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Weakly Star Semi Closed Sets in Topological Spaces

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Abstract

After the introduction of the concept of generalized closed (briefly *g*-closed) sets in topological spaces, numerous researchers studied several classes of generalized closed sets. In this paper, we are introducing a new class of closed sets termed as Weakly star semi closed sets in topological spaces. Also, we examine some of its properties.

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1 Introduction

The concept of generalized closed (briefly *g*-closed) sets was introduced by N. Levine[12] in(i) 1970. In 2000, the concept of weakly closed (briefly *w*-closed) sets was introduced by M. Sheik(ii) John[9]. In 2017, Veeresha A. Sajjanar[2] introduced the concept of weakly semi closed(iii) (briefly *ws*-closed) sets in topological spaces. And, In this paper, the concept of weakly star semi closed (briefly *w*s*-closed) sets is introduced and some of its properties are examined. Preliminaries needed to introduce this new class of closed sets are given in section 2. In section 3, the(vi) concept of *w*s*-closed set is studied and a figure describing the relationship of this closed set with other generalized closed sets was included. In Section 4, the closure of the newly introduced closed set was discussed. Section 5 contains the conclusion and at the end references were included.

2 Preliminaries

Throughout this paper *X* and *Y* represents the topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset *A* of a topological space *X*, *clA* and *intA* denote the closure of *A* and interior of *A*(i) respectively. *X* \ *A* denotes the complement of *A* in *X.*

Definition 2.1: A subset *A* of a space *X* is called pre-open [6] if $A \subseteq int\ cl\ A$ and pre-closed if $cl\ intA \subseteq A$.
semi-open [11] if $A \subseteq cl\ intA$ and semi-closed if $int\ cl\ A \subseteq A$.
semi-pre-open [4] if $A \subseteq cl\ int\ cl\ A$ and semi-pre-closed if $int\ cl\ intA \subseteq A$.
(iv) α-open [13] if $A \subseteq int\ cl\ intA$ and α-closed if $cl\ int\ clA \subseteq A$.
(v) regular open [10] if $A = int\ clA$ and regular closed if $cl\ intA = A$.
π-open [15] if *A* is the union of regular open sets and π-closed if *A* is the intersection of regular closed sets.
(vii) *b*-open [6] if $A \subseteq cl\ intA \cup int\ cl\ A$ and *b*-closed if $cl\ intA \cap int\ clA \subseteq A$.
The semi-closure (resp. pre-closure, resp. semi-pre-cloure, resp. α-closure, resp. *b*-closure) of a subset *A* of *X* is the intersection of all semi-closed (resp. pre-closed, resp. semi-pre-closed, resp. α-closed, resp. *b*-closed) sets containing *A* and is denoted by *sclA* (resp. *pclA*, resp. *spclA,* resp. αclA, resp. *bclA*).

Definition 2.2: A subset *A* of a space *X* is called generalized closed [12] (briefly *g*-closed) if $cl(A) \subseteq U$, whenever $A \subseteq U$ and *U* is open in *X*.

(ii) regular generalized closed [8] (briefly *rg*-closed) if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(iii) α-generalized closed [15] (briefly αg-closed) if α*cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in *X*.

(iv) generalized α-closed [15] (briefly *g*α-closed) if α*cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is α-open in *X*.

(v) generalized semi-closed [4] (briefly *gs*-closed) if *scl(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in *X*.

(vi) generalized pre-closed [4] (briefly *gp*-closed) if *pcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in X.

(vii) generalized semi-pre-closed [4] (briefly *gsp*-closed) if *spcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in *X*.

(viii) π-generalized closed [5] (briefly πg-closed) if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is π-open in *X*.

(ix) weakly generalized closed [8] (briefly *wg*-closed) if *cl int(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in *X*.

(x) $g^{\#}$-closed [15] if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is αg-open in *X*.

(xi) $g^{\#}p^{\#}$-closed [15] if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is $g^{\#}$-open in *X*.

(xii) generalized b-closed [1] (briefly *gb*-closed) if *bcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is open in *X*.

(xiii) generalized αb-closed [15] (briefly $g\alpha b$-closed) if *bcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is α-open in *X*.

(xiv) regular generalized b-closed [6] (briefly *rgb*-closed) if *bcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(xv) weakly closed [9] (briefly *w*-closed) if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is semi open in *X*.

(xvi) weakly semi closed [2] (briefly *ws*-closed) if *scl(A)*⊆*U*, whenever *A*⊆*U* and *U* is w-open in *X*.

(xvii) strongly generalized closed [15] (briefly *g**-closed) if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is *g*-open in *X*.

(xviii) semi generalized closed [1](briefly sg-closed) if *scl(A)*⊆*U*, whenever *A*⊆*U* and *U* is semi open in *X*.

(xix) (*gsp*)*-closed [8] if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is *gsp*-open in *X*.

(xx) generalized pre regular closed [4] (briefly gpr-closed) if *pcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(xxi) alpha generalized regular closed [15] (briefly α*gr*-closed) if α*cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(xxii) semi generalized *b*-closed [6] (briefly *sgb*-closed) if *bcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is semi open in *X*.

(xxiii) $\hat{g}$-closed [14] if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is semi open in *X*.

(xxiv) **g*-closed [14] if *cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is $\hat{g}$-open in *X*.

