Journal of Harbin Engineering University **Vol 44 No. 7**
ISSN: 1006-7043 **July 2023**

Partial Isometries and EP Matrices in Minkowski Space

D. Krishnaswamy[1] and J. Prabha[2]

[1]Department of Mathematics, Annamalai University, Annamalai Nagar, Tamil Nadu,(E-mail: krishna_swamy2004@yahoo.co.in)

[2]Department of Mathematics, Annamalai University, Annamalai Nagar, Tamil Nadu,(E-mail: krithika0392@gmail.com)

Abstract
In this paper, we discuss about the characterizations of partial isometries, range her-mitian, normal matrices and star-dagger in Minkowski space.

2020 AMS Classification: 15A09, 15A24, 15A57.

Keywords: EP Matrix, Minkowski Adjoint, Minkowski Inverse, Minkowski space, Partial isometry, Range Hermitian.

1. Introduction

Throughout we shall deal with $C^{n\times n}$, the space of n x n complex matrices. Let C^n be the space of complex n-tuples, we shall index the components of a complex vector in C^n from 0 to n-1, that is $u = (u_0, u_1, u_2, \ldots, u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, \ldots -u_{n-1})$. Clearly the Minkowski metrix matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}. \quad (1)$$

$G = G^*$ and $G^2 = I_n$. In [11], Minkowski inner product on C^n is defined by $(u, v) = [u, Gv]$, where $[.,.]$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called the Minkowski space and denoted as $\mathscr{M}$. For $A \in C^{n\times n}$, $x, y \in C^n$ by using (1),

$$(Ax, y) = [Ax, Gy] = [x, A^*Gy]$$

$$=[x, G(GA^*G)y] = [x, GA^\sim y] = (x, A^\sim y).$$

Where $A^\sim = GA^*G$. The matrix $A^\sim$ is called the minkowski adjoint of A in $\mathscr{M}$. Naturlly we call a matrix $A \in C^{n\times n}$ $\mathscr{M}$.-symmetric in M. if $A = A^\sim$.

For $A \in C^{n\times n}$, let A^*, $A^\sim$, $A^{\textcircled{m}}$ $A^\dagger$, R(A) and N(A) denote the conjugate transpose, Minkowski adjoint, Minkowaki inverse, Moore-Penrose inverse, range space and null space of a matrix A respectively. I_n denote the identity matrix of order $n X n$

Generalized inverses of matrices have important roles in theoretical and numerical methods of linear algebra. The most significant fact is that we can use generalized inverse of matrices, in the case when ordinary inverses do not exists, in order to solve some matrix equations. Similar rea- soning can be applied to linear (bounded or unbounded) operators on Banach and Hilbert spaces.

2. Preliminaries

Definition 2.1 *[7]* A^g is said to be generalized inverse of A, if $AA^gA = A$.

Definition 2.2 *[7]* A^r is said to be a reflexive generalized inverse of A if $AA^rA = A$ and $A^rAA^r = A^r$.

Definition 2.3 *[7]* For $A \in C^{n\times n}$, $A^{\textcircled{m}}$ is the Minkowski inverse of A if

$AA^{\textcircled{m}}A = A$, $A^{\textcircled{m}}AA^{\textcircled{m}} = A^{\textcircled{m}}$, $AA^{\textcircled{m}}$ and $A^{\textcircled{m}}A$ are m-symmetric.

Definition 2.4 *[7]* An operator $E \in C^{n\times n}$ is said to be EP if there exists $E^{\textcircled{m}}$ and $EE^{\textcircled{m}} = E^{\textcircled{m}}E$.

Definition 2.5 *[3]* An operator $E \in C^{n\times n}$ is said to be normal if $EE^\sim = E^\sim E$.

Definition 2.6 *[12]* Let A be an $m \times n$ complex matrix. Then range space of A is defined by⊂ (A) = $\{y \in C^m : y = Ax \text{ for some } x \in C^n\}$.

Definition 2.7 *[12]* Let A be an $m \times n$ complex matrix. Then null space of A is defined by

$N(A) = \{x \in C^n : Ax = 0\}$.

