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# Vertex Geodetic Number of a Fuzzy Graph

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**Abstract**

In this paper, vertex geodetic number of a fuzzy graph is introduced. Let $x$ be a vertex of a connected non-trivial fuzzy graph $G:(V,\sigma,\mu)$. A set $S$ of vertices of $G$ is $x-geodetic\ set$ if each vertex $u$ of $G$ lies on $x-y$ geodesic in $G$ for some element $y$ in $S$. The minimum cardinality of $x-geodetic\ set$ of $G$ is defined as $x-geodetic\ number$ of $G$ and is denoted by $gn_x(G)$. A $x-geodetic\ set$ of cardinality $gn_x(G)$ is called $gn_x(G)-set$ of $G$.

**Keywords:** geodesic path, geodesic number, vertex geodetic number.
**2020 Mathematics Subject Classification:** 05C72, 05C12.

## 1. Introduction:

Zadeh in 1965[15] developed a mathematical phenomenon for describing the uncertainties prevailing in day-to-day life situations by introducing the concept of fuzzy sets. The theory of fuzzy graphs was later on developed by Rosenfeld in the year 1975[12]. A fuzzy graph is a triplet $G:(V,\sigma,\mu)$ where $V$ is a vertex set, $\sigma$ is a fuzzy subset on $V$ and $\mu$ is a fuzzy relation on $\sigma$ such that $\mu(x,y)\leq\sigma(x)\wedge\sigma(y)\forall x,y\in V$. We assume that $V$ is finite and nonempty, $\mu$ is reflexive and symmetric. In all the examples $\sigma$ is chosen suitably. Also we denote the underlying crisp graph by $G^*:(\sigma^*,\mu^*)$ where $\sigma^*=\{x\in V:\sigma(x)>0\}$ and $\mu^*=\{(x,y)\in V\times V:\mu(x,y)>0\}$. Here we take $\sigma^*=V$. For basic fuzzy graph theoretic terminology we refer to Nagoorgani and Chandrasekaran VT [11]. A fuzzy graph $G:(V,\sigma,\mu)$ is a complete fuzzy graph if $\mu(x,y)\leq\sigma(x)\wedge\sigma(y)$ for every $x,y\in\sigma^*$.

A path $P$ of length $n$ is sequence of distinct vertices$u_0,u_1,\ldots.,u_n$ such that $\mu(u_{i-1},u_i)>0, i=1,2,\ldots,n$ and the degree of membership of a weakest arc is defined as its strength. The path becomes cycle if $u_0=u_n, n\geq 3$ and a cycle is called a fuzzy cycle if it contains more than one weakest arc. The strength of connectedness between two vertices $x$ and $y$ is defined as the maximum of the strength of all path between $x$ and $y$ and it is denoted by $CONN_G(x,y)$. An arc of a fuzzy graph is called strong if its weight is at least as great as the connectedness of its end vertices when it is deleted and a $x-y$ path is called a strong path if $P$ contains only strong arcs. A connected fuzzy graph $G:(V,\sigma,\mu)$ is called fuzzy tree if it has a spanning fuzzy sub graph $F:(V,\sigma,\upsilon)$, which is a tree such that for all arcs $x,y$ not in $F$, $CONN_F(x,y)>\mu(x,y)$. A fuzzy star is a fuzzy tree whose unique maximum spanning tree is a star. A fuzzy graph $G:(V,\sigma,\mu)$ is a fuzzy bipartite if it has a spanning fuzzy sub graph $H:(V,\tau,\pi)$ which is bipartite where for all edge $(x,y)$ not in $H$, weight of $(x,y)$ in $G$ is strictly less than the strength of pair $(x,y)$ in $H$. A fuzzy cut vertex $w$ is a vertex in $G:(V,\sigma,\mu)$ whose removal reduces the strength of connectedness between some pair of vertices in $G:(V,\sigma,\mu)$. If $\mu(x,y)>0$, then $u$ and $v$are called neighbors. A fuzzy bipartite graph $G$ with fuzzy bipartite $(V_1,V_2)$ is said to be a complete fuzzy bipartite if for each node of $V_1$, every vertex of $V_2$ is a strong neighbor. A vertex $u$ in a fuzzy graph $G:(V,\sigma,\mu)$ is extreme in $G:(V,\sigma,\mu)$if $N_G[u]$ is a complete fuzzy graph. A vertex in a fuzzy graph $G:(V,\sigma,\mu)$ having only one neighbor is called a pendent vertex. A vertex $v$ is called a fuzzy end of $G:(V,\sigma,\mu)$ if it has at most one strong neighbor in $G:(V,\sigma,\mu)$.