(xxv) generalized semi pre regular closed [4] (briefly *gspr*-closed) if *spcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(xxvi) pre generalized pre regular closed [7] (briefly *pgpr*-closed) if *pcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is *rg*-open in *X*.

(xxvii) g*-pre closed [14] (briefly *g*p*-closed) if *pcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is *g*-open in *X*.

(xxviii) π generalized α-closed [3] (briefly $\pi g\alpha$-closed) if α*cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is π-open in *X*.

(xxix) π generalized pre closed [3] (briefly πgp-closed) if *pcl(A)*⊆*U*, whenever *A*⊆*U* and *U* is π-open in *X*.

(xxx) regular weakly generalized closed [14] (briefly rwg-closed) if *cl int(A)*⊆*U*, whenever *A*⊆*U* and *U* is regular open in *X*.

(xxxi) alpha weakly semi closed [15] (briefly α*ws*-closed) if α*cl(A)*⊆*U*, whenever *A*⊆*U* and *U* is *ws*-open in *X*.

The complements of the above mentioned closed sets are their respective open sets. For example a subset *B* of a space *X* is generalized open (briefly *g*-open) if *X* \ *B* is *g*-closed.

Lemma 2.3:[15] In an extremally disconnected space *X*,

(i) *pcl(A)=spcl(A)*

(ii) α*cl(A)=scl(A)*

(iii) *scl(A*∪*B)=scl(A)*∪*scl(B)*

Lemma 2.4:[15] In an extremally disconnected submaximal space X, $cl(A)$ = α*cl(A)* = *scl(A)* = *pcl(A)* = *spcl(A)*.

Lemma 2.5:[15] For any subset *A* of *X*, the following results hold:

(i) *sintA* = *A*∩*cl int A*
(ii) *pintA* = *A*∩*int cl A*
(iii) *scl A* = *A*∪*int cl A*
(iv) *pcl A* = *A*∪*cl int A*

3 Weakly star semi closed sets

In this section, we introduce a new type of closed set namely *w*s*-closed sets in topological spaces and study some of their properties.

Definition 3.1: A subset *A* of a topological space *(X,* τ*)* is called weakly star semi closed (briefly *w*s*-closed) if *scl(A)*⊆*U* whenever *A*⊆*U* and *U* is *ws*-open. The set of all *w*s*-closed sets in *(X,* τ*)* is denoted by *W*SC*⊆(X, τ).

Proposition 3.2:

(i) Every closed set in *X* is *w*s*-closed.
(ii) Every α-closed set in *X* is *w*s*-closed.
(iii) Every semi-closed set in *X* is *w*s*-closed.
(iv) Every (*gsp*)*-closed set in *X* is *w*s*-closed.
(v) Every π-closed set in *X* is *w*s*-closed.
(vi) Every regular-closed set in *X* is *w*s*-closed.
(vii) Every αws-closed set in *X* is *w*s*-closed.

Proof:

(i) Let *A* be any closed set in *X.* Assume *U* to be any ws-open set in *X* such that *A*⊆*U*. Since *A* is closed in *X, cl(A)=A*⊆*U*. Since every closed set is semi-closed, *scl(A)* ⊆*cl(A)* ⊆*U*. Therefore *A* is *w*s*-closed.

(ii) Let *A* be any α-closed set in *X.* Assume *U* to be any *ws*-open set in *X* such that *A*⊆*U*. Since *A* is α-closed in *X,* α*cl(A)=A*⊆*U*. Since every α-closed set is semi-closed, *scl(A)*⊆ α*cl(A)*⊆*U*. Therefore *A* is *w*s*-closed.

(iii) Let *A* be any semi-closed set in *X.* Assume *U* to be any *ws*-open in *X* such that A⊆*U*. Since *A* is semi-closed in *X, scl(A)=A*⊆*U*. Therefore *A* is *w*s*-closed.

(iv) Let *A* be any *(gsp)**-closed set in *X.* Assume *U* to be any *ws*-open set in *X* such that *A*⊆*U*. Since every *ws*-open set is *gsp*-open and since *A* is *(gsp)**-closed in *X, cl(A)*⊆*U*. But *scl(A)* ⊆*cl(A)*. This implies *scl(A)*⊆*U.* Therefore *A* is *w*s*-closed.

(v) Let *A* be any π-closed set in *X.* Since every π-closed set is closed and by (i), *A* is *w*s*-closed.

(vi) Let *A* be any regular-closed set in *X.* Since every regular-closed set is closed and by (i), *A* is *w*s*-closed.

(vii) Let *A* be any αws-closed set in *X.* Assume U to be any *ws*-open set in *X* such that *A*⊆*U*. Since *A* is αws-closed in *X,* α*cl(A)*⊆*U*. But *scl(A)*⊆ α*cl(A)*. This implies that *scl(A)*⊆*U*. Therefore *A* is *w*s*-closed.

Remark 3.3: For the above theorem the converse need not hold as seen from the following example.

Example 3.4: Consider X=*{a,b,c}* having the topology $\tau = \{\Phi, X, \{a\}, \{c\}, \{a, c\}\}$. In this example,

(i) {a} and {c} are *w*s*-closed but not closed.
(ii) {a} and {c} are *w*s*-closed but not α-closed.