Theorem 2.8 *[3]* Let X be a Banach space and consider $E \in L(X)$ such that $E^{\textcircled{m}}$ exists and $E \in U(L(X))$ in M. Then the following statements hold.

(i)$R(E^\sim) \subseteq R(E)$ if and only if $E = EEE^{\textcircled{m}}$ in $\mathscr{M}$.

(ii) $N(E) \subseteq N(E^\sim)$ if and only if $E = E^{\textcircled{m}}EE$ in $\mathscr{M}$.

In addition, if the condition of statements (i) and (ii) are satisfied, then E is an EP operator in $\mathscr{M}$.

Proof: (i) Let us assume that $R(E^\sim) \subseteq R(E)$

To prove that $E = EEE^{\textcircled{m}}$.

since $R(E) = R(EE^{\textcircled{m}}) = N(I - EE^{\textcircled{m}})$

$R(E^\sim) \subseteq R(E)$ is equivalent to $E^\sim = EE^{\textcircled{m}}E^\sim$

Next consider $U, V \in H(L(X))$ such that $E = U + iV$

$E^{\sim} = U^{\sim} - iV^{\sim}$
$= U - iV$ (U, V are M-symmetry $U^{\sim} = U$, $V^{\sim} = V$)

Adding E and $E^{\sim}$, we get
$E + E^{\sim} = 2U$
$EE^{ⓜ}E + EE^{ⓜ}E^{\sim} = 2U$
$EE^{ⓜ}(E + E^{\sim}) = 2U$ $EE^{ⓜ}2U = 2U$
$EE^{ⓜ}U = U.$

$E - E^{\sim} = 2Vi$
$EE^{ⓜ}E - EE^{ⓜ}E^{\sim} = 2Vi$
$EE^{ⓜ}(E - E^{\sim}) = 2Vi$
$EE^{ⓜ}\ 2Vi = 2Vi$
$EE^{ⓜ}V = V.$
Therefore $U = EE^{ⓜ}U$ and $V = EE^{ⓜ}V$
In addition,
$UEE^{ⓜ} = EE^{ⓜ}U$ and $VEE^{ⓜ} = EE^{ⓜ}V$
$E = U + iV$
$= UEE^{ⓜ} + iVEE^{ⓜ}$
$(UEE^{ⓜ} - EE^{ⓜ}U) + i(VEE^{ⓜ} - EE^{ⓜ}V) = 0$ (2)
However, since $V, EE^{ⓜ} \in H(L(X))$,
$i(VEE^{ⓜ} - EE^{ⓜ}V) \in H(L(X))$
In addition,
$(UEE^{ⓜ} - EE^{ⓜ}VU) = -i(VEE^{ⓜ} - EE^{ⓜ}V) \in H(L(X))$.
Multiplying by to –i equation (2), we obtain
$(VEE^{ⓜ} - EE^{ⓜ}V) + i(EE^{ⓜ}U - UEE^{ⓜ}) = 0$

Hence $(VEE^{m} - EE^{ⓜ}V) \in H(L(N(E)))$.
However, $UE\ E^{m} = E\ E^{ⓜ}U$ and $VEE^{ⓜ} = E\ E^{ⓜ}V$.
Consequently, since $EE^{ⓜ}$ is an idempotent, $R(E) = R(EE^{ⓜ})$ and
$N(E^{ⓜ}) = N(EE^{ⓜ})$ are closed invarient subspaces both for U and V.
Consider $U' = U \in L(N(E^{ⓜ}))$ and $V' = V \in L(N(E^{ⓜ}))$.

$U', V'\ H(L(N(E^{ⓜ})))$.