The geodetic number of a crisp graph was introduced in [5] and further studied in [2,3]. This concept was extended to fuzzy graphs using geodetic distance by N. T. Suvarna and M. S. Sunitha in [14] and using $\mu$ −distance by J. P. Linda and M. S. Sunitha in [7]. The concept of vertex geodomination number of crisp graph was introduced in [13]. Let $x$ be a vertex of a connected crisp graph $G$. A set $S$ of vertices of $G$ is an $x-geodominating\ set$ of $G$ if each vertex $v$ of $G$ lies on $x-y$ geodesic in $G$ for some element $y$ in $S$. The minimum cardinality of an $x-geodominating\ set$ of $G$ is defined as the $x-geodomination\ number$ of $G$ and is denoted by $g_x(G)$. An $x-geodominating\ set$ of cardinality $g_x(G)$ is called $g_x-set$. The concept of geodesics in fuzzy graph was introduced in [1].

Table 1

| Vertex | Minimum vertex geodetic sets | Vertex geodetic number |
|---|---|---|
| $u_1$ | $\{u_4,u_6\}$ | 2 |
| $u_2$ | $\{u_4,u_6\}$ | 2 |
| $u_3$ | $\{u_4,u_6\}$ | 2 |
| $u_4$ | $\{u_6\}$ | 1 |
| $u_5$ | $\{u_4,u_6\}$ | 2 |
| $u_6$ | $\{u_4\}$ | 1 |

In this paper we introduce the concept of vertex geodetic number of fuzzy graph. The following theorem will be used in sequel.

**Theorem 1.1 [11]** Let $G:(V,\sigma,\mu)$ be ay graph connected fuzzy graph and let $x\in V$. The following are equivalent:

1. $x$ is a cut vertex of $G:(V,\sigma,\mu)$.
2. There exists a vertices $y$ and $z$ distinct from $x$ such that $x$ is on every strongest path between $y$ and $z$.
3. There exists a partition of the set of vertices $V-\{x\}$ into subsets $Y,Z$ and $X$ such that for all $y\in Y, z\in Z$, the vertex $x$ is on every strongest path between $y$ and $z$.
4.

## 2. Vertex Geodetic Number of a Fuzzy Graph [$gn_x(G)$]

In this section, we introduced vertex geodetic number of fuzzy graph.

**Definition 2.1**

Let $x$ be a vertex of a connected non-trivial fuzzy graph $G:(V,\sigma,\mu)$. A set $S$ of vertices of $G-x$ is $x-geodetic\ set$ if each vertex $r$ of $G$ lies on $x-y$ geodesic in $G$ for some element $y$ in $S$. The minimum cardinality of $x-geodetic\ set$ of $G$ is defined as $x-geodetic\ number$ of $G$ and is denoted by $gn_x(G)$. A $x-geodetic\ set$ of cardinality $gn_x(G)$ is called $gn_x(G)-set$ of $G$.

**Example 2.2**

For the fuzzy graph $G:(V,\sigma,\mu)$given in Fig. 2.1, the minimum vertex geodetic sets and the vertex geodetic numbers are given in Table 1.