Example 3.5: Consider X=*{a,b,c}* having the topology $\tau = \{\Phi, X, \{c\}, \{a, b\}\}$. In this example,

(i) {a}, {b},{b, c} and {a, c} are *w*s*-closed but not semi-closed.
(ii) {a}, {b},{b,c} and {a,c} are *w*s*-closed but not (*gsp*)*-closed.

Example 3.6: Consider X=*{a,b,c}* having the topology $\tau = \{\Phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. In this example, the subsets {c} and {b,c} are *w*s*-closed but not π-closed.

Example 3.7: Consider X=*{a,b,c}* having the topology $\tau = \{\Phi, X, \{a\}, \{b, c\}\}$. In this example, the subsets {b}, {c},{a, b} and {a, c} are *w*s*-closed but not regular-closed.

Example 3.8: Consider X=*{a,b,c}* having the topology $\tau = \{\Phi, X, \{a\}, \{b\}, \{a, b\}\}$. In this example, the subsets {a} and {b} are *w*s*-closed but not αws-closed.

Proposition 3.9:

(i) Every *w*s*-closed set in *X* is *gs*-closed.
(ii) Every *w*s*-closed set in *X* is *rgb*-closed.
(iii) Every *w*s*-closed set in *X* is *gb*-closed.
(iv) Every *w*s*-closed set in *X* is *gsp*-closed.
(v) Every *w*s*-closed set in *X* is $g\alpha b$-closed.
(vi) Every *w*s*-closed set in *X* is *sgb*-closed.
(vii) Every *w*s*-closed set in *X* is *gspr*-closed.

Proof:

(i) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any open set in *X* such that *A*⊆*U*. Since every open set is *ws*-open and since *A* is *w*s*-closed*, scl(A)*⊆*U*. Thus *A* is *gs*-closed.

(ii) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any regular-open set in *X* such that *A*⊆*U*. As we know that, "Every regular-open set is *ws*-open" and since *A* is *w*s*-closed*, scl(A)*⊆*U*. But *bcl(A)*⊆*scl(A).* This implies *bcl(A)*⊆*U*. Thus *A* is *rgb*-closed.

(iii) Let *A* be any *w*s*-closed set in *X.* Assume *A* to be any open set in *X* such that *A*⊆*U*. As we know that "Every open set is *ws*-open" and since *A* is *w*s*-closed*, scl(A)*⊆*U*. But *bcl(A)*⊆*scl(A)*. This implies *bcl(A)*⊆*U*. Thus *A* is *gb*-closed.

(iv) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any open set in *X* such that *A*⊆*U*. As we know that, "Every open set is *ws*-open" and since *A* is *w*s*-closed*, scl(A)*⊆*U*. But *spcl(A)*⊆*scl(A).* This implies *spcl(A)*⊆*U*. Thus *A* is *gsp*-closed.

(v) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any α-open set in *X* such that *A*⊆*U*. As we know that, "Every α-open set is *ws*-open" and since *A* is *w*s*-closed in*, scl(A)*⊆*U*. But *bcl(A)*⊆*scl(A).* This implies *bcl(A)*⊆*U*. Thus *A* is $g\alpha b$-closed.

(vi) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any semi-open set in *X* such that *A*⊆*U*. Since every semi-open set is *ws*-open and since *A* is *w*s*-closed in*, scl(A)*⊆*U.* But *bcl(A)*⊆*scl(A).* This implies *bcl(A)*⊆*U*. Thus *A* is *sgb*-closed.

(vii) Let *A* be any *w*s*-closed set in *X.* Assume *U* to be any regular-open set in *X* such that *A*⊆*U*. As we know that "Every regular-open set is *ws*-open" and since *A* is *w*s*-closed*, scl(A)*⊆*U*. But *spcl(A)*⊆*scl(A).* This implies *spcl(A)*⊆*U*. Thus *A* is *gspr*-closed.

Remark 3.10: For the above theorem the converse need not hold as seen from the following example.

Example 3.11: Consider *X={a,b,c,d}* having the topology $\tau = \{\Phi, X, \{a\}, \{a,b\}\}$. In this example,

(i) {a, c}, {a, d}, {a, b, c}, {a, c, d} and {a, b, d} are *gs*-closed but not *w*s*-closed.

(ii) {a}, {a, b},{a, c}, {a,d}, {a, b, c}, {a, c, d} and {a, b, d} are *rgb*-closed but not *w*s*-closed.

Example 3.12: Consider *X={a,b,c}* having the topology $\tau = \{\Phi, X, \{a\}, \{a,b\}\}$. In this example,

(i) {a, c} is *gb*-closed but not *w*s*-closed.

(ii) {a, c} is *gsp*-closed but not *w*s*-closed.

Example 3.13: Consider *X={a,b,c,d}* having the topology $\tau = \{\Phi, X, \{a,b\}, \{a,b,c\}\}$. In this example,

(i) {a}, {b}, {b, c}, {a, c}, {a, d} and {b, d} are $g\alpha b$-closed but not *w*s*-closed.

(ii) {a}, {b}, {b, c}, {a, c}, {a, d} and {b, d} are *sgb*-closed but not *w*s*-closed.