If E' the restriction of E to $N(E^{ⓜ})$ then it is clear that $E' = U' + iV'$.
To prove that $N(E) \subseteq N(E^{\sim})$

from we know from $E = EE^{ⓜ}E$

$E + E^{\sim} = U + iV + U^{\sim} - iV^{\sim}$
$= U + iV + U - iV$

Substracting E and $E^{\sim}$, we get
$E - E^{\sim} = U + iV - U^{\sim} + iV^{\sim}$
$= U + iV - U + iV$ (U, V are M-symmetry $U^{\sim} = U$, $V^{\sim} = V$)

$= (U + iV)EE^{ⓜ}$
$= EEE^{ⓜ}$.
Conversely,

Let us assume that $E = EEE^{ⓜ}$. To prove that $R(E^{\sim} \subseteq R(E)$.
If $E = EEE^{ⓜ}$ then $U + iV = (U + iV)EE^{ⓜ} = EE^{ⓜ}(U + iV)$.
In particular

However, since $E = EE\ E^{ⓜ}$, $N(E^{ⓜ}) \subseteq N(E)$ which according toimplies that $U' = V' = 0$.
In particular, $E^{\sim}(N(E^{ⓜ})) = 0$. It is clear that $E^{\sim}(R(E)) \subseteq R(E)$.
Therefore $R(E^{\sim})$ $R(E)$.

(ii) Let us assume that $N(E) \subseteq N(E^{\sim})$.
To prove that $E = .\ E^{ⓜ}EE$
Since $N(E) = N(E^{ⓜ}E) = R(I - E^{ⓜ}E)$, $N(E) \subseteq N(E^{\sim})$ is equivalent to $E^{\sim} = E^{\sim}E^{ⓜ}E$.
Now as in the proof of statement (i), if E = U+i V and $E^{\sim}$= U-i V with U,V $\in H(L(N(E)))$
Adding and subtracting E and $E^{\sim}$ and using $E = EE^{ⓜ}E$, it is then clear that $U = UE^{ⓜ}E$ and $V = VE^{ⓜ}E$
In particular, $UE^{ⓜ}E$, $VE^{ⓜ}E\ H(L(N(E)))$. Again
$UE^{ⓜ}E = E^{ⓜ}EU$ and $VE^{ⓜ}E = E^{ⓜ}EV$.
$E = U + iV$
$= UE^{O}E + iVE^{ⓜ}E$
$= E^{ⓜ}E(U + iV)$
$= E^{ⓜ}EE$.
Conversely,

Let us assume that $E = E^{ⓜ}EE$.

By using statement (i) but considering

that $UE^{\textcircled{\#}}E = E^{\textcircled{\#}}EU$ and $VE^{\textcircled{\#}}E = E^{\textcircled{\#}}EV$.
As a result, since $E^{\textcircled{\#}}E$ is an idempotent such that $N(E^{\textcircled{\#}}E) = N(E)$.
Hence $N(E)$ is a closed invarient subspaces for U
However, since $E(N(E)) = 0$, $\tilde{U}=\widetilde{V} = 0$.
Consequently, $E^{\sim}(N(E)) = 0$ equivalently $N(E) \subseteq N(E^{\sim})$.
Notice that $R(E^{\sim}) \subseteq R(E)$ is equvalient to $E^{\sim} = EE^{\textcircled{\#}}E^{\sim}$,

by $R(E) = R(EE^{\textcircled{\#}}) = N(IEE^{\textcircled{\#}})$.

3. Partial Isometry and EP Matrices

In this section, we establish some characterizations of partial isometrices in Minkowski space.

Theorem 3.1 *Let E* be a unital Banach algebra and consider $E \in U(E)$ such that $E^{\textcircled{\#}}$ and $E^{\#}$ exists in $\mathscr{M}$. then the following statements are equivalent
(i)E is a partial isometry in $\mathscr{M}$
(ii)$E^{\#}E^{\sim} E = E^{\#}$
(iii)$EE^{\sim} E^{\#} = E$