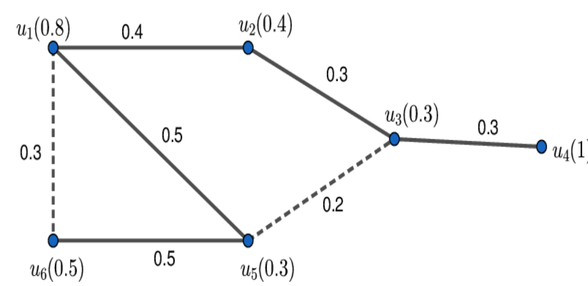


Fig. 2.1 (with $\delta-arcs$)

For the fuzzy graph $G:(V,\sigma,\mu)$given in Fig. 2.2, the minimum vertex geodetic sets and the vertex geodetic numbers are given in Table 2.

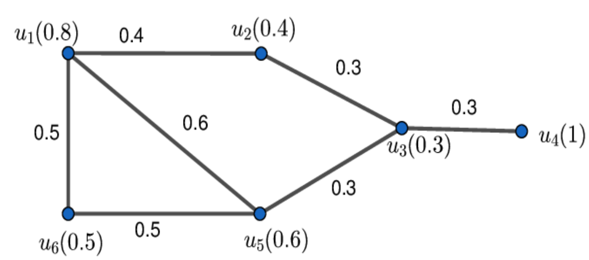


Fig 2.2 (with out $\delta-arcs$)

Table 2

| Vertex | Minimum vertex geodetic sets | Vertex geodetic number |
|---|---|---|
| $u_1$ | $\{u_4, u_6\}$ | 2 |
| $u_2$ | $\{u_4, u_5, u_6\}$ | 3 |
| $u_3$ | $\{u_1, u_4, u_6\}$ | 3 |
| $u_4$ | $\{u_1, u_6\}$ | 2 |
| $u_5$ | $\{u_2, u_4, u_6\}$ | 3 |
| $u_6$ | $\{u_2, u_4\}$ | 2 |

**Note 2.3** Each vertex in an $x-y$ geodesic is $x$ −geodetic to the other vertex via the vertex $y$. The vertex $x$ and the internal vertices of an $x-y$ geodesic do not belong to a " $gn_x - set$ " because a " $gn_x - set$" is minimal by definition.

**Theorem 2.4** For any connected fuzzy graph $G:(V,\sigma,\mu)$on $n$ vertices containing no $\delta-arcs$, $gn_x(G) = n-1$ if and only if $x$ is a vertex of $G^*$ of degree $n-1$.

**Proof:**

Let $gn_x(G) = n-1$. Assume that $x$ is a $G^*$ vertex with a degree lower than n-1. Then, in $G$, there is a vertex $u$ that is not adjacent to $x$. There is a geodesic with a length of at least 2 from $x$ to $u$ say $P$ since $G$ is a connected fuzzy graph without $\delta - arcs$. According to Note 2.3, $x$ and the internal vertices of $P$ do not belong to the " $gn_x - set$," hence the statement " $gn_x \le n-1$" is contradictory.

Conversely, all additional vertices of $G$ that are strong neighboring to vertex $x$ form the " $gn_x - set$" if vertex $x$ is a vertex of degree $n-1$ in the graph $G^*$. Therefore, $gn_x(G) = n-1$.

**Remark 2.5** For any connected fuzzy graph $G:(V,\sigma,\mu)$on $n$ vertices containing $\delta - arcs$ such that $x$ is a vertex of $G^*$ of degree $n-1$, then $gn_x(G)$ need not be $n-1$.

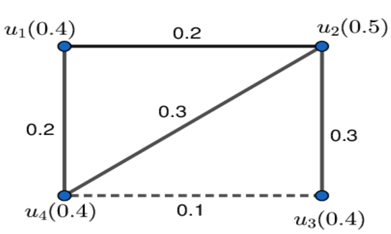


Fig 2.3

For example, the fuzzy graph given in Fig. 2.2 on 4 vertices, here $u_2, u_4$ are the vertices of $G^*$ of degree $n-1$. But in this graph the arc $(u_3, u_4)$ is a $\delta - arcs$. So the vertex $u_4$ contains only two strong neighbors, so $\{u_1, u_3\}$ is a $gn_{u_4}(G)$- set of $G$. So $gn_{u_4}(G) = 2 \neq 3$.