Example 3.14: Consider *X={a,b,c,d}* having the topology $\tau = \{\Phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. In this example, the subsets {b}, {a, b}, {b,d}, {a, b, c} and {a, b, d} are *gspr*-closed but not *w*s*-closed.

Independence Theorems:

The concept "*w*s*-closed" is independent from the concepts *g*-closed, *g**-closed, *rg*-closed, αg-closed, $g^{\#}p^{\#}$-closed, πg-closed, **g*-closed, αgr-closed, $g\alpha$-closed, *gp*-closed, *wg*-closed, *gpr*-closed, *pgpr*-closed, *g*p*-closed, $\pi g\alpha$-closed, πgp-closed and *rwg*-closed.

Example 3.15: Consider *X*={*a, b, c*} having the topology $\tau = \{\Phi, X, \{a\}, \{a,b\}\}$.

(i) {b} is *w*s*-closed but not *g*-closed and {a,c} is *g*-closed but not *w*s*-closed.

(ii) {b} is *w*s*-closed but not *g**-closed and {a,c} is *g**-closed but not *w*s*-closed.

Example 3.16: Consider *X*={*a, b, c, d*} having the topology $\tau = \{\Phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$.

(i) {a} is *w*s*-closed but not *rg*-closed and {a,b} is *rg*-closed but not *w*s*-closed.

(ii) {a} is *w*s*-closed but not αg-closed and {b, d} is αg-closed but not *w*s*-closed.

(iii) {a} is *w*s*-closed but not $g^{\#}p^{\#}$-closed and {b, d} is $g^{\#}p^{\#}$-closed but not *w*s*-closed.

(iv) {a} is *w*s*-closed but not πg-closed and {b, d} is πg-closed but not *w*s*-closed.

(v) {a} is *w*s*-closed but not **g*-closed and {b, d} is **g*-closed but not *w*s*-closed.

(vi) {a} is *w*s*-closed but not αgr-closed and {a, b} is αgr-closed but not *w*s*-closed.
(vii) {a} is *w*s*-closed but not $g\alpha$-closed and {b, d} is $g\alpha$-closed but not *w*s*-closed.
(viii) {a} is *w*s*-closed but not *gp*-closed and {b, d} is *gp*-closed but not *w*s*-closed.
(ix) {a} is *w*s*-closed but not *wg*-closed and {b, d} is *wg*-closed but not *w*s*-closed.
(x) {a} is *w*s*-closed but not *gpr*-closed and {a, b} is *gpr*-closed but not *w*s*-closed.
(xi) {a} is *w*s*-closed but not *pgpr*-closed and {a, b, d} is *pgpr*-closed but not *w*s*-closed.
(xii) {a} is *w*s*-closed but not *g*p*-closed and {b, d} is *g*p*-closed but not *w*s*-closed.
(xiii) {a} is *w*s*-closed but not $\pi g\alpha$-closed and {b, d} is $\pi g\alpha$-closed but not *w*s*-closed.
(xiv) {a} is *w*s*-closed but not πgp-closed and {b, d} is πgp-closed but not *w*s*-closed.
(xv) {a} is *w*s*-closed but not *rwg*-closed and {a, b} is *rwg*-closed but not *w*s*-closed.

The above discussions lead to the following figure. In this figure, "A→B" means A implies B but not conversely and "A↮B" means A and B are independent of each other.

