).P(\beta)$
(ii) $P(\alpha^{-1}) \leq P(\alpha)$
(iii) $P(\alpha\beta) = P(\beta\alpha)$
To exemplify the above notion following example is worthy.

Example 2.1. Let G be the Klein four-group. Then $G = \{e, a, b, ab\}$, where $a^2 = b^2 = e$ and $ab = ba = e$. Now, we define a fuzzy subset $P: G \to [0,1]$ by the setting

$$P(e) = 0, P(a) = 0.2, P(b) = P(c) = 0.4$$

Evidently, P is an anti-fuzzy subgroup of G. Here, observe that $\forall\, \alpha, \beta, P(\alpha\beta) = P(\beta\alpha)$. Hence P is an anti-fuzzy normal subgroup of G.
Based on the above-redefined notion of anti-fuzzy normal subgroups, some valuable results have been presented in this section.

3. Some Propositions and Theorems:

Proposition 3.1. Union of any collection of anti-fuzzy subgroups of Group G is an anti-fuzzy subgroup of G.

Theorem 3.1. Union of any collection of anti-fuzzy normal subgroups of Group G is an anti-fuzzy normal subgroup of G.

Proof: Let $\{P_i\}_{i\in I}$ be the family of anti-fuzzy normal subgroups of group G. Where I is an index set and is such that $\forall\, i \in I$, A_i is an anti-fuzzy normal subgroups of group G.
Let $P = \underset{i\in I}{\cup} P_i$ be the union of the given family of anti-fuzzy normal subgroups of G. Clearly $P = \underset{i\in I}{\cup} P_i$ is an anti-fuzzy subgroup of G. It only remains to show that P is an anti-fuzzy normal subgroup of group G. For all $\alpha, \beta \in G$, we have

$$\begin{aligned} P(\alpha\beta) &= \underset{i\in I}{\cup} P_i(\alpha\beta) \\ &= \inf\{P_i(\alpha\beta)\} \\ &= \inf\{P_i(\beta\alpha)\} \\ &= \underset{i\in I}{\cup} P_i(\beta\alpha) \\ &= P(\beta\alpha) \end{aligned}$$

i.e., $P(\alpha\beta) = P(\beta\alpha)$ for all $\alpha, \beta \in G$.
Hence P is an anti-fuzzy normal subgroup of group G. That is, the union of any collection of anti-fuzzy normal subgroups of Group G is an anti-fuzzy normal subgroup of group G.

Theorem 3.2. Show that every anti-fuzzy subgroup of an abelian group is an anti-fuzzy normal subgroup.

Proof: Let G be an abelian group and P be an anti-fuzzy subgroup of group G. Let α, β be any two elements of group G. Then, we have
$P(\alpha\beta) = P(\beta\alpha)$, since $\alpha\beta = \beta\alpha$ in G
Hence, P is an anti-fuzzy normal subgroup of G.
Now we will introduce the notion of the product ($*$) between any two anti-fuzzy sets.

4. On Product of Anti-Fuzzy Normal Subgroups:

Definition 4.1. Let $U \neq \emptyset$ be a set, and P and Q are anti-fuzzy sets on U. Then, we define the product ($*$) between any two anti-fuzzy sets as follows:

$$(P * Q)(u) = P(u) + Q(u) - P(u).Q(u)\ \forall\, u \in U$$

Based on the idea of the product of the anti-fuzzy normal subgroup, we now establish some interesting results based on the notion mentioned above.

Theorem 4.1 . If P and Q are two anti-fuzzy normal subgroups of group G, then $P * Q$ (the * product of two fuzzy normal subgroups) will also be an anti-fuzzy normal subgroup of G.

Proof: Let us assume that P and Q be any two anti-fuzzy normal subgroups of group G. Let α, β be any two elements of the group G. Then the product of two anti-fuzzy normal subgroups be defined as follows

$$(P * Q)(\alpha) = P(\alpha) + Q(\alpha) - P(\alpha).Q(\alpha) \quad \text{.......(i)}$$

Then we claim that $P * Q$ is an anti-fuzzy normal subgroup of the group G. Now let α, β be any two elements of the group G. Then, we have

$$\begin{aligned} (P * Q)(\alpha\beta) &= P(\alpha\beta) + Q(\alpha\beta) - P(\alpha\beta).Q(\alpha\beta) \\ &\leq \big(P(\alpha) + P(\beta) - P(\alpha)P(\beta)\big) + \end{aligned}$$
$$\big(Q(\alpha) + Q(\beta) - Q(\alpha)Q(\beta)\big) -$$