**Theorem 2.6** Let $G:(V,\sigma,\mu)$ be a connected non-trivial fuzzy graph.

(i) Each $gn_x - set$ contains every extreme vertex of G except for the vertex $x$(whether or not $x$ is an extreme vertex).

(ii) Eccentric vertices of any vertex $x$ are a part of the $gn_x - set$.

(iii) No cut vertex of $G$ belongs to any $gn_x - set$.

**Proof:**

(i) Let $x$ represent any of $G:(V,\sigma,\mu)'s$ vertex. By Note 2.3, $x$ is excluded from the $gn_x - set$. Let $S_x$ be $gn_x - set$ $G:(V,\sigma,\mu)$ and let $a \neq x$ be an extreme vertex. Let's say $a \notin S_x$. Then, for some $y \in S_x$, $a$ is an internal vertex of an $x-y$ geodetic, let's say $P$. Assume that $b$ and $c$ are $a's$ neighbors on $P$, preventing them from being adjacent. As a result, $a$ is not an extreme vertex, which is incongruous.

(ii) Y should be $a's$ eccentric vertex for $d_f(x,y) = e_f(x)$d_f (x,y). Assume that $y$ is not a member of the $G's$ $gn_x - set$. Then $c$ in the $gn_x - set$ has a vertex $y$ that is an internal vertex of an $x-c$ geodesic. As a result, $e_f(x) \le d_f(x,c)$, which is a contradiction, $d_f(x,y) < d_f(x,c)$.

(iii) Assume that $v$ is the cut vertex of $G:(V,\sigma,\mu)$. According to Theorem 1.1, the set of vertices $V-\{v\}$ may be divided into subsets $Y, Z$and $X$ such that the vertex $v$ is on each of the strongest paths connecting $y$ and $z$ for all values of $y \in Y, z \in Z$. Since $v$ is an internal vertex of an $x-z$ geodesic if $x \in Y$, $v$ lies on every $x-Y$ strong path for each vertex $z$in $Z$. As a result, v is not a member of the $gn_x - set$. By note 2.3, $v$ does not belong to the "$gn_x - set$" if $x = v$.

**Corollary 2.7** The $x$ −geodetic number of a complete fuzzy graph $G:(V,\sigma,\mu)$ on $n$ vertices is $n-1$.

**Proof:** Let $x \in G:(V,\sigma,\mu)$. Since each vertex in a complete fuzzy graph $G:(V,\sigma,\mu)$ is an extreme vertex. Then by Theorem 2.5(ii), every extreme vertex of $G$ other than the vertex $x$ belongs to every $gn_x - set$ . Hence $gn_x(G) = n - 1$.

**Corollary 2.8** Let $G = FS_n$ be a fuzzy star with $n$ vertices. Then $gn_x(G) = \begin{cases} n-2 & if\ x\ is\ an\ end\ vertex \\ n-1 & if\ x\ an\ intenal\ vertex. \end{cases}$

**Proof:** Let $x \in G$.

**Case (i):** $x$ is an internal vertex of $G$.

By definition of fuzzy star, the remaining $n - 1$ vertices are fuzzy end vertices of $G$. By Note [2.3] , $gn_x(G) \geq n - 1$. Let $S$ be the set of all fuzzy end vertices of $G$. Then $S$ is a $gn_x - set$ of $G$. So that $gn_x(G) = n - 1$.

**Case (ii):** $x$ is an end vertex of $G$.

Then $S' = S - \{x\}$ be the set of all fuzzy end vertices of $G$. Then by Note [2.3] , $S'$ is a subset of $gn_x - set$ of $G$. Also $gn_x(G) \geq n - 2$. Since the internal vertex $y$ is a fuzzy cut vertex of $G$. By Theorem[2.6] (iii), $y$ does not belong to any $gn_x - set$ of $G$. Now $S'$ is a $gn_x - set$ of $G$. So that $gn_x(G) = n - 2$.