Proof: (i)⟹(ii):
Since E is a partial isometry in $\mathscr{M}$, $EE^{\sim}E = E$
To prove that $E^{\#}E^{\sim}E = E^{\#}$.Now,
$E^{\#}E^{\sim}E = (E^{\#})^2EE^{\sim}E$
$= (E^{\#})^2E$
$= E^{\#}$.
(ii) ⇒ (i):
Let us assume that $E^{\#}E^{\sim} E = E^{\#}$.
To prove that E is a partialisometry in $\mathscr{M}$.
Consider,
$EE^{\sim}E = E^2(E^{\#}E^{\sim}E)$
$= E^2E^{\#}$
$= E$.
(i) ⇒ (iii):
Let us assume that E is a partial isometry in $\mathscr{M}$, $EE^{\sim}E = E$
To prove that $EE^{\sim}E^{\#} = E^{\#}$.Now,
$EE^{\sim}E^{\#} = EE^{\sim}E(E^{\#})^2$
$= E(E^{\#})^2$
$= E^{\#}$.
(iii) ⇒ (i):
Let us assume that $EE^{\sim}E^{\#} = E^{\#}$.
To prove that E is a partial isometry in $\mathscr{M}$

and V.
Consider $\tilde{U}= U$, $\tilde{V}= V \in L(N(E))$.
$\tilde{U}, \tilde{V} \in H(L(N(E)))$.
~ ~

The condition $N(E) \subseteq N(E^{\sim})$. Is equivalent to $E^{\sim} =.E^{\sim} E^{\textcircled{\#}}E.$ because

$$N(E)=N(E^{\textcircled{\#}}E)=R(I- E^{\textcircled{\#}}E)$$

$EE^{\sim}E = (EE^{\sim}E^{\#})E^2$
$= E^{\#}E^2$
$= E.$

Theorem 3.2 Let X ba a Banach space and consider $E \in \mathscr{L}(X)$ such that $E^{\textcircled{\#}}$ and $E^{\#}$ exists and let $E \in U(\mathscr{L}(X))$in $\mathscr{M}$ then the following statements are equivalent :
(i) E is a partial isometry and EP in $\mathscr{M}$
(ii) E is a partial isometry and normal in $\mathscr{M}$
(iii) $E^{\sim} = E^{\#}$,
(iv) $EE^{\sim}= E^{\textcircled{\#}}E$ and $E=EEE^{\textcircled{\#}}$,
(v) $E^{\sim}E = EE^{\textcircled{\#}}$ and $E=E^{\textcircled{\#}}EE$,
(vi) $EE^{\sim}= EE^{\#}$ and $E =EEE^{\textcircled{\#}}$,
(vii) $E^{\sim}E = EE^{\#}$ and $EE^{\textcircled{\#}}EE$,
(vii) $E^{\sim}E^{\textcircled{\#}} = E^{\textcircled{\#}}E^{\#}$,
(ix) $E^{\textcircled{\#}}E^{\sim} = E^{\#}E^{\textcircled{\#}}$,
(x) $E^{\textcircled{\#}}E^{\sim} = E^{\textcircled{\#}}E^{\#}$ and $E = EEE^{\textcircled{\#}}$,
(xi) $E^{\sim}E^{\textcircled{\#}} = E^{\#}E^{\textcircled{\#}}$ and $E = E^{\textcircled{\#}}EE$,
(xii) $E^{\sim}E^{\#} = E^{\#}E^{\textcircled{\#}}$ and $E = E^{\textcircled{\#}}EE$,
(xiii) $E^{\sim}E^{\textcircled{\#}} = E^{\#}E^{\#}$ and $E = E^{\textcircled{\#}}EE$,
(xvi) $E^{\sim}E^{\#} = E^{\#}E^{\#}$ and $E = E^{\textcircled{\#}}EE$,
(xv) $EE^{\sim}E^{\#} = E^{\textcircled{\#}}$ and $E = E^{\textcircled{\#}}EE$,
(xvi) $E^{\sim}E^2 = E$ and $E = E^{\textcircled{\#}}EE$,
(xvii) $E^2E^{\sim} = E$ and $E = EEE^{\textcircled{\#}}$,
(xviii) $EE^{\textcircled{\#}}E^{\sim} = E^{\#}$ and $E = EEE^{\textcircled{\#}}$,
(xix) $E^{\sim}E^{\textcircled{\#}}E=E^{\#}andE=E^{\textcircled{\#}}EE$

Proof: (i) ⇒(ii):
Let us assume that E is a partial isometry and EP in $\mathscr{M}$ .It is enough to prove that E is normal in $\mathscr{M}$ alone.