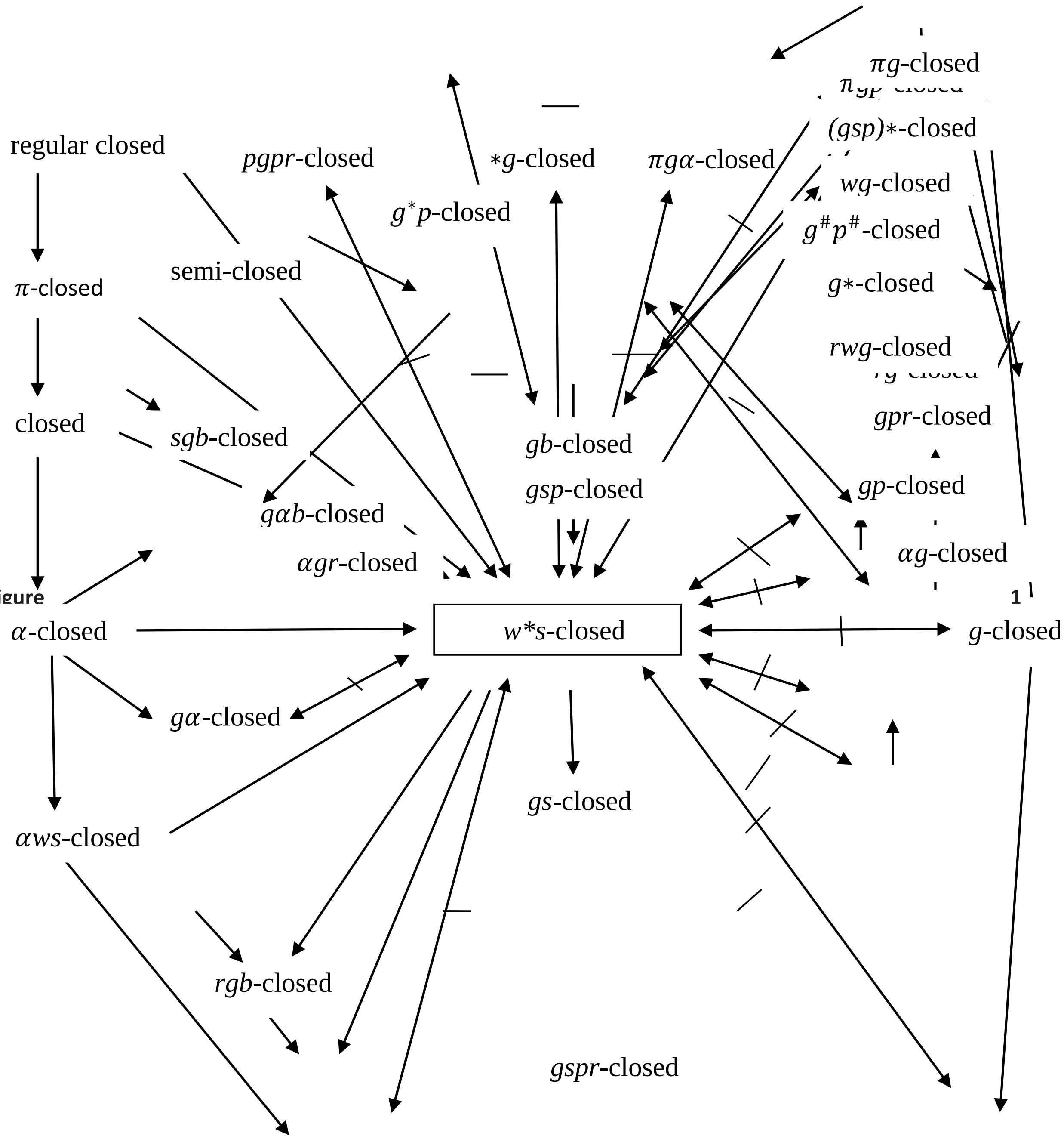

Figure 1

Theorem 3.17: If *A* is a subset of *X* which is w*s- hen *scl(A)\A* does not contain any non-empty *ws*-closed set in *X*.

Proof: Consider a *w*s*-closed set *A* in *X*. Take a *ws*-closed subset *F* of *scl(A)\A*. Then *F⊆scl(A)∩(X\A)*. This implies *F⊆scl(A)* and *F⊆X\A*. And hence *A⊆X\F*. Since *A* is *w*s*-closed and since *X\F* is *ws*-open, then *scl(A) ⊆X\F⊆X\scl(A).* Already we have, *F⊆scl(A).* Therefore, *F⊆scl(A)∩(X\scl(A)) =* Φ. Thus, *F⊆* Φ. And hence, *F=*Φ. Hence, *scl(A)\A* does not contain a non-empty *ws*-closed set. (i)

Theorem 3.18: If *A* and *B* are *w*s*-closed sets, then *A∩B* is *w*s*-closed.

Proof: Let *A* and *B* be *w*s*-closed sets. Let *A∩B⊆U* and *U* be *ws*-open. Then *A⊆U* and *B⊆U.* Since *A*(ii) and *B* are *w*s*-closed sets, *scl(A) ⊆U* and *scl(B) ⊆U.* Therefore, *scl(A)∩scl(B) ⊆U.* But, *scl(A∩B) ⊆scl(A)∩scl(B)*[28]. Therefore, *scl(A∩B) ⊆U*. Hence *A∩B* is *w*s*-closed.

Corollary 3.19: Let *A* be a *w*s*-closed set.

(i) If *F* is semi-closed, then *A∩F* is *w*s*-closed.

(ii) If *F* is regular-closed, then *A∩F* is *w*s*-closed.

Proof:

(i) Since *F* is semi-closed, by Theorem 3.2(iii), *F* is *w*s*-closed. Since *A* is *w*s*-closed, by Theorem 3.18, *A∩F* is *w*s*-closed.

(ii) Since *F* is regular-closed, by Theorem 3.2(vi), *F* is *w*s*-closed. Since *A* is *w*s*-closed, by Theorem 3.18, *A∩F* is *w*s*-closed.

Theorem 3.20: Let *A* be *w*s*-closed. Then *A* is semi-closed if and only if *scl(A)\A* is *ws*-closed.

Proof: Let *A* be a semi-closed set. Then *scl(A)=A*. This implies *scl(A)\A=*Φ*,* which is *ws*-closed. Conversely, suppose that *scl(A)\A* is *ws*-closed. Since *A* is *w*s*-closed, by Theorem 3.17, *scl(A)\A=*Φ. That is, *scl(A)=A*. Hence *A* is semi-closed.

Theorem 3.21: Let *A* be *w*s*-closed and *ws*-open.(i) Then *A* is semi-closed. (ii)

Proof: Since *A* is *w*s*-closed and *ws*-open, *A⊆A*. This implies *scl(A)⊆A*. Hence *A* is semi-closed.

Theorem 3.22: If *A* is a *w*s*-closed subset of *X* and *A⊆B⊆scl(A)*, then

(i) *B* is also *w*s*-closed.

(ii) *scl(B)\B* contains no non empty *ws*-closed set.