**Corollary 2.9** Let $P_n$ be a non-trivial fuzzy path .

Then $gn_x(P_n) = \begin{cases} 1 & if\ x\ is\ an\ extreme\ vertex \\ 2 & Otherwise. \end{cases}$

**Proof:** Let $P_n$ be a non-trivial fuzzy path. Since there exist exactly two extreme vertices which are in two ends of the fuzzy path. Let $x, y$ be the two extreme vertices of $P_n$.

**Case (i):** Let $x \in P_n$ be an extreme vertex. Then by Note 2.3, $x$ does not belong to $gn_x - set$. Also by Theorem 2.5(i), $\{y\}$ is the only $gn_x - set$ of $P_n$. Hence $gn_x(P_n) = 1$.

**Case (ii):** Let $w \in P_n$ be any other vertex which is not an extreme vertex. Also $w$ is an internal vertex of $x - y$ geodesic. It is easily verified that there exists exactly two paths $x - w$ and $w - y$, which covers all the vertex $v \in P_n$. Thus $\{x, y\}$ is the only $gn_w - set$ of $P_n$. Therefore $gn_w(P_n) = 2$.

**Corollary 2.10** The $x$ −geodetic number of a fuzzy cycle $G: (V, \sigma, \mu)$ on $n$ vertices, $(n \geq 3)$ is given by $gn_x(G) = \begin{cases} 1 & when\ n\ is\ even \\ 2 & when\ n\ is\ odd. \end{cases}$

**Proof:** Let $x \in G: (V, \sigma, \mu)$.

**Case (i):** Let n be even.

Let $y$ be eccentric vertex of $x$. Then $\{y\}$ is a $x$ −geodetic set of $G$ so that $gn_x(G) = 1$.

**Case (ii):** Let n be odd.

It is easily verified that no singleton subset of $G$ is not a $x$ −geodetic set of $G$. Also $gn_x(G) \geq 2$. Let $y, z$ be the two eccentric vertices of $x$. Then $\{y, z\}$ is a $x$ − geodetic set of $G$ so that$gn_x(G) = 1$.

**Corollary 2.11** Let $T$ be a fuzzy tree with number of fuzzy end vertices $t$.

Then $gn_x(T) = \begin{cases} t-1 & if\ x\ is\ end\ vertex \\ t & Otherwise. \end{cases}$

**Theorem 2.12** Let $G: (V, \sigma, \mu)$ be a connected fuzzy graph containing no $\delta - arcs$.

(i) For the wheel fuzzy graph $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $gn_x(W_n) = n - 1$ $or$ $n - 4$ according as $x$ is$K_1$ or $x$ is in $C_{n-1}$.

(ii) Let $K_{\sigma_1,\sigma_2} = (V_1 \cup V_2, \sigma, \mu)$ be a complete fuzzy bipartite. Then

(1) $gn_x(K_{\sigma_1,\sigma_2}) = 1$, if $|V_1| = |V_2| = 1$.

(2) $gn_x(K_{\sigma_1,\sigma_2}) = \begin{cases} 1 & if\ x\ is\ in\ V_2 \\ 2 & if\ x\ is\ in\ V_1 \end{cases}$ ,

where$|V_1| = 1, |V_1| \geq 2$ .

(3) $gn_x(K_{\sigma_1,\sigma_2}) = \begin{cases} m-1 & if\ x\ is\ in\ V_1 \\ n-1 & if\ x\ is\ in\ V_2 \end{cases}$ ,

where $|V_1| = m$ and $|V_2| = n$ , $(m,n) \geq 2$.