Since $E$is EP $E = EEE^{\textcircled{\#}}$, $E^{\sim} = EE^{\textcircled{\#}}E^{\sim}$.
since E is a partial isometry, we have
$EE^{\sim}E^{\#} = EE^{\sim}E(E^{\#})^2$ $[E^{\#} = E(E^{\#})^2]$

$= (EE^{\sim}E)(E^{\#})^2$
$= E(E^{\#})^2$
$= E^{\#}$
$E^{\sim}E^{\#}E = EE^{\dagger} E^{\sim}E^{\#}$ E
$= E^{\#}$.

Thus $EE^{\sim}E^{\#} = E^{\sim}E^{\#}E$ and $EE^{\dagger}EE \Rightarrow E$ is normal
(ii)⇒(iii):

Let us assume that E is a partial isometry and normal in $\mathscr{M}$.

⇒

.To prove that $E^{\sim} = E^{\#}$.
The condition E is normal ⇒ $E^{\sim} = EE^{\sim}E^{\#}$.
Because E is a partial isometry, we have
$E^{\sim} = EE^{\sim}E^{\#}$
$= EE^{\sim}E(E^{\#})^2$
$EE^{\sim} = EE^{\#}$
$= E^{\#}E$ [E is an EP, $EE^{\dagger} = E^{\dagger}E$]

$= E^{\sim}E$
Therefore $EE^{\sim} = E^{\sim}E$
$EE^{\sim}E = EE^{\#}E$ [$E^{\sim} = E^{\#}$]
$= E$.
E is normal gives E is EP. The condition (i) is satisfied. ⇒
(ii) ⇒ (iv):
Let us assume that E is a partial isometry and normal in $\mathscr{M}$.
.To prove that $EE^{\sim} = E^{\dagger}E$ and $E = EEE^{\dagger}$.
E is normal gives $EE^{\sim}E^{\#} = E^{\#}EE^{\sim}$ and $E =$
(iv) ⇒ (vi):
Let us assume that $EE = E^{\dagger}E$
To prove that $EE^{\sim} = EE^{\#}$ and $E = EEE^{\dagger}$
Consider,
$E^{\#}(EE^{\sim}) = E^{\#}E^{\dagger}$
$= (E^{\#})^2EE^{\dagger}E$ [$E^{\#} = (E^{\#})^2E$]
$= E^{\#}$
Similarly,
$EE^{\sim}E^{\#} = E(EE^{\sim}E^{\#})E^{\#}$ [$E^{\sim} = EE^{\sim}E^{\#}$]
$E(E^{\#}EE^{\sim})E^{\#}$ [$EE^{\sim}E^{\#}$
E is normal and E is EP.
Hence $EE^{\sim} = E^{\dagger}E = EE^{\dagger} = EE^{\#}$
Let us assume that $EE^{\sim} = EE^{\#}$ and $E = EEE^{\dagger}$
To prove that E is a partial isometry and normal in $\mathscr{M}$.
Now,
$E(EE^{\sim}) = EEE^{\#}$

$= E^{\dagger}EE^{\sim}EE^{\#}$ [E is an EP, $EE^{\dagger} = E^{\dagger}E$]
$= E$ $(EE^{\sim}E)E^{\#}$
$= E$ $EE^{\#}$
$= E^{\#}EE^{\#}$
.