$[(P(\alpha)+P(\beta)-P(\alpha)P(\beta))(Q(\alpha)+Q(\beta)-Q(\alpha)Q(\beta)]$
$=(P(\alpha)+Q(\alpha)-P(\alpha).Q(\alpha))+(P(\beta)+Q(\beta)-P(\beta).Q(\beta))-$
$(P(\alpha)+Q(\alpha)-P(\alpha).Q(\alpha))(P(\beta)+Q(\beta)-P(\beta).Q(\beta))$
$=(P*Q)(\alpha)+(P*Q)(\beta)-(P*Q)(\alpha).(P*Q)(\beta)$

i.e., $(P*Q)(\alpha\beta)\leq(P*Q)(\alpha)+(P*Q)(\beta)-(P*Q)(\alpha).(P*Q)(\beta)$

Again

$(P*Q)(\alpha^{-1})=P(\alpha^{-1})+Q(\alpha^{-1})-P(\alpha^{-1}).Q(\alpha^{-1})$
$\leq P(\alpha)+Q(\alpha)-P(\alpha).Q(\alpha)$
$=(P*Q)(\alpha)$

i.e., $(P*Q)(\alpha^{-1})\leq(P*Q)(\alpha)$

Finally,

$(P*Q)(\alpha\beta)=P(\alpha\beta)+Q(\alpha\beta)-P(\alpha\beta).Q(\alpha\beta)$
$=P(\beta\alpha)+Q(\beta\alpha)-P(\beta\alpha).(\beta\alpha)$
$=(P*Q)(\beta\alpha)$

Hence, $P*Q$ is an anti-fuzzy normal subgroup of G. That is the product of two anti-fuzzy normal subgroup of group G, is again an anti-fuzzy normal subgroup of G.

Theorem 4.2: Show that the product $(*)$ of an anti-fuzzy subgroup P and an anti-fuzzy normal subgroup Q of the abelian group G is an anti-fuzzy normal subgroup of G.

Proof: Let G be an abelian group. Suppose P is an anti-fuzzy subgroup and Q is any anti-fuzzy normal subgroup of the abelian group G. Then we have to prove that $P*Q$ is an anti-fuzzy normal subgroup of G.

Since Q is an anti-fuzzy normal subgroup of G; therefore Q is an anti-fuzzy subgroup of G. Again, P is an anti-fuzzy subgroup of G; therefore $P*Q$ is an anti-fuzzy subgroup of G. Then it only remains to show that $P*Q$ is an anti-fuzzy normal subgroup of G. Let α,β be any two elements of the group G. Then, we have

$(P*Q)(\alpha\beta)=P(\alpha\beta)+Q(\alpha\beta)-P(\alpha\beta).Q(\alpha\beta)$
$=P(\alpha\beta)+Q(\beta\alpha)-P(\alpha\beta).(\beta\alpha)$

[Since Q is an AFNSG]

$=P(\beta\alpha)+Q(\beta\alpha)-P(\beta\alpha).(\beta\alpha)$

[Since G is an abelian]

$=(P*Q)(\beta\alpha)$

Hence, $P*Q$ is an anti-fuzzy normal subgroup of G.

References

[1] Biswas, R., "Fuzzy Subgroup and Anti Fuzzy Subgroups", Fuzzy Sets and Systems, 35 (1990) 121- 124.

[2] Gayen S., Jha S., Singh M**.,** Prasad A., "On Anti-Fuzzy Subgroup" Yugoslav Journal of Operations Research xx (2020).

[3] Mukherjee and Bhattacharya, Fuzzy Normal Subgroups and Fuzzy Cosets Information Sciences 34,225-239 (1984).

[4] Zadeh, L.A. "Fuzzy Sets" , Information and control, 128 (1965) 338-353.

[5] Rosenfeld, A., "Fuzzy Groups'', Journal of Mathematical Analysis and Applications, 35 (1971) 512-517.

[6] Das, P.S., "Fuzzy Groups and level Subgroups'' Journal of Mathematical Analysis and Applications, 84 (1981) 264-269.

[7] Singh, M., "Theory of Fuzzy Structures and Applications", Lambert Academic Publishing, Germany, 2010.