**Proof:**

(i) Let x represent the $K_1$ vertex. Then, $x$ is a vertex of $G^*$of degree $x$ because it is adjacent to every other vertex of $W_n$. Theorem 2.4 states that $gn_x(W_n) = n - 1$

Let the cycle of $W_n$ be $C_{n-1}: u_1, u_2, u_3, \ldots., u_{n-1}, u_1$. Let $x$ represent any vertex in $C_{n-1}$. Take $x = u_1$ without losing generality. Since the eccentric vertices of $x$ are set at the diameter of $d = 2, \{u_3, u_4, \ldots, u_{n-2}\}$. If $K_1$ is $z$, then the vertices $u_2, z$ and $u_{n-1}$ are, respectively, located on the geodesics $x, u_2, u_3$; $x, z, u_3$ and $x, u_{n-1}, u_{n-2}$. Since $gn_x(W_n) = n - 4$ as a result of Theorem 2.6(ii), the $gn_x - set$ of $W_n$ is $\{u_3, u_4, \ldots, u_{n-2}\}$.

(ii) (1) It follows from Corollary 2.7.

(2) It follows from Corollary 2.11.

(3) Let $x \in V_1$ be any vertex. Clearly $V_1 - \{x\}$ is the set of eccentric vertices of $x$. Let $v$ be any vertex of $V_2$. Clearly $v$ lies on the geodesic $x, v, u$ for every vertex $u$ in $V_1 - \{x\}$. By Theorem 2.6(ii), the $gn_x - set$ of $K_{\sigma_1,\sigma_2}$ is $V_1 - \{x\}$ and hence $gn_x(K_{\sigma_1,\sigma_2}) = m - 1$. Similarly $gn_x(K_{\sigma_1,\sigma_2}) = n - 1$ if $x$ is in $V_2$.

**Note 2.13** Even if $x$ is an extreme vertex of $G$, Note 2.3 states that $x$ does not belong to the $gn_x - set$.

**Theorem 2.14** For any vertex $x$ in a connected non-trivial fuzzy graph $G: (V, \sigma, \mu)$, $gn_x - set$ is unique and it is contained in every $x$ −geodetic set of $G$.

**Proof:** Let's imagine there are two $gn_x - sets$, $S_1$ and $S_2$. Give $u$ the role of a vertex in $G$ such that $u \in S_1$ and $u \notin S_2$. There is a vertex $v \neq u$ in $G$ such that $v \in S_2$ and $v \notin S_1$ because $S_2$ is a $gn_x - set$ and $|S_2| = |S_1|$ . There is a vertex $w \in S_1$ such that $v \in I[x, w]$ exists since $S_1$ is a $gn_x - set$.

**Case 1:** Let's say $w \in S_2$. Since $v$ is an internal vertex of an $x - w$ geodesic and $S_2$ is a $gn_x - set$, this results in a contradiction that $v$ is not in $S_2$

**Case 2:** Let's say $w \notin S_2$. If $S_2$ is a $gn_x - set$, then there is an element $y \in S_2$ such that $w$ is on a $x - y$ geodesic say $P$. Say $Q$ because $v$ is located on an $x - w$ geodesic. So that $v \in I[x, y]$, the geodesic formed by the union of the geodesic $Q$ from $x$ to $w$ and the $w - y$ section of the geodesic $P$ is an $x - y$ geodesic. As a result, $v$ is an internal geodesic $x - y$ vertex. A contradiction results from the fact that $v$ is not in $S_2$ since $S_2$ is a $gn_x - set$.

Now assert that every $x$ −geodetic set of $G$ contains the $gn_x - set$. Let $y$ be a component of the $gn_x - set$ of $G$, let's say $S$. Since $S$ is the smallest, no other vertex of $G$ can make $y$ as $x$ −geodetic. It is contradictory for $y$ to be $x$ −geodetic by the vertex $v$ in $G$ if there is an $x$ −geodetic set, let's say $S'$, such that $y$ is $S'$. This is because $y$ lies on an $x - v$ geodesic for some $v$ in $S'$.

## 3. Conclusion

In this article, we studied the concept of vertex geodetic number of fuzzy graph. We extend the concept of other distance related parameters in fuzzy graph for future work.

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