$= E(E^{\#})^2$
$= E^{\#}$
(iii) (i):
Let us assume that $E^{\sim} = E^{\#}$.
To prove that E is a partial isometry and EP in $\mathscr{M}$.
Since $E^{\sim} = E^{\#}$, we get

$EEE^{\dagger}$
Now, $EE^{\sim} = E(E^{\#}EE^{\sim})$
$= E(EE^{\sim}E^{\#})$ [$EE^{\sim}E^{\#} = E^{\#}EE^{\sim}$]
$= (EEE^{\sim})E^{\#}$ [$EEE^{\sim} = EE^{\sim}E$]
$= (EE^{\sim}E)E^{\#}$
$= EE^{\#}$.
since E is normal E is EP, then
$EE^{\sim} = EE^{\#}$
$= E^{\#}E$ [since E is normal, $EE^{\#} = E^{\#}E$]
$= E^{\dagger} E$. [$E^{\#} = E^{\dagger}$]

and $E = EEE^{\dagger}$.
m

E

$= E^{\#}EE^{\sim}$]
$= EE^{\#}E^{\#}$
$= E(E^{\#})^2$
$= E^{\#}$. [$E^{\#} = E(E^{\#})^2$]
and $E^{\#}EE^{\sim} = EE^{\sim}E^{\#}$

$= E^2E^{\#}$
$= E$ [$E = E^2E^{\#}$]
$= (EE^{\#})E$
$= EE^{\sim}E$
Hence E is a partial isometry and normal.

(ii) ⇒ (v) ⇒ (vii) ⇒ (ii): these implications can be proved in the same manner as(ii) ⇒ (iv) ⇒ (vi) ⇒

(viii) ⇒ (xi): suppose that, $E^{\sim}E^{\textcircled{m}}$

To prove that $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\textcircled{m}}$ and $E = E^{\textcircled{m}}EE$.

Consider,

$EE^{\#} = EE(E^{\#})^2$ $[E^{\#} = E(E^{\#})^2]$

$= EE\ E^{\textcircled{m}}E(E^{\#})^2$

$= EE(E^{\textcircled{m}}E^{\#})$ $[E^{\#} = E(E^{\#})^2]$

$= EEE^{\sim}E^{\textcircled{m}}$

$= EE(E^{\sim}E^{\textcircled{m}})EE^{\textcircled{m}}$

$= EEE^{\textcircled{m}}E^{\#}EE^{\textcircled{m}}$

(xi) ⇒ (xvi):

Let us assume that

$E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\textcircled{m}}$ and $E = E^{\textcircled{m}}EE$

To prove that $E^{\sim}E^2 = E$ and $E = E^{\textcircled{m}}EE$.

The assumptions $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\textcircled{m}}$ and $E = E^{\textcircled{m}}EE$.

Now,

$E^{\sim}E^2 = (E^{\sim}\ E^{\textcircled{m}})EE^2$

$= E^{\#}E^{\textcircled{m}}EE^2$

$= (E^{\#})^2EE^{\textcircled{m}}EE^2$ $[E^{\#} = (E^{\#})^2E]$

$= E^{\#}E^{\textcircled{m}}EE^2$

$= E^{\#}E^{\textcircled{m}}EEE$

$= E^{\#}EE$

$= E^{\#}E^2$

$= E.$ $[E = E^{\#}E^2]$ ⇒

(xvi) ⇒ (xiv):

Let us assume that $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\#}$ and $E = E^{\textcircled{m}}EE$

To prove that $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\textcircled{m}}$ and $E = E^{\textcircled{m}}EE$

Multiplying the equality $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\#}$ by $EE^{\textcircled{m}}$

(xii) ⇒ (vii):

Let us assume that $E^{\sim}E^{\#} = E^{\#}E^{\textcircled{m}}$ and $E =$

(ii)

(i)⇒ (viii):is an immediate consequence

$= E^{\textcircled{m}}E^{\#}$

$= EEE^{\textcircled{m}}EE^{\#}E^{\textcircled{m}}$ $[EE^{\#}E^{\textcircled{m}} = E^{\textcircled{m}}EE^{\#}]$

$= EE^{\textcircled{m}}$ $[E^{\textcircled{m}} = E^{\textcircled{m}}EE^{\#}]$

Hence, $EE^{\#}$ is hermitian and E is EP.Now condition (xi) is satisfied by

$E^{\sim}E^{\textcircled{m}} = E^{\textcircled{m}}E^{\#}$

$= E^{\#}E^{\textcircled{m}}$

$[E^{\textcircled{m}}E^{\#} = E^{\#}E^{\textcircled{m}}]$

and $E^{\textcircled{m}}EE = EE^{\textcircled{m}}E = E$.