Proof: Let *A* be a *w*s*-closed set in *X* such that *A⊆B⊆scl(A).*

To prove: *B* is also *w*s*-closed set in *X*. It is enough to prove that *scl(B)⊆U*. Let *U* be a *ws*-open set in *X* such that *B⊆U*. Since *A⊆B, A⊆U*. Also, since *A* is *w*s*-closed, *scl(A)⊆U*. We have, *B⊆scl(A).* This implies *scl(B)⊆scl(sclA))=scl(A)⊆U*. That is, *scl(B)⊆U*. Therefore, *B* is *w*s*-closed.

Since *B* is *w*s*-closed, by Theorem 3.17, scl(B)\B does not contain any non-empty *ws*-closed set.

Theorem 3.23: Let *X* and *Y* be topological spaces and *A⊆Y⊆X.* Then a *w*s*-closed set *A* in *X* is *w*s*-closed relative to *Y.*

Proof: Given *A⊆Y⊆X* and *A* is *w*s*-closed in *X*. To prove that, *A* is *w*s*-closed relative to *Y*. Let *A⊆Y∩U*, where *U* is *ws*-open in X. Since *A* is *w*s*-closed, then by Definition 3.1, *scl(A)⊆U.* This implies *Y∩scl(A)⊆Y∩U*, where *Y∩scl(A)* is the semi-closure of *A* in *Y* and *Y∩U* is *ws*-open in *Y*. Therefore, *scl(A)⊆Y∩U* in *Y*. Hence *A* is *w*s*-closed set relative to *Y*.

Theorem 3.24: If *A* is *w*s*-closed set in *X,* and *B* is closed set in *X*, then *A∩B* is *w*s*-closed.

Proof: Let *A* be a *w*s*-closed set and *B* be a closed set in *(X, τ)*. Let *A∩B⊆U* and *U* be *ws*-open. Then *U* ∪ *(X\B)* is a *ws*-open set containing *A*. That is *A⊆ U* ∪ *(X\B)*. Since *A* is *w*s*-closed, *scl(A)⊆U* ∪ *(X\B).* Now, *scl(A∩B)⊆scl(A)∩scl(B)⊆scl(A)∩cl(B)=scl(A)∩B⊆(U* ∪*(X\B))∩B =U∩B⊆U*. Therefore, *A∩B* is *w*s*-closed.

Theorem 3.25: Let *A* be a *w*s*-closed set. Then

sint A is *w*s*-closed.

If *A* is regular open, then *pintA* and *scl(A)* are also *w*s*-closed.

(iii) If *A* is regular closed, then *pcl(A)* is *w*s*-closed

Proof:

(i) Let *A* be a *w*s*-closed set. Since *cl int A* is closed, *cl int A* and *A* are *w*s*-closed sets. By Theorem 3.18, *A*∩ *(cl int A)* is also *w*s*-closed set. Then by Lemma 2.5(i), *sint A* is *w*s*-closed.

(ii) Let *A* be *w*s*-closed and regular open. Since *A* is regular open, *A=int cl (A).* By Lemma 2.5(iii), *scl(A)=A*∪*int cl(A).* This implies *scl(A)=A*∪*A=A.* Thus, *scl(A)* is *w*s*-closed. Also, we know that, *pint A =A*∩*int cl A=A*∩*A=A.* Thus *pint A* is also *w*s*-closed.

(iii) Let *A* be a *w*s*-closed set in *X.* Since *A* is regular closed, *A= cl int A*. By Lemma 2.5(iv), *pcl(A)=A*∪*cl int A=A*∪*A=A.* Thus, *pcl(A)* is *w*s*-closed.

Theorem 3.26: For every point *x* in a space , $X-\{x\}$ is *w*s*-closed or *ws*-open.

Proof:

Suppose *X*-{*x*} is not *ws*-open. Then *X* is the only *ws*-open set containing *X*-{*x*}. Then by definition: 3.1, scl(*X*-{*x*})⊆ X. Hence *X*-{*x*} is *w*s*-closed.

Theorem 3.27: In a topological space *X* if SO(*X*)={X, Φ}, then every subset of *X* is *w*s*-closed set.

Proof: Given that *X* is a topological space and SO(*X*)={*X*, Φ}. Let *A* be a subset of *X*. Suppose $A = \Phi$. Then Φ is a *w*s*-closed set. Suppose *A*≠ Φ. Then *X* is the only semi-open set containing *A* and hence *scl(A)*⊆*X*. Hence *A* is a *w*s*-closed set in *X*.

Theorem 3.28: Let *A* and *B* be *w*s*-closed sets in *X*, such that *cl(A)=scl(A)* and *cl(B)=scl(B),* then *A*∪*B* is *w*s*-closed.

Proof: Let *A*∪*B*⊆*U* and *U* be *ws*-open. Then *A*⊆*U* and *B*⊆*U*. Since *A* and *B* are *w*s*-closed sets, *scl(A)* ⊆*U* and *scl(B)* ⊆*U*. Now,(i) *cl(A*∪*B)=cl(A)*∪*cl(B)= scl(A)*∪*scl(B)* ⊆*U*. But,(ii) *scl(A*∪*B)* ⊆*cl(A*∪*B).* Therefore, *scl(A*∪*B)*⊆(iii) *cl(A*∪*B)* ⊆*U*. Thus, *scl(A*∪*B)*⊆*U*, whenever *A*∪*B* ⊆*U*(iv) and *U* is *ws*-open. Hence *A*∪*B* is *w*s*-closed.

Theorem 3.29: In an extremally disconnected space *X*, the union of two *w*s*-closed sets is *w*s*-closed.

Proof: Let *A* and *B* be two *w*s*-closed subsets of an extremally disconnected space *X*. Let *A*∪*B*⊆*U* and *U* be *ws*-open. Then *A*⊆*U* and *B*⊆*U*. Since *A* and *B* are *w*s*-closed sets, *scl(A)* ⊆*U* and *scl(B)* ⊆*U*. Therefore, *scl(A)*∪*scl(B)* ⊆*U*. Since *X* is extremally disconnected space, by Lemma 2.3(iii), *scl(A*∪*B)=scl(A)*∪*scl(B).* This implies *scl(A*∪*B)*⊆*U*. Hence *A*∪*B* is also *w*s*-closed.

Theorem 3.30: In an extremally disconnected submaximal space *X*, every *w*s*-closed set is *w*-closed.

Proof: Let *X* be an extremally disconnected submaximal space and *A* be a *w*s*-closed subset of *X*. Let *A*⊆*U* and *U* is semi-open. Since every semi-open set is *ws*-open and since *A* is *w*s*-closed, *scl(A)*⊆*U*. Since *X* is extremally disconnected submaximal space, by Lemma 2.4, *cl(A)*⊆*U.* Hence by Definition 2.2(xv), *A* is *w*-closed.

Theorem 3.31: Let *A* be a locally closed set in *X.* Then *A* semi-closed if and only if *A* is *w*s*-closed.

Proof: Suppose that *A* is semi-closed. By Theorem 3.2(iii), *A* is *w*s*-closed. Conversely, suppose that *A* is *w*s*-closed. Since *A* is locally closed, *A*∪*(X*∖*cl(A))* is open in *X*. Since every open set is *ws*-open, *A*∪*(X*∖*cl(A))* is *ws*-open in *X*. Clearly, *A*⊆ *A*∪*(X*∖*cl(A)).* Since *A* is *w*s*-closed, *scl(A)* ⊆ *A*∪*(X*∖*cl(A)).* Since *cl(A)*∩*(X*∖*cl(A))=*Φ and *scl(A)*⊆*cl(A),* we have *scl(A)*∩*(X*∖*cl(A))=*Φ. Therefore, *scl(A)*⊆***A*.** Hence *A* is semi-closed.

Theorem 3.32: In an extremally disconnected space *X*,

Every *w*s*-closed set is $g\alpha$-closed.

Every *w*s*-closed set is αg-closed.

Every *w*s*-closed set is *gpr*-closed.

Every *w*s*-closed set is *gp*-closed.

Proof:

(i) Let *A* be a *w*s*-closed subset of *X*. Let *A*⊆*U* and *U* be α-open. Since every α-open set is *ws*-open and since *A* is *w*s*-closed, *scl(A)*⊆*U*. Since *X* is extremally disconnected, by

Lemma 2.3(ii), α*cl(A)⊆U*. Hence by Definition 2.2(iv), *A* is *g*α-closed.

(ii) Let *A* be a *w*s*-closed subset of *X*. Let *A⊆U* and *U* be open. Since every open set is *ws*-open and since *A* is *w*s*-closed, *scl(A)⊆U*. Since *X* is extremally disconnected, by Lemma 2.3(ii), α*cl(A)⊆U*. Hence by Definition 2.2(iii), *A* is αg-closed.

(iii) Let *A* be a *w*s*-closed subset of *X*. Let *A⊆U*and *U* be regular-open. Since every regular-open set is *ws*-open and since *A* is *w*s*-closed, *scl(A)⊆U*. Since *X* is extremally disconnected, by Lemma 2.3(ii), α*cl(A)⊆U*. But *pcl(A)⊆* α*cl(A).* Therefore *pcl(A)⊆U*. Hence by Definition 2.2(x), *A* is *gpr*-closed.

(iv) Let *A* be a *w*s*-closed subset of *X.* Let *A⊆U* and *U* be open. Since every open set is *ws*-open and since *A* is *w*s*-closed, *scl(A)⊆U*. Since *X* is extremally disconnected, by Lemma 2.4(ii), α*cl(A)⊆U*. But *pcl(A)⊆* α*cl(A).* Therefore *pcl(A)⊆U*. Hence by Definition 2.2(vi), *A* is *gp*-closed.

4 Closure of *w*s*-closed set:

*w*s*-closure in topological spaces by using the notion of *w*s*-closed sets is introduced and its characteristics are investigated.

Definition 3.33: For a subset *A* of a space *X*, *w*s*-closure is defined as follows: *w*s-cl(A)=∩{F: A⊆F* and *F* is *w*s*-closed in *X*}.

Remark 3.34:

(i) *w*s*-closure of a set is always a *w*s*-closed set.