Let us assume that

$E^{\sim}E^2 = E$ and $E = EE^{\textcircled{m}}E$.

To prove that

$E^{\sim}E^{\#} = E^{\#}E^{\#}$ and

$E = EE^{\textcircled{m}}E$.

Multiplying $E^{\sim}E^2 = E$ by $(E^{\#})^3$ from the right side, we get

$E^{\sim}E^2(E^{\#})^3 = E(E^{\#})^3$

$E^{\sim}E^2(E^{\#})^2E^{\#} = E(E^{\#})^2E^{\#}$

$E^{\sim}EE(E^{\#})^2E^{\#} = E^{\#}E^{\#}$

$E^{\sim}EE^{\#}E^{\#} = E^{\#}E^{\#}$ $[E^{\#} = E(E^{\#})^2]$

$E^{\sim}E(E^{\#})^2 = E^{\#}E^{\#}$

$E^{\sim}E^{\#} = E^{\#}E^{\#}$.

Hence E satisfies condition (xiv).

(xiv) (xii):

Let us assume that $E^{\sim}E^{\#} = E^{\#}E^{\#}$ and $E = E^{\textcircled{m}}EE$

$E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\#}\ EE^{\textcircled{m}}$

$= (E^{\#})^2EE^{\textcircled{m}}$

$= E^{\#}E^{\textcircled{m}}$ $[E^{\#} = (E^{\#})^2E]$

Hence $E^{\sim}E^{\textcircled{m}} = E^{\#}E^{\textcircled{m}}$. So, we deduce that condition (xi) holds.

$E^{\textcircled{m}}EE$

To prove that $E^{\sim}E = EE^{\#}$ and $E = E^{\dagger}EE$
Now,
$E^{\sim}E = (E^{\sim}E^{\#})E^2$ [$E = E^{\#}E^2$] m
$= E^{\#}(E^{0}E^2)$ [$E = E^{\dagger}E^2$]
$= E^{\#}E.$
(i) ⇒ (ix) ⇒ (x) ⇒ (xvii): similarly as (i) ⇒ (viii) ⇒ (xi)=⇒ (xvi).
(xiii)⇒ (xi):
Let us assume that $E^{\sim}E^{\dagger} = E^{\#}E^{\#}$ and $E = E^{\dagger}EE$
To prove that $E^{\sim}E^{\dagger} = E^{\#}E^{\dagger}$ and $E = E^{\dagger}EE$
Multiplying the equality $E^{\sim}E^{\dagger} = E^{\#}E^{\#}$ by $EE^{\dagger}$ from right side, we get
$E^{\sim}E^{\dagger} = E^{\#}E^{\#}\ EE^{\dagger}$
$= (E^{\#})^2EE^{\dagger}$
$= E^{\#}E^{\dagger}$ [$E^{\#}=(E^{\#})^2E$]
Hence $E^{\sim}E^{\dagger} = E^{\#}E^{\dagger}$

So we deuce that condition(xi) holds.

(xi)⇒ (xiii):
By (xi), we have that E is EP and condition (xiii) is satisfied.

(xv)⇒ (i):
Let us assume that $EE^{\sim}E^{\#} = E^{\dagger}$ and $E = E^{\dagger}EE$.
To prove that E is partial isometry and EP in $\mathscr{M}$
Now,
(xviii) ⇒ (iii):
Let us assume that $EE^{\dagger}$

To prove that $E^{\sim} = E^{\#}$.
Now,
$E^{\sim} = EE^{\dagger}E^{\sim}$
$= E^{\#}$.
(iii) ⇒ (xviii):
Let us assume that $E^{\sim} = E^{\#}$
To prove that $EE^{\dagger}E^{\sim} = E^{\#}$ and $E = EEE^{\dagger}$
Now,

(xvii) (vi):
Let us assume that $E^2E^{\sim} = E$ and $E = EEE^{\dagger}$
To prove that
$EE^{\sim} = EE^{\#}$ and $E = EE\ E^{\dagger}$
Consider,
$EE^{\sim} = E^{\#}E^2E^{\sim}$
$= E^{\#}E.$
and the condition (vi) is satisfied.

$EE^{\sim}E = (EE^{\sim}E^{\#})\ E^2$
$= E^{\dagger}EE$
=E
$E^{\dagger} = EE^{\sim}E^{\#}$
$= E^{\#}E^2E^{\sim}E(E^{\#})^2$ [E= $E^{\#}E^2$]
$= E^{\#}E(EE^{\sim}E)(E^{\#})^2$ [since E is a partial isometry, $EE^{\sim}E=E$]
$= E^{\#}EE(E^{\#})^2$
$= E^{\#}EE^{\#}$ [$E^{\#} = E(E^{\#})^2$]
$= E^{\#}$.
Therefore, E is a partial isometry and EP in $\mathscr{M}$

(i) (xv):
Let us assume that E is a partial isometry and EP in $\mathscr{M}$To prove that $EE^{\sim}E^{\#} = E^{\dagger}$ and $E = E^{\dagger}EE$.
The hypothesis E is EP gives $E = E^{\dagger}EE$ and because (i) implies (iii).
$EE^{\sim}E^{\#} = EE^{\#}E^{\#}$
$= E(E^{\#})^2$
$= E^{\#}$ [$E^{\#} = E(E^{\#})^2$]
$= E^{\dagger}$

$E^{\sim} = E^{\#}$ and $E = EEE^{\dagger}$.

$EE^{\dagger}E^{\sim} = EE^{\dagger}E^{\#}$
$= EE^{\dagger}\ E(E^{\#})^2$ [$E^{\#} = E(E^{\#})^2$]
$= E^{\#}$.
and E is EP $E = EEE^{\dagger}$
It is clear that
(iii) ⇔ (xix): Analogy as (iii) ⇔ (xviii)

References

[1] A. Ben-Israel and T. N. E. Greville, Generalized inverse : Theory and applications, *wiley and Sons*, New York, 1974. doi:10.1112/plms/s3-24.1.101
[2] E. Berkson, Hermitian projections and orthogonality in Banach spaces, *Proc.London Math.Soc.*,**24**,(3)(1972), 101–118. doi:10.48550/arxiv.1307.0750
[3] E. Boasso and V.Rakocevic, Characterizations of EP and normal Banach algebra elements, *Linear Algebra and Its Applications* **435**,(2011), 342–353.
[4] H. R. Dowson, Spectral theory of linear operators, *Academic press Inc. London-New york- sanfrancisco*, (1978).
[5] A.R. Meenakshi, Generalized inverses of matrices in Minkowski space, *in Proc. Nat. SeminarAlg. Appln. Annamalai University, Annamalainagar* (2000), 1–14 .
[6] A.R. Meenakshi and D. Krishnaswamy, On Sums of Range Symmetric Matrices in Minkowski space, *Bull. Malays. Math. Sci. Soc* **25**, (2)(2002), 137–148. S
[7] A.R. Meenakshi, Range Symmetric Matrices in Minkowski space, *Bull. Malays. Math. Sci.Soc (1), Bull. Malays. Math. Sci. Soc (2)*, (2000), 45– 52.
[8] A.R. Meenakshi and D. Krishnaswamy, Product of Range symmetric Block matrices inMinkowski Space, *Bull. Malays. Math. Sci. Soc* **29** (1),(2006), 59–68.
[9] D. Mosic, D.S. Djordjevic, Partial isometrices and EP elements in rings with involution, *Electron.J. Linear Algebra* **1**,(2009), 761–772.
[10] D. Mosic, D.S. Djordjevic, Partial isometrices and EP elements in Banach algebras, *Abstract and Applied Analysis*,(2011).
[11] M. Renardy, Singular value decomposition in Minkowski space , *Linear Algebra Appl.*,**236**,(1996), 53– 58.
[12] M. Radojevic, Nis, New Results for EP Matrices in indefinite inner product spaces, *Czechoslovak Mathematical Journal*, **139**, (2014), 91– 103