(ii) *A* is a *w*s*-closed set if and only if *w*s-cl(A)=A.*

Lemma 3.35: Let *A* and *B* be subsets of *X*. Then

(i) *w*s-cl*(Φ)=Φ and *w*s-cl(X)=X.*

(ii) If *A⊆B*, then *w*s-cl(A)⊆ w*s-cl(B).*

(iii) *A⊆ w*s-cl(A).*

Theorem 3.36: Let *x∈X.* Then *x∈ w*s-cl(A)* if and only if *V∩A≠* Φ for every *w*s*-open set *V* containing the point *x.*

Proof: Assume that *x∈ w*s-cl(A).* Suppose that there exists a *w*s*-open set *V* containing the point *x* such that *V∩A=* Φ. Now *V∩A=* Φ implies *A⊆X\V.* By Lemma 3.35(ii), *w*s-cl(A)⊆ w*s-cl(X\V).* Since *X\V* is *w*s*-closed, by Remark 3.34(ii), *w*s-cl(X\V)= X\V*. Thus *w*s-cl(A)⊆ w*s-cl(X\V) = X\V*. This implies *w*s-cl(A)⊆ X\V*. Therefore *x∉ w*s-cl(A),* which is a contradiction to our assumption. Hence *V∩A≠* Φ for every *w*s*-open set *V* containing the point *x.* Conversely, assume that *V∩A≠* Φ for every *w*s*-open set *V* containing the point *x*. Suppose that *x∉ w*s-cl(A).* Then by Definition 3.33, there exists a *w*s*-closed subset *F* containing *A* such that *x∉F*. Therefore *x∈X\F*. Since *A⊆F*, *(X\F)∩A=* Φ. This is a contradiction, since *X\F* is a *w*s*-open set containing the point *x*. Therefore *x∈ w*s-cl(A).*

Proposition 3.37: If *W*SC(X,* τ*)* be closed under finite union, then *w*s-cl(A∪B)=* (*w*s-clA*) ∪ (*w*s-clB*) for every *A, B∈ W*SC(X,* τ*).*

Proof: Since *A* and *B* are *w*s*-closed, by Remark 3.34(ii), *A= w*s-cl(A)* and *B=w*s-cl(B).* Since *W*SC(X,* τ*)* is closed under finite union, *A∪B* is *w*s*-closed. Therefore by Remark 3.34(ii), *w*s-cl(A∪B)=A∪B.* This implies *w*s-cl(A∪B)= w*s-cl(A)∪ w*s-cl(B).*

Theorem 3.38: If *SC(X,* τ*)* be closed under finite union, then *W*SC(X,* τ*)* is closed under finite union.

Proof: Let *A*, *B∈ W*SC(X,* τ*).* Let *A∪B⊆U* and *U* be *ws*-open in *(X,* τ*).* Then *A⊆U* or *B⊆U.* Since *A* and *B* are *w*s*-closed sets, by Definition 3.1, *scl(A) ⊆U* or *scl(B) ⊆U*. This implies *scl(A)∪scl(B)⊆U*. Since *SC(X,* τ*)* is closed under finite union, *scl(A∪B)=scl(A)∪scl(B).* Therefore *scl(A∪B))⊆U*. Thus *scl(A∪B)⊆U,* whenever *A∪B⊆U* and *U* is *ws*-open in *(X,* τ*).* Hence *A∪B* is *w*s*-closed in *(X,* τ*).* Therefore *A∪B∈ W*SC(X,* τ*).*

Lemma 3.39: Let *A* and *B* be subsets of *(X,* τ*).* Then

(i) *w*s-cl*(*A*) = *w*s-cl*(*w*s-cl(A)*).

(ii) (*w*s-clA*)∪(*w*s-clB*) ⊆ *w*s-cl(A∪B).*

(iii) *w*s-cl(A∩B)* ⊆ (*w*s-clA)∩(w*s-clB).*

Proof:

(i) Since *A⊆ w*s-cl(A),* by Lemma 3.35(ii), *w*s-cl(A)⊆ w*s-cl*(*w*s-cl(A)).* Suppose that *x∈ w*s-cl*(*w*s-cl(A)).* Let us assume that *V* be a *w*s*-open set containing the point *x*. Then by Theorem 3.36, *V∩*(*w*s-cl(A))≠* Φ. Take *y* ∈ *V∩*(*w*s-cl(A)).* This implies *y ∈V* and *y* ∈ *w*s-*

cl(A). Again by using Theorem 3.36, *V*∩*A*≠ *Φ*. Therefore *x*∈ *w*s-cl(A).* Thus we get, *w*s-cl(w*s-cl(A))* ⊆ *w*s-cl(A).* Therefore *w*s-cl(A)*= *w*s-cl*(*w*s-cl(A)).*

(ii) Since *A*⊆*A*∪*B* and *B*⊆*A*∪*B,* by Lemma 3.35(ii), we have, *w*s-cl(A)*⊆ *w*s-cl(A*∪*B)* and *w*s-cl(B)*⊆ *w*s-cl(A*∪*B).* Thus (*w*s-clA*)∪(*w*s-clB*) ⊆ *w*s-cl(A*∪*B).*

(iii) Since *A*∩*B*⊆*A* and *A*∩*B*⊆*B*, by Lemma 3.35(ii), we have, *w*s-cl(A*∩*B)*⊆ *w*s-cl(A)* and *w*s-cl(A*∩*B)*⊆ *w*s-cl(B).* Thus *w*s-cl(A*∩*B)* ⊆ (*w*s-clA*)∩(*w*s-clB*).

5 Conclusion

In this paper, we have concentrated on weakly star semi closed sets in topological spaces and found some important properties. In future this concept can be extended to other topological spaces.

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